RIEMANN-STIELTJES INTEGRAL INEQUALITIES RELATED TO STEKLOFF'S RESULT

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Abstract. In this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|^2 du(t)$ in the case that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $\int_a^b f(t)\,dt = 0$, $f' \in L_2[a,b]$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing. Applications for Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

1. Introduction

It is well known that, see for instance [5], or [12], if $u \in C^1([a,b],\mathbb{R})$, namely u is continuous on $[a,b]$ and has a derivative that is continuous on (a,b) and satisfies $u(a) = u(b) = 0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K \sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [10]-[14].

In 1901, W. Stekloff, [16], proved that, if $u \in C^1([a,b],\mathbb{R})$ and $\int_a^b u(t)\,dt = 0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2} \int_a^b [u'(x)]^2\,dx. \tag{1.3}$$

In addition, if $u(a) = u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2} \int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h : [a,b] \to \mathbb{C}$.

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Theorem 1. *If $h \in C^1([a,b],\mathbb{C})$ and $\int_a^b h(t)\,dt = 0$, then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.5}$$

In addition, if $h(a) = h(b)$, then

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u = \operatorname{Re} h$ and $u = \operatorname{Im} h$ and by adding the corresponding inequalities.

In the recent paper we obtained the following weighted version of the above results:

Theorem 2. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{C})$. If $\frac{f'}{\sqrt{w}} \in L_2[a,b]$ and $\int_a^b f(t)\,w(t)\,dt = 0$, then*

$$\int_a^b |f(t)|^2\,w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.7}$$

In addition, if $f(a) = f(b)$, then we have the better inequality

$$\int_a^b |f(t)|^2\,w(t)\,dt \le \frac{1}{4\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.8}$$

Motivated by the above results, in this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|^2\,du(t)$ in the case that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $\int_a^b f(t)\,dt = 0$, $f' \in L_2[a,b]$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing. Applications for Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

2. Main Results

We have the following Riemann-Stieltjes integral inequality:

Theorem 3. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $\int_a^b f(t)\,dt = 0$ and $f' \in L_2[a,b]$. If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then*

$$\begin{aligned} &\int_a^b |f(t)|^2\,du(t) \\ &\le \left[\frac{1}{12}(b-a)[u(b)-u(a)] + \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)^2 du(t)\right]\int_a^b |f'(s)|^2\,ds \\ &= \left[\frac{1}{3}(b-a)[u(b)-u(a)] - \frac{2}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b |f'(s)|^2\,ds. \end{aligned} \tag{2.1}$$

The inequality is sharp.

Proof. We use Montgomery identity

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{b-a}\int_a^b k(t,s)\,f'(s)\,ds, \tag{2.2}$$

where

$$k(t,s) = \begin{cases} s-a, & s \in [a,t], \\ s-b, & s \in (t,b]. \end{cases}$$

Since $\int_a^b f(s)\,ds = 0$, hence by Cauchy-Bunyakowsky-Schwarz inequality

$$|f(t)|^2 = \frac{1}{(b-a)^2}\left|\int_a^b k(t,s) f'(s)\,ds\right|^2 \leq \frac{1}{(b-a)^2}\int_a^b |k(t,s)|^2\,ds \int_a^b |f'(s)|^2\,ds. \tag{2.3}$$

Observe that

$$\int_a^b |k(t,s)|^2\,ds = \int_a^t (s-a)^2\,ds + \int_t^b (s-b)^2\,ds = \frac{1}{3}\left[(t-a)^3 + (b-t)^3\right],$$

then by (2.3) we get

$$|f(t)|^2 \leq \frac{1}{3(b-a)^2}\int_a^b |f'(s)|^2\,ds \left[(t-a)^3 + (b-t)^3\right]$$

for $t \in [a,b]$.

