L_1-NORM RIEMANN-STIELTJES INTEGRAL INEQUALITIES RELATED TO WIRTINGER'S RESULT

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Abstract. In this paper we obtain, among others, sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|\,du(t)$ in the case that $f:[a,b]\to\mathbb{C}$ is absolutely continuous with $f(a)=f(b)=0$ and $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing. Applications for Trapezoid and Grüss' type inequalities are provided.

1. Introduction

It is well known that, see for instance [5], or [14], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then we have Wirtinger's inequality

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $h\in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a)=h(b)=0$, then $\operatorname{Re}h(a)=\operatorname{Re}h(b)=0$ and $\operatorname{Im}h(a)=\operatorname{Im}h(b)=0$ and by writing (1.1) for $\operatorname{Re}h$ and $\operatorname{Im}h$ and adding the obtained inequalities, we get

$$\int_a^b |h(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$h(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K\in\mathbb{C}$.

Similarly, if $h\in C^1([a,b],\mathbb{C})$ with $h(a)=0$, then by (1.2) we have

$$\int_a^b |h(t)|^2\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Wirtinger's inequality, Trapezoid inequality, Grüss' inequality, Riemann-Stieltjes integral.

RGMIA Res. Rep. Coll. 24 (2021), Art. 61, 14 pp. Received 19/05/21

and the equality holds if and only if

$$h(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [11]-[16].

In the recent paper [8] we obtained the following weighted version of Wirtinger results:

Theorem 1. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.5}$$

The equality holds in (1.5) iff

$$f(t) = K\sin\left[\pi\int_a^t w(s)\,ds\right],\ K \in \mathbb{C}.$$

If $f(a) = 0$, then

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt \tag{1.6}$$

with equality iff

$$f(t) = K\sin\left[\frac{1}{2}\pi\int_a^t w(s)\,ds\right],\ K \in \mathbb{C}.$$

More recently, we obtained the following inequality for Riemann-Stieltjes integral [9]:

Theorem 2. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = f(b) = 0$ and $f' \in L_2[a,b]$. If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then*

$$\begin{aligned}
&\int_a^b |f(t)|^2 du(t) \\
&\le \frac{1}{2}\int_a^b \left[(t-a)\int_a^t |f'(s)|^2 ds + (b-t)\int_t^b |f'(s)|^2 ds\right] du(t) \\
&\le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t)\right]\int_a^b |f'(s)|^2 ds \\
&= \frac{1}{2}\left[(b-a)[u(b)-u(a)] - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b |f'(s)|^2 ds.
\end{aligned}$$

The inequalities are sharp in (2.1).

Motivated by the above results, in this paper we obtain new sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|\,du(t)$ in the case that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = 0$ or $f(b) = 0$ or $f(a) = f(b) = 0$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing. Applications for trapezoid and Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

2. Main Results

Theorem 3. *Assume that* $f : [a,b] \to \mathbb{C}$ *is absolutely continuous and* $u : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing.*

(i) *If* $f(a) = 0$, *then*

$$\int_a^b |f(t)|\, du(t) \le \int_a^b [u(b) - u(t)] |f'(t)|\, dt. \tag{2.1}$$

The inequality is sharp.

(ii) *If* $f(b) = 0$, *then*

$$\int_a^b |f(t)|\, du(t) \le \int_a^b [u(t) - u(a)] |f'(t)|\, dt. \tag{2.2}$$

The inequality is sharp.

(iii) *If* $f(a) = f(b) = 0$, *then*

$$\int_a^b |f(t)|\, du(t) \le \int_a^b \left| u(t) - u\left(\frac{a+b}{2}\right)\right| |f'(t)|\, dt. \tag{2.3}$$

The inequality is sharp.

