L_1-NORM RIEMANN-STIELTJES INTEGRAL INEQUALITIES RELATED TO STEKLOFF'S RESULT

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Abstract. In this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|\,du(t)$ in the case that $f:[a,b]\to\mathbb{C}$ is absolutely continuous with $\int_a^b f(t)\,dt=0$ and $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing. Applications for Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

1. Introduction

It is well known that, see for instance [5], or [12], if $u\in C^1([a,b],\mathbb{R})$, namely u is continuous on $[a,b]$ and has a derivative that is continuous on (a,b) and satisfies $u(a)=u(b)=0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [10]-[14].

In 1901, W. Stekloff, [16], proved that, if $u\in C^1([a,b],\mathbb{R})$ and $\int_a^b u(t)\,dt=0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.3}$$

In addition, if $u(a)=u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h:[a,b]\to\mathbb{C}$.

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Theorem 1. *If* $h \in C^1([a,b],\mathbb{C})$ *and* $\int_a^b h(t)\,dt = 0$, *then*

$$\int_a^b |h(x)|^2 \, dx \le \frac{(b-a)^2}{\pi^2} \int_a^b |h'(x)|^2 \, dx. \tag{1.5}$$

In addition, if $h(a) = h(b)$, *then*

$$\int_a^b |h(x)|^2 \, dx \le \frac{(b-a)^2}{4\pi^2} \int_a^b |h'(x)|^2 \, dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u = \operatorname{Re} h$ and $u = \operatorname{Im} h$ and by adding the corresponding inequalities.

In the recent paper we obtained the following weighted version of the above results:

Theorem 2. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds = 1$ *and* $f \in C^1([a,b],\mathbb{C})$. *If* $\frac{f'}{\sqrt{w}} \in L_2[a,b]$ *and* $\int_a^b f(t)\,w(t)\,dt = 0$, *then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{\pi^2} \int_a^b \frac{|f'(t)|^2}{w(t)} dt. \tag{1.7}$$

In addition, if $f(a) = f(b)$, *then we have the better inequality*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{4\pi^2} \int_a^b \frac{|f'(t)|^2}{w(t)} dt. \tag{1.8}$$

Motivated by the above results, in this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|\,du(t)$ in the case that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $\int_a^b f(t)\,dt = 0$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing. Applications for Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

2. Main Results

Our main result is as follows:

Theorem 3. *Assume that* $f : [a,b] \to \mathbb{C}$ *is absolutely continuous with* $\int_a^b f(t)\,dt = 0$. *If* $u : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}
&\int_a^b |f(t)|\,du(t) \\
&\le \frac{1}{b-a} \int_a^b \{(t-a)[u(b)-u(t)] + (b-t)[u(t)-u(a)]\} |f'(t)|\,dt \\
&\le \begin{cases} \frac{u(b)-u(a)}{b-a} \int_a^b \left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] |f'(t)|\,dt \\ \\ \int_a^b \left[\frac{1}{2}[u(b)-u(a)] + \left|u(t) - \frac{u(a)+u(b)}{2}\right|\right] |f'(t)|\,dt \end{cases} \\
&\le [u(b)-u(a)] \int_a^b |f'(t)|\,dt.
\end{aligned} \tag{2.1}$$

The first inequality in (2.1) is sharp.

Proof. We use Montgomery identity

$$f(t)=\frac{1}{b-a}\int_a^b f(s)\,ds+\frac{1}{b-a}\int_a^b k(t,s)\,f'(s)\,ds, \tag{2.2}$$

where

$$k(t,s)=\begin{cases} s-a, & s\in[a,t], \\ s-b, & s\in(t,b]. \end{cases}$$

Using (2.2) we get

$$\begin{aligned}|f(t)| &= \frac{1}{b-a}\left|\int_a^b k(t,s)\,f'(s)\,ds\right| \\ &= \frac{1}{b-a}\left|\int_a^t (s-a)\,f'(s)\,ds+\int_t^b (s-b)\,f'(s)\,ds\right| \\ &\le \frac{1}{b-a}\left[\int_a^t (s-a)\,|f'(s)|\,ds+\int_t^b (b-s)\,|f'(s)|\,ds\right]\end{aligned}$$

for all $t\in[a,b]$.