Taking the Riemann-Stieltjes integral we get

$$\int_a^b |f(t)|^2\,du(t) \leq \frac{1}{3(b-a)^2}\int_a^b |f'(s)|^2\,ds \int_a^b \left[(t-a)^3 + (b-t)^3\right] du(t). \tag{2.4}$$

Since

$$\frac{1}{3}\left[\left(\frac{t-a}{b-a}\right)^3 + \left(\frac{b-t}{b-a}\right)^3\right] = \frac{1}{12} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2,$$

hence

$$\frac{1}{3}\int_a^b \left[\left(\frac{t-a}{b-a}\right)^3 + \left(\frac{b-t}{b-a}\right)^3\right] du(t) = \frac{1}{12}\left[u(b)-u(a)\right] + \frac{1}{(b-a)^2}\int_a^b \left(t-\frac{a+b}{2}\right)^2 du(t)$$

which, by (2.4), proves the first part of (2.1).

Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\int_a^b \left[(t-a)^3 + (b-t)^3\right] du(t) \\ &= \left[(t-a)^3 + (b-t)^3\right] u(t)\Big|_a^b - 3\int_a^b \left[(t-a)^2 - (b-t)^2\right] u(t)\,dt \\ &= (b-a)^3 u(b) - (b-a)^3 u(a) - 3(b-a)\int_a^b (2t-a-b)\,u(t)\,dt \\ &= (b-a)^3\left[u(b)-u(a)\right] - 6(b-a)\int_a^b \left(t-\frac{a+b}{2}\right) u(t)\,dt. \end{aligned} \tag{2.5}$$

If we use (2.4) and (2.5) we derive the desired result (2.1).

The sharpness of the inequality follows by Corollary 2 below. □

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $\int_a^b f(t)\,dt = 0$ *and* $f'\in L_2[a,b]$. *If* $w:[a,b]\to(0,\infty)$ *is integrable on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$, *then*

$$\begin{aligned}(2.6)\qquad &\int_a^b |f(t)|^2 w(t)\,dt \\ &\le \left[\frac{1}{12}(b-a)+\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)^2 w(t)\,dt\right]\int_a^b |f'(s)|^2\,ds.\end{aligned}$$

Remark 1. *Assume that* $g:[a,b]\to\mathbb{R}$ *is absolutely continuous and* $g'\in L_2[a,b]$. *By taking*

$$f=g-\frac{1}{b-a}\int_a^b g(s)\,ds$$

in (2.1) we obtain

$$\begin{aligned}(2.7)\quad &\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 du(t) \\ &\le\left[\frac{1}{12}(b-a)[u(b)-u(a)]+\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)^2 du(t)\right]\int_a^b|g'(s)|^2\,ds \\ &=\left[\frac{1}{3}(b-a)[u(b)-u(a)]-\frac{2}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b|g'(s)|^2\,ds,\end{aligned}$$

provided that $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing.*

Using Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic integrators, we have

$$\begin{aligned}&\left|\int_a^b g(t)\,du(t)-\frac{u(b)-u(a)}{b-a}\int_a^b g(s)\,ds\right|^2 \\ &\le[u(b)-u(a)]\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 du(t).\end{aligned}$$

Therefore by (2.7) we derive

$$\begin{aligned}(2.8)\qquad &\left|\int_a^b g(t)\,du(t)-\frac{u(b)-u(a)}{b-a}\int_a^b g(s)\,ds\right|^2 \\ &\le\left[\frac{1}{12}(b-a)[u(b)-u(a)]+\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)^2 du(t)\right] \\ &\times\int_a^b|g'(s)|^2\,ds.\end{aligned}$$

If $w:[a,b]\to(0,\infty)$ *is integrable on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$, *then we have the following result comparing the weighted integral mean with the integral mean,*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,w(t)\,dt-\frac{1}{b-a}\int_a^b g(s)\,ds\right| \tag{2.9}$$
$$\leq\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 w(t)\,dt$$
$$\leq\left[\frac{1}{12}+\frac{1}{(b-a)^2}\int_a^b\left(t-\frac{a+b}{2}\right)^2 w(t)\,dt\right]\int_a^b\left|g'(s)\right|^2 ds.$$

Corollary 2. *Assume that* $g:[a,b]\to\mathbb{R}$ *is absolutely continuous and* $g'\in L_2[a,b]$, *then*

$$\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2\leq\frac{1}{12}(b-a)\int_a^b\left|g'(s)\right|^2 ds. \tag{2.10}$$