Proof. (i) Since $f(a) = 0$, hence

$$\begin{aligned}
\int_a^b |f(t)|\, du(t) &= \int_a^b \left|\int_a^t f'(s)\, ds\right| du(t) \le \int_a^b \left(\int_a^t |f'(s)|\, ds\right) du(t) \\
&= \left(\int_a^t |f'(s)|\, ds\right) u(t)\Big|_a^b - \int_a^b |f'(t)|\, u(t)\, dt \\
&= \left(\int_a^b |f'(s)|\, ds\right) u(b) - \int_a^b |f'(t)|\, u(t)\, dt \\
&= \int_a^b [u(b) - u(t)] |f'(t)|\, dt,
\end{aligned}$$

which proves the desired inequality (2.1).

If we take $f(t) = t - a$ and $u(t) = t$, then we get

$$\int_a^b |f(t)|\, du(t) = \int_a^b (t-a)\, dt = \frac{1}{2}(b-a)^2$$

and

$$\int_a^b [u(b) - u(t)] |f'(t)|\, dt = \int_a^b (b-t)\, dt = \frac{1}{2}(b-a)^2,$$

which shows that the inequality (2.1) is sharp.

(ii) Since $f(b)=0$, hence

$$\begin{aligned}\int_a^b |f(t)|\,du(t) &= \int_a^b \left|\int_t^b f'(s)\,ds\right| du(t) \le \int_a^b \left(\int_t^b |f'(s)|\,ds\right) du(t)\\ &= \left(\int_t^b |f'(s)|\,ds\right) u(t)\bigg|_a^b + \int_a^b |f'(t)|\,u(t)\,dt\\ &= -\left(\int_a^b |f'(s)|\,ds\right) u(a) + \int_a^b |f'(t)|\,u(t)\,dt\\ &= \int_a^b [u(t)-u(a)]\,|f'(t)|\,dt,\end{aligned}$$

which proves the desired inequality (2.2).

If we take $f(t)=b-t$ and $u(t)=t$, then we get

$$\int_a^b |f(t)|\,du(t) = \int_a^b (b-t)\,dt = \frac{1}{2}(b-a)^2$$

and

$$\int_a^b [u(t)-u(a)]\,|f'(t)|\,dt = \int_a^b (t-a)\,dt = \frac{1}{2}(b-a)^2,$$

which shows that the inequality (2.2) is sharp.

(iii) If we use the inequality (2.1) in the interval $\left[a,\frac{a+b}{2}\right]$, then we have

$$\int_a^{\frac{a+b}{2}} |f(t)|\,du(t) \le \int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right)-u(t)\right]|f'(t)|\,dt. \tag{2.4}$$

Also, by (2.2) on the interval $\left[\frac{a+b}{2},b\right]$, then

$$\int_{\frac{a+b}{2}}^b |f(t)|\,du(t) \le \int_{\frac{a+b}{2}}^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right]|f'(t)|\,dt. \tag{2.5}$$

By adding (2.4) with (2.5) we derive

$$\begin{aligned}&\int_a^b |f(t)|\,du(t)\\ &\le \int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right)-u(t)\right]|f'(t)|\,dt + \int_{\frac{a+b}{2}}^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right]|f'(t)|\,dt\\ &= \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\left[u(t)-u\left(\frac{a+b}{2}\right)\right]|f'(t)|\,dt\\ &= \int_a^b \left|u(t)-u\left(\frac{a+b}{2}\right)\right| |f'(t)|\,dt\end{aligned}$$

and the inequality (2.3) is proved.

Consider the function

$$f(t) = \begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right],\\ b-t, & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Then for $u(t) = t$, $t \in [a,b]$

$$\int_a^b |f(t)| \, du(t) = \int_a^{\frac{a+b}{2}} (t-a) \, dt + \int_{\frac{a+b}{2}}^b (b-t) \, dt = \frac{1}{4}(b-a)^2$$

and

$$\int_a^b \left| u(t) - u\left(\frac{a+b}{2}\right) \right| |f'(t)| \, dt = \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{1}{4}(b-a)^2 .$$

This proves that the inequality (2.3) is sharp. □

Corollary 1. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous and $w : [a,b] \to (0,\infty)$ is integrable.*

If $f(a) = 0$, then

$$\int_a^b |f(t)| \, w(t) \, dt \le \int_a^b \left(\int_t^b w(s) \, ds \right) |f'(t)| \, dt. \tag{2.6}$$

The inequality is sharp.