Taking the Riemann-Stieltjes integral and using integration by parts for Riemann-Stieltjes integrals, we get

$$\begin{aligned}&\int_a^b |f(t)|\,du(t) \\ &\le \frac{1}{b-a}\int_a^b\left[\int_a^t (s-a)\,|f'(s)|\,ds+\int_t^b (b-s)\,|f'(s)|\,ds\right]du(t) \\ &= \frac{1}{b-a}\left[\left[\int_a^t (s-a)\,|f'(s)|\,ds+\int_t^b (b-s)\,|f'(s)|\,ds\right]u(t)\Big|_a^b\right. \\ &\left.-\int_a^b [(t-a)\,|f'(t)|-(b-t)\,|f'(t)|]\,u(t)\,dt\right] \\ &= \frac{1}{b-a}\left[u(b)\int_a^b (s-a)\,|f'(s)|\,ds-u(a)\int_a^b (b-s)\,|f'(s)|\,ds\right. \\ &\left.-\int_a^b [(t-a)\,|f'(t)|-(b-t)\,|f'(t)|]\,u(t)\,dt\right] \\ &= \frac{1}{b-a}\int_a^b \{(t-a)\,[u(b)-u(t)]+(b-t)\,[u(t)-u(a)]\}\,|f'(t)|\,dt,\end{aligned} \tag{2.3}$$

that proves the first inequality in (2.1).

Observe that

$$\begin{aligned}
&(t-a)[u(b)-u(t)]+(b-t)[u(t)-u(a)]\\
&\leq\begin{cases}\max\{t-a,b-t\}[u(b)-u(a)]\\ \\ \max\{u(b)-u(t),u(t)-u(a)\}(b-a)\end{cases}\\
&=\begin{cases}\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right][u(b)-u(a)]\\ \\ (b-a)\left[\frac{1}{2}[u(b)-u(a)]+\left|u(t)-\frac{u(a)+u(b)}{2}\right|\right]\end{cases}\\
&\leq(b-a)[u(b)-u(a)],
\end{aligned}$$

which proves the last part of (2.1).

The sharpness of the first inequality in (2.1) is proven in Remark 2 below. □

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $\int_a^b f(t)\,dt=0$. *If* $w:[a,b]\to(0,\infty)$ *is integrable with* $\int_a^b w(s)\,ds=1$, *then*

$$\begin{aligned}
&\int_a^b |f(t)|\,w(t)\,dt \qquad (2.4)\\
&\leq\frac{1}{b-a}\int_a^b\left\{(t-a)\int_t^b w(s)\,ds+(b-t)\int_a^t w(s)\,ds\right\}|f'(t)|\,dt\\
&\leq\begin{cases}\frac{1}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|f'(t)|\,dt\\ \\ \int_a^b\left[\frac{1}{2}+\left|\int_a^t w(s)\,ds-\frac{1}{2}\right|\right]|f'(t)|\,dt\end{cases}\\
&\leq\int_a^b|f'(t)|\,dt.
\end{aligned}$$

It follows by (2.1) by taking $u(t)=\int_a^t w(s)\,ds$, $t\in[a,b]$.

Corollary 2. *Assume that* $g:[a,b]\to\mathbb{R}$ *is absolutely continuous. If* $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}
&\left|\int_a^b g(t)\,du(t)-\frac{u(b)-u(a)}{b-a}\int_a^b g(s)\,ds\right| \qquad (2.5)\\
&\leq\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|du(t)\\
&\leq\frac{1}{b-a}\int_a^b\{(t-a)[u(b)-u(t)]+(b-t)[u(t)-u(a)]\}|g'(t)|\,dt\\
&\leq\begin{cases}\frac{u(b)-u(a)}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|g'(t)|\,dt\\ \\ \int_a^b\left[\frac{1}{2}[u(b)-u(a)]+\left|u(t)-\frac{u(a)+u(b)}{2}\right|\right]|g'(t)|\,dt\end{cases}\\
&\leq[u(b)-u(a)]\int_a^b|f'(t)|\,dt.
\end{aligned}$$

If we take in (2.5) $u(t)=\int_a^t w(s)\,ds,\ t\in[a,b]$, then we get the weighted inequality

$$\left|\int_a^b g(t)\,w(t)\,dt-\frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| w(t)\,dt \tag{2.6}$$

$$\le \frac{1}{b-a}\int_a^b \left\{(t-a)\int_t^b w(s)\,ds+(b-t)\int_a^t w(s)\,ds\right\}|g'(t)|\,dt$$
$$\le \begin{cases} \frac{1}{b-a}\int_a^b \left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|g'(t)|\,dt \\ \int_a^b \left[\frac{1}{2}+\left|\int_a^t w(s)\,ds-\frac{1}{2}\right|\right]|g'(t)|\,dt \end{cases}$$
$$\le \int_a^b |f'(t)|\,dt.$$

Remark 1. *If we take $u(t)=t$ in (2.5), then we get*

$$\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \le \frac{2}{b-a}\int_a^b (t-a)(b-t)\,|g'(t)|\,dt, \tag{2.7}$$

where $g:[a,b]\to\mathbb{C}$ is absolutely continuous.