The constant $\frac{1}{12}$ *is best possible in (2.10).*

Proof. We use the inequality (2.1) for the nondecreasing function $u(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$. Integrating by parts in the Riemann-Stieltjes integral,

$$\int_a^b f^2(t)\,du(t)$$
$$=f^2(t)\,u(t)\big|_a^b-2\int_a^b f(t)\,f'(t)\,u(t)\,dt$$
$$=f^2(b)\,u(b)-f^2(a)\,u(a)-2\left(-\int_a^{\frac{a+b}{2}} f(t)\,f'(t)\,dt+\int_{\frac{a+b}{2}}^b f(t)\,f'(t)\,dt\right)$$
$$=f^2(b)+f^2(a)+2\left(\int_a^{\frac{a+b}{2}} f(t)\,f'(t)\,dt-\int_{\frac{a+b}{2}}^b f(t)\,f'(t)\,dt\right).$$

Since

$$\int_a^{\frac{a+b}{2}} f(t)\,f'(t)\,dt=\frac{1}{2}\int_a^{\frac{a+b}{2}} d\left(f^2(t)\right)=\frac{1}{2}\left[f^2\left(\frac{a+b}{2}\right)-f^2(a)\right]$$

and

$$\int_{\frac{a+b}{2}}^b f(t)\,f'(t)\,dt=\frac{1}{2}\int_{\frac{a+b}{2}}^b d\left(f^2(t)\right)=\frac{1}{2}\left[f^2(b)-f^2\left(\frac{a+b}{2}\right)\right],$$

hence

$$\int_a^b f^2(t)\,du(t)$$
$$=f^2(b)+f^2(a)+f^2\left(\frac{a+b}{2}\right)-f^2(a)-f^2(b)+f^2\left(\frac{a+b}{2}\right)$$
$$=2f^2\left(\frac{a+b}{2}\right).$$

Also

$$\frac{1}{3}(b-a)[u(b)-u(a)]-\frac{2}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)u(t)\,dt$$
$$=\frac{2}{3}(b-a)-\frac{2}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{2}{3}(b-a)-\frac{1}{2}(b-a)$$
$$=\frac{1}{6}(b-a).$$

Then by (2.1) we get

$$f^2\left(\frac{a+b}{2}\right)\le\frac{1}{12}(b-a)\int_a^b|f'(s)|^2\,ds.$$

Finally, if we take $f=g-\frac{1}{b-a}\int_a^b g(s)\,ds$, then we get the desired result (2.10).

Consider the function

$$f_0(t):=\begin{cases}\frac{1}{2}(t-a)^2, & t\in\left[a,\frac{a+b}{2}\right],\\[2mm] \frac{1}{2}(t-b)^2, & t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Then

$$f_0\left(\frac{a+b}{2}\right)=\frac{1}{8}(b-a)^2$$

and

$$f_0'(t):=\begin{cases}t-a, & t\in\left(a,\frac{a+b}{2}\right),\\[2mm] t-b, & t\in\left(\frac{a+b}{2},b\right).\end{cases}$$

We have

$$\int_a^b f_0(t)\,dt=\frac{1}{2}\int_a^{\frac{a+b}{2}}(t-a)^2\,dt+\frac{1}{2}\int_{\frac{a+b}{2}}^b(t-b)^2\,dt=\frac{(b-a)^3}{24},$$

$$\int_a^b|f'(t)|^2\,dt=\int_a^{\frac{a+b}{2}}(t-a)^2\,dt+\int_{\frac{a+b}{2}}^b(t-b)^2\,dt=\frac{(b-a)^3}{12}$$

and

$$(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt=\frac{1}{8}(b-a)^3-\frac{(b-a)^3}{24}=\frac{(b-a)^3}{12}.$$

Therefore

$$\left|(b-a)f\left(\frac{a+b}{2}\right)-\int_a^b f(t)\,dt\right|^2=\frac{(b-a)^6}{144}$$

and

$$\frac{1}{12}(b-a)^3\int_a^b|f'(t)|^2\,dt=\frac{(b-a)^6}{144},$$

which proves the sharpness of the constant $\frac{1}{12}$. □

The following lemma was obtained by the author in 2007, [7] and is of interest in itself as well (see also [6]):

Lemma 1. *If $p:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dV(t) \le \left(\int_a^b |p(t)|^p\,dV(t)\right)^{1/p}\left(\bigvee_a^b(v)\right)^{1/q} \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{2.11}$$

where $V(t):=\bigvee_a^t(v)$ is the total variation of v on $[a,t]$ with $t\in[a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a)=0$ and $V(b)=\bigvee_a^b(v)$.