If $f(b) = 0$, then

$$\int_a^b |f(t)| \, w(t) \, dt \le \int_a^b \left(\int_a^t w(s) \, ds \right) |f'(t)| \, dt. \tag{2.7}$$

The inequality is sharp.

If $f(a) = f(b) = 0$, then

$$\int_a^b |f(t)| \, w(t) \, dt \le \int_a^b \left| \int_{\frac{a+b}{2}}^t w(s) \, ds \right| |f'(t)| \, dt. \tag{2.8}$$

The inequality is sharp.

The proof follows by taking $u(t) = \int_a^t w(s) \, ds$, $t \in [a,b]$.

The following lemma was obtained by the author in 2007, [7] and is of interest in itself as well (see also [6]):

Lemma 1. *If $p : [a,b] \to \mathbb{C}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\begin{aligned} \left| \int_a^b p(t) \, dv(t) \right| &\le \int_a^b |p(t)| \, dV(t) \\ &\le \left(\int_a^b |p(t)|^p \, dV(t) \right)^{1/p} \left(\bigvee_a^b (v) \right)^{1/q} \\ &\le \max_{t \in [a,b]} |p(t)| \bigvee_a^b (v), \end{aligned} \tag{2.9}$$

where $V(t) := \bigvee_a^t (v)$ is the total variation of v on $[a,t]$ with $t \in [a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a) = 0$ and $V(b) = \bigvee_a^b (v)$.

Corollary 2. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous and $u : [a,b] \to \mathbb{C}$ is of bounded variation.*

If $f(a)=0,$ *then*

$$\left|\int_a^b f(t)\,dv(t)\right| \le \int_a^b \left(\bigvee_t^b (v)\right) |f'(t)|\,dt \tag{2.10}$$

If $f(b)=0,$ *then*

$$\left|\int_a^b f(t)\,dv(t)\right| \le \int_a^b \left(\bigvee_a^t (v)\right) |f'(t)|\,dt. \tag{2.11}$$

If $f(a)=f(b)=0,$ *then*

$$\left|\int_a^b f(t)\,dv(t)\right| \le \int_a^b \left|\bigvee_{\frac{a+b}{2}}^t (v)\right| |f'(t)|\,dt. \tag{2.12}$$

The inequalities (2.10)-(2.12) are sharp.

Proof. By (2.9) we have

$$\left|\int_a^b f(t)\,dv(t)\right| \le \int_a^b |f(t)|\,dV(t).$$

If $f(a)=0$, then by (2.1),

$$\int_a^b |f(t)|\,dV(t) \le \int_a^b [V(b)-V(t)]\,|f'(t)|\,dt = \int_a^b \left(\bigvee_t^b (v)\right) |f'(t)|\,dt,$$

which proves (2.10).

The rest follow in a similar way. □

A complementary result is as follows:

Theorem 4. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous,* $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing and* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

(i) *If* $f(a)=0,$ *then*

$$\int_a^b |f(t)|\,du(t) \le \left(\int_a^b [u(b)-u(t)]\,dt\right)^{1/q} \left(\int_a^b |f'(t)|^p [u(b)-u(t)]\,dt\right)^{1/p}. \tag{2.13}$$

The inequality is sharp.

(ii) *If* $f(b)=0,$ *then*

$$\int_a^b |f(t)|\,du(t) \le \left(\int_a^b [u(t)-u(a)]\,dt\right)^{1/q} \left(\int_a^b |f'(t)|^p [u(t)-u(a)]\,dt\right)^{1/p} \tag{2.14}$$

The inequality is sharp.