Theorem 4. *Assume that $g:[a,b]\to\mathbb{C}$ is absolutely continuous with and $g'\in L_2[a,b]$. If $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b g(t)\,dv(t) - \frac{v(b)-v(a)}{b-a}\int_a^b g(s)\,ds\right|^2 \le \left[\frac{1}{3}(b-a)\bigvee_a^b(v) - \frac{2}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \times \bigvee_a^b(v)\int_a^b |g'(s)|^2\,ds. \tag{2.12}$$

Proof. Using Lemma 1 and the Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\left|\int_a^b g(t)\,dv(t) - \frac{v(b)-v(a)}{b-a}\int_a^b g(s)\,ds\right|^2 \\ &= \left|\int_a^b\left(g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right)dv(t)\right| \\ &\le \left(\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dV(t)\right)^2 \\ &\le [V(b)-V(a)]\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dV(t) \\ &= \bigvee_a^b(v)\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dV(t) \end{aligned} \tag{2.13}$$

From (2.7) we get

$$\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^2 dV(t) \tag{2.14}$$
$$\leq \left[\frac{1}{12}(b-a)V(b) + \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)^2 dV(t)\right]\int_a^b |g'(s)|^2 ds$$
$$= \left[\frac{1}{3}(b-a)V(b) - \frac{2}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)V(t)\,dt\right]\int_a^b |g'(s)|^2 ds$$

which proves the desired result (2.12). □

Corollary 3. *Assume that* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $g'\in L_2[a,b]$. *If* $h:[a,b]\to\mathbb{C}$ *is continuous on* $[a,b]$, *then*

$$\left|\frac{1}{b-a}\int_a^b g(t)h(t)\,dt - \frac{1}{b-a}\int_a^b g(s)\,ds \frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 \tag{2.15}$$
$$\leq \frac{1}{2}\left[\int_a^b |h(s)|\,ds - \frac{1}{(b-a)^2}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\left(\int_a^t |h(s)|\,ds\right)dt\right]$$
$$\times \int_a^b |h(s)|\,ds \int_a^b |g'(s)|^2 ds.$$

It follows by (2.12) for $v(t) = \int_a^t h(s)\,ds$, $t\in[a,b]$.

3. Applications for Selfadjoint and Unitary Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [11, p. 256]:

Theorem 5 (Spectral Representation Theorem)**.** *Let* A *be a bounded selfadjoint operator on the Hilbert space* H *and let* $a=\min\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\min\operatorname{Sp}(A)$ *and* $b=\max\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\max\operatorname{Sp}(A)$. *Then there exists a family of projections* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, *called the spectral family of* A, *with the following properties*

a) $E_\lambda \leq E_{\lambda'}$ *for* $\lambda\leq\lambda'$;
b) $E_{a-0}=0, E_b=1_H$ *and* $E_{\lambda+0}=E_\lambda$ *for all* $\lambda\in\mathbb{R}$;

c) *We have the representation*

$$A=\int_{a-0}^{b}\lambda dE_{\lambda}.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A)\in\mathcal{B}(H)$ such that for every $\varepsilon>0$ there exists a $\delta>0$ satisfying the inequality

$$\left\|\varphi(A)-\sum_{k=1}^{n}\varphi\left(\lambda_k'\right)\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\le\varepsilon$$

whenever

$$\begin{cases}\lambda_0<a=\lambda_1<...<\lambda_{n-1}<\lambda_n=b,\\ \lambda_k-\lambda_{k-1}\le\delta \text{ for } 1\le k\le n,\\ \lambda_k'\in[\lambda_{k-1},\lambda_k] \text{ for } 1\le k\le n\end{cases}$$

this means that

$$\varphi(A)=\int_{a-0}^{b}\varphi(\lambda)\,dE_{\lambda}, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4. *With the assumptions of Theorem 5 for A, E_λ and φ we have the representations*

$$\varphi(A)x=\int_{a-0}^{b}\varphi(\lambda)\,dE_{\lambda}x \quad \text{for all } x\in H$$

and

$$\langle\varphi(A)x,y\rangle=\int_{a-0}^{b}\varphi(\lambda)\,d\langle E_{\lambda}x,y\rangle \quad \text{for all } x,\ y\in H. \tag{3.3}$$