(iii) *If $f(a)=f(b)=0$, then*

$$\int_a^b |f(t)|\,du(t) \le \left(\int_a^b \left|u\left(\frac{a+b}{2}\right)-u(t)\right| dt\right)^{1/q} \times \left(\int_a^b |f'(t)|^p \left|u\left(\frac{a+b}{2}\right)-u(t)\right| dt\right)^{1/p}. \tag{2.15}$$

The inequality is sharp.

Proof. (i). Let $f(a)=0$. Using Hölder's integral inequality, we have

$$\int_a^b |f(t)|\,du(t) = \int_a^b \left|\int_a^t f'(s)\,ds\right| du(t) \le \int_a^b \left(\int_a^t |f'(s)|\,ds\right) du(t) \le \int_a^b (t-a)^{1/q}\left(\int_a^t |f'(s)|^p\,ds\right)^{1/p} du(t). \tag{2.16}$$

Also, by Hölder's integral inequality for Riemann-Stieltjes integral with monotonic integrators, we have

$$\begin{aligned}&\int_a^b (t-a)^{1/q}\left(\int_a^t |f'(s)|^p\,ds\right)^{1/p} du(t)\\ &\le \left(\int_a^b \left[(t-a)^{1/q}\right]^q du(t)\right)^{1/q}\left(\int_a^b \left[\left(\int_a^t |f'(s)|^p\,ds\right)^{1/p}\right]^p du(t)\right)^{1/p}\\ &= \left(\int_a^b (t-a)\,du(t)\right)^{1/q}\left(\int_a^b \left(\int_a^t |f'(s)|^p\,ds\right) du(t)\right)^{1/p}.\end{aligned} \tag{2.17}$$

Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^b (t-a)\,du(t) &= (t-a)\,u(t)\big|_a^b - \int_a^b u(t)\,dt\\ &= (b-a)\,u(b) - \int_a^b u(t)\,dt = \int_a^b [u(b)-u(t)]\,dt\end{aligned}$$

and

$$\begin{aligned}\int_a^b \left(\int_a^t |f'(s)|^p\,ds\right) du(t) &= \left(\int_a^t |f'(s)|^p\,ds\right) u(t)\Big|_a^b - \int_a^b |f'(t)|^p\,u(t)\,dt\\ &= \left(\int_a^b |f'(s)|^p\,ds\right) u(b) - \int_a^b |f'(t)|^p\,u(t)\,dt\\ &= \int_a^b |f'(t)|^p\,[u(b)-u(t)]\,dt.\end{aligned}$$

By making use of (2.16) and (2.17) we get (2.13).

Now, if we take $f(t)=t-a$ and $u(t)=t$, then

$$\int_a^b |f(t)|\,du(t) = \frac{1}{2}(b-a)^2,$$

and

$$\left(\int_a^b [u(b)-u(t)]\,dt\right)^{1/q}\left(\int_a^b |f'(t)|^p [u(b)-u(t)]\,dt\right)^{1/p}$$
$$=\left(\int_a^b (b-t)\,dt\right)^{1/q}\left(\int_a^b (b-t)\,dt\right)^{1/p}=\int_a^b (b-t)\,dt=\frac{1}{2}(b-a)^2,$$

which proves the sharpness of the inequality.

(ii). The proof is similar and we omit the details.

(iii). If we use the inequality (2.13) in the interval $\left[a,\frac{a+b}{2}\right]$, then we have

$$\int_a^{\frac{a+b}{2}} |f(t)|\,du(t)\le\left(\int_a^{\frac{a+b}{2}}\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt\right)^{1/q}$$
$$\times\left(\int_a^{\frac{a+b}{2}}|f'(t)|^p\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt\right)^{1/p},$$

while from (2.14) we have

$$\int_{\frac{a+b}{2}}^b |f(t)|\,du(t)\le\left(\int_{\frac{a+b}{2}}^b\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/q}$$
$$\times\left(\int_{\frac{a+b}{2}}^b|f'(t)|^p\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/p}.$$

If we add these two inequalities, then we get

$$\int_a^b |f(t)|\,du(t) \tag{2.18}$$
$$\le\left(\int_a^{\frac{a+b}{2}}\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt\right)^{1/q}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^p\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt\right)^{1/p}$$
$$+\left(\int_{\frac{a+b}{2}}^b\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/q}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^p\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/p}.$$

If we use the elementary inequality

$$ab+cd\le(a^q+c^q)^{1/q}(b^p+d^p)^{1/p},\ a,b,c,d\ge0,$$

where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

(2.19)

$$\begin{aligned}
&\left(\int_a^{\frac{a+b}{2}}\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt\right)^{1/q}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^p\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt\right)^{1/p}\\
&+\left(\int_{\frac{a+b}{2}}^b\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/q}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^p\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/p}\\
&\leq\left(\int_a^{\frac{a+b}{2}}\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt+\int_{\frac{a+b}{2}}^b\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/q}\\
&\times\left(\int_a^{\frac{a+b}{2}}|f'(t)|^p\left[u\left(\frac{a+b}{2}\right)-u(t)\right]dt+\int_{\frac{a+b}{2}}^b|f'(t)|^p\left[u(t)-u\left(\frac{a+b}{2}\right)\right]dt\right)^{1/p}\\
&=\left(\int_a^b\left|u\left(\frac{a+b}{2}\right)-u(t)\right|dt\right)^{1/q}\left(\int_a^b|f'(t)|^p\left|u\left(\frac{a+b}{2}\right)-u(t)\right|dt\right)^{1/p}.
\end{aligned}$$

By making use of (2.18) and (2.19) we deduce (2.15).

Consider the function

$$f(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right],\\ b-t, & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Then for $u(t)=t,\ t\in[a,b]$

$$\int_a^b|f(t)|\,du(t)=\int_a^{\frac{a+b}{2}}(t-a)\,dt+\int_{\frac{a+b}{2}}^b(b-t)\,dt=\frac{1}{4}(b-a)^2$$

and

$$\begin{aligned}
&\left(\int_a^b\left|u\left(\frac{a+b}{2}\right)-u(t)\right|dt\right)^{1/q}\left(\int_a^b|f'(t)|^p\left|u\left(\frac{a+b}{2}\right)-u(t)\right|dt\right)^{1/p}\\
&=\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{4}(b-a)^2.
\end{aligned}$$

This proves the sharpness of (2.15). □

Corollary 3. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous and $v:[a,b]\to\mathbb{C}$ is of bounded variation.*

If $f(a)=0$, then

$$\left|\int_a^b f(t)\,dv(t)\right|\leq\left(\int_a^b\left(\bigvee_t^b(v)\right)dt\right)^{1/q}\left(\int_a^b|f'(t)|^p\left(\bigvee_t^b(v)\right)dt\right)^{1/p}. \tag{2.20}$$

If $f(b)=0$, then

$$\left|\int_a^b f(t)\,dv(t)\right|\leq\left(\int_a^b\left(\bigvee_a^t(v)\right)dt\right)^{1/q}\left(\int_a^b|f'(t)|^p\left(\bigvee_a^t(v)\right)dt\right)^{1/p}. \tag{2.21}$$

If $f(a)=f(b)=0$, then

$$\left|\int_a^b f(t)\,dv(t)\right| \leq \left(\int_a^b \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/q} \left(\int_a^b |f'(t)|^p \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/p}. \tag{2.22}$$

The inequalities (2.20)-(2.22) are sharp.