In particular,

$$\langle\varphi(A)x,x\rangle=\int_{a-0}^{b}\varphi(\lambda)\,d\langle E_{\lambda}x,x\rangle \quad \text{for all } x\in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2=\int_{a-0}^{b}|\varphi(\lambda)|^2\,d\|E_{\lambda}x\|^2 \quad \text{for all } x\in H.$$

We have the following result:

Theorem 6. *Assume that $f:I\to\mathbb{C}$ is locally absolutely continuous with $[a,b]\subset\mathring{I}$ (the interior of I) and $f'\in L_2[a,b]$. Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\min\operatorname{Sp}(A)$ and $b=\max\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\max\operatorname{Sp}(A)$. Then*

$$\begin{aligned}&\left|f(A)-\frac{1}{b-a}\left(\int_a^b f(s)\,ds\right)1_H\right|^2 \\ &\le\left(\int_a^b|f'(s)|^2\,ds\right)\left[\frac{1}{12}(b-a)1_H+\left(A-\frac{a+b}{2}1_H\right)^2\right]\end{aligned} \tag{3.4}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of A, $x\in H$ and small $\varepsilon>0$. Consider the function

$$f_\varepsilon(t):=\begin{cases} f(a)\,, & t\in[a-\varepsilon,a) \\ \\ f(t)\,, & t\in[a,b]\,. \end{cases},$$

Then by (2.7) on the interval $[a-\varepsilon,b]$ we get

$$\begin{aligned}(3.5)\qquad &\int_{a-\varepsilon}^{b}\left|f_\varepsilon(t)-\frac{1}{b-a+\varepsilon}\int_{a-\varepsilon}^{b}f_\varepsilon(s)\,ds\right|^2 d\langle E_t x,x\rangle \\ &\le\left[\frac{1}{12}(b-a+\varepsilon)\left[\langle E_b x,x\rangle-\langle E_{a-\varepsilon}x,x\rangle\right]\right. \\ &\left.+\frac{1}{b-a+\varepsilon}\int_{a-\varepsilon}^{b}\left(t-\frac{a-\varepsilon+b}{2}\right)^2 d\langle E_t x,x\rangle\right]\int_a^b\left|f_\varepsilon'(s)\right|^2 ds.\end{aligned}$$

By taking the limit over $\varepsilon\to 0+$ in (3.5) and using Corollary 4, then we get

$$\begin{aligned}&\left\langle\left|f(A)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 x,x\right\rangle \\ &\le\left[\frac{1}{12}(b-a)\langle x,x\rangle+\left\langle\left(A-\frac{a+b}{2}1_H\right)^2 x,x\right\rangle\right]\int_a^b\left|f'(s)\right|^2 ds,\end{aligned}$$

namely

$$\left\langle|f(A)|^2 x,x\right\rangle\le\left\langle\int_a^b|f'(s)|^2 ds\left[\frac{1}{12}(b-a)1_H+\left(A-\frac{a+b}{2}1_H\right)^2\right]x,x\right\rangle,$$

for all $x\in H$, which is equivalent, in the operator order, to (3.4). □

We say that the bounded linear operator $U:H\to H$ on the Hilbert space H is *unitary* iff $U^*=U^{-1}$.

It is well known that (see for instance [11, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda\le E_\mu$ for $0\le\lambda\le\mu\le 2\pi$;
b) $E_0=0$ and $E_{2\pi}=1_H$ (the *identity operator* on H);
c) $E_{\lambda+0}=E_\lambda$ for $0\le\lambda<2\pi$;
d) $U=\int_0^{2\pi}e^{i\lambda}dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda=E_\lambda$ for all $\lambda\in[0,2\pi]$.