3. Trapezoid and Grüss' Type Inequalities

We have the following equalities:

Lemma 2. *Let $f,\, v:[a,b]\to\mathbb{C}$ be such that one is continuous and the other is of bounded variation. Then*

$$\begin{aligned} &T(f,v;[a,b]) \\ &:= \int_a^b f(t)\,dv(t) \\ &- f(b)\left[v(b)-\frac{1}{b-a}\int_a^b v(t)\,dt\right] - f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt - v(a)\right] \\ &= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t) \\ &= \int_a^b \left[f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t). \end{aligned} \tag{3.1}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\int_a^b \left[f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t) \\ &\quad= \int_a^b f(t)\,dv(t) - \int_a^b \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\,dv(t) \\ &= \int_a^b f(t)\,dv(t) - \left.\frac{f(a)(b-t)+f(b)(t-a)}{b-a}v(t)\right|_a^b \\ &+ \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\ &= \int_a^b f(t)\,dv(t) - f(b)v(b) + f(a)v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\ &= \int_a^b f(t)\,dv(t) \\ &- f(b)\left[v(b)-\frac{1}{b-a}\int_a^b v(t)\,dt\right] - f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt - v(a)\right]. \end{aligned}$$

Integrating by parts again, we also have

$$\begin{aligned} &\int_a^b f(t)\,dv(t) - f(b)v(b) + f(a)v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\ &= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t). \end{aligned}$$

These prove the required identities. □

Theorem 5. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous. If* $v:[a,b]\to\mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$|T(f,v;[a,b])| \leq \left(\int_a^b \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/q} \left(\int_a^b \left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/p} \tag{3.2}$$

for $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let

$$h(t):=f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a},\ t\in[a,b].$$

Observe that $h(a)=h(b)=0$, $h'\in L_2[a,b]$ and

$$h'(t)=f'(t)-\frac{f(b)-f(b)}{b-a},\ t\in(a,b).$$

By making use of the inequality (2.22), we have

$$\begin{aligned}&\left|\int_a^b \left[f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t)\right|\\ &\leq \left(\int_a^b \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/q} \left(\int_a^b \left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/p},\end{aligned}$$

which proves the desired result. □

Remark 1. *Since*

$$\begin{aligned}&\left(\int_a^b \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/q} \left(\int_a^b \left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p \left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right| dt\right)^{1/p}\\ &\leq \sup_{t\in[a,b]}\left|\bigvee_{\frac{a+b}{2}}^{t}(v)\right|(b-a)^{1/q}\left(\int_a^b \left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p dt\right)^{1/p}\\ &= \max\left\{\bigvee_{\frac{a+b}{2}}^{b}(v),\bigvee_{a}^{\frac{a+b}{2}}(v)\right\}(b-a)^{1/q}\left(\int_a^b \left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p dt\right)^{1/p}\end{aligned}$$

hence we derive the simpler inequality

$$\begin{aligned}|T(f,v;[a,b])| &\leq (b-a)\left(\frac{1}{b-a}\int_a^b \left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p dt\right)^{1/p}\\ &\quad\times\max\left\{\bigvee_{\frac{a+b}{2}}^{b}(v),\bigvee_{a}^{\frac{a+b}{2}}(v)\right\}\end{aligned} \tag{3.3}$$

for $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 4. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous. If $w:[a,b]\to(0,\infty)$ is integrable with $\int_a^b w(s)\,ds=1$, then*

$$\left|\int_a^b f(t)\,w(t)\,dt-\frac{f(a)(b-E(w,[a,b]))+f(b)(E(w,[a,b])-a)}{b-a}\right|^2 \tag{3.4}$$
$$\le\left(\int_a^b\left|\int_{\frac{a+b}{2}}^t w(s)\,ds\right|dt\right)^{1/q}$$
$$\times\left(\int_a^b\left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p\left|\int_{\frac{a+b}{2}}^t w(s)\,ds\right|dt\right)^{1/p},$$

where $E(w,[a,b]):=\int_a^b tw(t)\,dt$.

We also have the simpler inequality

$$\left|\int_a^b f(t)\,w(t)\,dt-\frac{f(a)(b-E(w,[a,b]))+f(b)(E(w,[a,b])-a)}{b-a}\right|^2 \tag{3.5}$$
$$\le(b-a)\left(\frac{1}{b-a}\int_a^b\left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^p dt\right)^{1/p}$$
$$\times\max\left\{\int_{\frac{a+b}{2}}^b w(s)\,ds,\int_a^{\frac{a+b}{2}} w(s)\,ds\right\}.$$

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{3.6}$$

In 1935, Grüss [12] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{3.7}$$

provided that there exists the real numbers m, M, n, N such that

$$m\le f(t)\le M\quad\text{and}\quad n\le g(t)\le N\quad\text{for a.e. } t\in[a,b]. \tag{3.8}$$

The constant $\frac{1}{4}$ is best possible in (3.7) in the sense that it cannot be replaced by a smaller quantity.

Theorem 6. *Assume that $h:[a,b]\to\mathbb{C}$ is integrable on $[a,b]$ and $v:[a,b]\to\mathbb{C}$ of bounded variation on $[a,b]$, then*

$$|C(h,v)|\le\left(\frac{1}{b-a}\int_a^b\left|\bigvee_{\frac{a+b}{2}}^t(v)\right|dt\right)^{1/q} \tag{3.9}$$
$$\times\left(\frac{1}{b-a}\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^p\left|\bigvee_{\frac{a+b}{2}}^t(v)\right|dt\right)^{1/p}.$$

We also have the simpler inequality

(3.10)

$$\left|C\left(h,v\right)\right|\leq\max\left\{\bigvee_{\frac{a+b}{2}}^{b}\left(v\right),\bigvee_{a}^{\frac{a+b}{2}}\left(v\right)\right\}\left(\frac{1}{b-a}\int_{a}^{b}\left|h\left(t\right)-\frac{1}{b-a}\int_{a}^{b}h\left(s\right)ds\right|^{p}dt\right)^{1/p}.$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_{a}^{b}\left(\int_{a}^{t}h\left(s\right)ds-\frac{t-a}{b-a}\int_{a}^{b}h\left(s\right)ds\right)dv\left(t\right)\\&=\left(\int_{a}^{t}h\left(s\right)ds-\frac{t-a}{b-a}\int_{a}^{b}h\left(s\right)ds\right)v\left(t\right)\Big|_{a}^{b}\\&-\int_{a}^{b}v\left(t\right)d\left(\int_{a}^{t}h\left(s\right)ds-\frac{t-a}{b-a}\int_{a}^{b}h\left(s\right)ds\right)\\&=-\int_{a}^{b}v\left(t\right)h\left(t\right)dt+\frac{1}{b-a}\int_{a}^{b}h\left(s\right)ds\int_{a}^{b}v\left(t\right)dt,\end{aligned}$$

which gives that

$$C\left(h,v\right)=\frac{1}{b-a}\int_{a}^{b}\left(\frac{t-a}{b-a}\int_{a}^{b}h\left(s\right)ds-\int_{a}^{t}h\left(s\right)ds\right)dv\left(t\right). \tag{3.11}$$

Consider

$$g\left(t\right):=\frac{t-a}{b-a}\int_{a}^{b}h\left(s\right)ds-\int_{a}^{t}h\left(s\right)ds,\ t\in\left[a,b\right],$$

then g is absolutely continuous, $g\left(a\right)=g\left(b\right)=0$,

$$g'\left(t\right):=\frac{1}{b-a}\int_{a}^{b}h\left(s\right)ds-h\left(t\right),\ t\in\left[a,b\right]$$

and by (2.22) we get

$$\begin{aligned}&\left|\int_{a}^{b}\left(\frac{t-a}{b-a}\int_{a}^{b}h\left(s\right)ds-\int_{a}^{t}h\left(s\right)ds\right)dv\left(t\right)\right|\\&\leq\left(\int_{a}^{b}\left|\bigvee_{\frac{a+b}{2}}^{t}\left(v\right)\right|dt\right)^{1/q}\\&\times\left(\int_{a}^{b}\left|\frac{1}{b-a}\int_{a}^{b}h\left(s\right)ds-h\left(t\right)\right|^{p}\left|\bigvee_{\frac{a+b}{2}}^{t}\left(v\right)\right|dt\right)^{1/p},\end{aligned} \tag{3.12}$$

which by (3.11), proves the desired result. □

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