Also, for every continuous complex valued function $g:\mathcal{C}(0,1)\to\mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$g(U)=\int_0^{2\pi}g\left(e^{i\lambda}\right)dE_\lambda \tag{3.6}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle g(U)x,y\rangle=\int_0^{2\pi}g\left(e^{i\lambda}\right)d\langle E_\lambda x,y\rangle \tag{3.7}$$

and

$$\|g(U)x\|^2 = \int_0^{2\pi} \left|g\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} \left|g\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle, \tag{3.8}$$

for any $x, y \in H$.

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \leq \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C}\setminus\{x+iy : x \leq 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \geq 1,\ z \in \mathbb{C}_\ell.$$

We also have the identity

$$\operatorname{Log}\left(e^{is}\right) = is \text{ for } -\pi < s \leq \pi.$$

Theorem 7. *Assume that $g : \mathcal{C}(0,1) \to \mathbb{C}$ is of class C^1 on the circle $\mathcal{C}(0,1)$. If $U : H \to H$ is unitary, then*

$$\begin{aligned} &\left|g(U) - \frac{1}{2\pi}\left(\int_0^{2\pi} g\left(e^{is}\right) ds\right) 1_H\right|^2 \\ &\leq \frac{1}{2}\int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds \left(\frac{1}{3}\pi 1_H + \frac{1}{\pi}\left|\operatorname{Log}(U)\right|^2\right). \end{aligned} \tag{3.9}$$

Proof. Let $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ be the *spectral family* of U and $x \in H$. Consider the function $f(t) = g\left(e^{it}\right)$, $t \in [0, 2\pi]$. By (2.7) we then have

$$\begin{aligned} &\int_0^{2\pi} \left|g\left(e^{it}\right) - \frac{1}{2\pi}\int_0^{2\pi} g\left(e^{is}\right) ds\right|^2 d\langle E_t x, x\rangle \\ &\leq \left[\frac{1}{6}\pi\left[\langle 1_H x, x\rangle\right] + \frac{1}{2\pi}\int_0^{2\pi} (t-\pi)^2 d\langle E_t x, x\rangle\right] \int_0^{2\pi} \left|g'(s)\right|^2 ds. \end{aligned} \tag{3.10}$$

Now, observe that

$$\begin{aligned} &\int_0^{2\pi} |t-\pi|^2 d\langle E_t x, x\rangle \\ &= \int_0^{2\pi} |i(t-\pi)|^2 d\langle E_t x, x\rangle = \int_0^{2\pi} \left|\operatorname{Log}\left(e^{i(t-\pi)}\right)\right|^2 d\langle E_t x, x\rangle \\ &= \int_0^{2\pi} \left|\operatorname{Log}\left(e^{it}\right)\right|^2 d\langle E_t x, x\rangle = \left\langle \left|\operatorname{Log}(U)\right|^2 x, x\right\rangle \end{aligned}$$

and by (3.10) we derive

$$\begin{aligned} &\left\|g(U)x - \left(\frac{1}{2\pi}\int_0^{2\pi} g\left(e^{is}\right) ds\right) x\right\|^2 \\ &\leq \left[\frac{1}{6}\pi\left[\langle 1_H x, x\rangle\right] + \frac{1}{2\pi}\left\langle \left|\operatorname{Log}(U)\right|^2 x, x\right\rangle\right] \int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds \end{aligned}$$

Since

$$\left\|g(U)x - \left(\frac{1}{2\pi}\int_0^{2\pi} g\left(e^{is}\right) ds\right) x\right\|^2 = \left\langle \left|g(U) - \left(\frac{1}{2\pi}\int_0^{2\pi} g\left(e^{is}\right) ds\right)\right|^2 x, x\right\rangle,$$

hence

$$\left\langle \left| g(U) - \left(\frac{1}{2\pi} \int_0^{2\pi} g\left(e^{is}\right) ds \right) \right|^2 x, x \right\rangle$$
$$\leq \left\langle \int_0^{2\pi} \left| g'\left(e^{is}\right) \right|^2 ds \left(\frac{1}{6}\pi 1_H + \frac{1}{2\pi} \left| \mathrm{Log}(U) \right|^2 \right) x, x \right\rangle$$

for all $x \in H$, which is, in the operator order, the desired inequality (3.9). □

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