From a different view point, we also have

Remark 2. *If we take $u(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right),\ t\in[a,b]$, then we have*

$$\begin{aligned}\int_a^b g(t)\,du(t) &= g(t)\,u(t)|_a^b-\int_a^b g'(t)\,u(t)\,dt\\ &= g(b)\,u(b)-g(a)\,u(a)+\int_a^{\frac{a+b}{2}} g'(t)\,dt-\int_{\frac{a+b}{2}}^b g'(t)\,dt\\ &= g(b)+g(a)+g\left(\frac{a+b}{2}\right)-g(a)-g(b)+g\left(\frac{a+b}{2}\right)\\ &= 2g\left(\frac{a+b}{2}\right),\end{aligned}$$

which gives

$$\int_a^b g(t)\,du(t)-\frac{u(b)-u(a)}{b-a}\int_a^b g(s)\,ds=2\left[g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(s)\,ds\right].$$

Also

$$\int_a^b \{(t-a)[u(b)-u(t)]+(b-t)[u(t)-u(a)]\}|g'(t)|\,dt$$
$$=\int_a^{\frac{a+b}{2}} \{(t-a)[u(b)-u(t)]+(b-t)[u(t)-u(a)]\}|g'(t)|\,dt$$
$$+\int_{\frac{a+b}{2}}^b \{(t-a)[u(b)-u(t)]+(b-t)[u(t)-u(a)]\}|g'(t)|\,dt$$
$$=2\int_a^{\frac{a+b}{2}} (t-a)|g'(t)|\,dt+2\int_{\frac{a+b}{2}}^b (b-t)|g'(t)|\,dt.$$

From (2.5) we get

$$\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(s)\,ds\right| \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} (t-a)|g'(t)|\,dt+\int_{\frac{a+b}{2}}^b (b-t)|g'(t)|\,dt\right], \tag{2.8}$$

where $g:[a,b]\to\mathbb{C}$ *is absolutely continuous. The inequality (2.8) is sharp.*

Indeed if we take $g_0(t)=\left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$ *then*

$$\left|g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|=\frac{1}{4}(b-a)$$

and

$$\int_a^{\frac{a+b}{2}} (t-a)|g'(t)|\,dt+\int_{\frac{a+b}{2}}^b (b-t)|g'(t)|\,dt$$
$$=\int_a^{\frac{a+b}{2}} (t-a)\,dt+\int_{\frac{a+b}{2}}^b (b-t)\,dt=\left.\frac{(t-a)^2}{2}\right|_a^{\frac{a+b}{2}}-\left.\frac{(b-t)^2}{2}\right|_{\frac{a+b}{2}}^b$$
$$=\frac{(b-a)^2}{8}+\frac{(b-a)^2}{8}=\frac{1}{4}(b-a)^2,$$

which give in both sides of (2.8) the same quantity $\frac{1}{4}(b-a)$.

The following lemma was obtained by the author in 2007, [7] and is of interest in itself as well (see also [6]):

Lemma 1. *If* $p:[a,b]\to\mathbb{C}$ *is continuous on* $[a,b]$ *and* $v:[a,b]\to\mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dV(t) \le \left(\int_a^b |p(t)|^p\,dV(t)\right)^{1/p}\left(\bigvee_a^b(v)\right)^{1/q} \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{2.9}$$

where $V(t) := \bigvee_a^t (v)$ *is the total variation of* v *on* $[a,t]$ *with* $t \in [a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a) = 0$ and $V(b) = \bigvee_a^b (v)$.

Corollary 3. *Assume that* $g : [a,b] \to \mathbb{C}$ *is absolutely continuous with. If* $v : [a,b] \to \mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$
\begin{aligned}
(2.10)\qquad & \left| \int_a^b g(t)\,dv(t) - \frac{v(b)-v(a)}{b-a} \int_a^b g(s)\,ds \right| \\
& \le \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right| dV(t) \\
& \le \frac{1}{b-a} \int_a^b \left\{ (t-a) \bigvee_t^b (v) + (b-t) \bigvee_a^t (v) \right\} |g'(t)|\,dt \\
& \le \begin{cases} \frac{1}{b-a} \int_a^b \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] |g'(t)|\,dt \bigvee_a^b (v), \\[2ex] \int_a^b \left[\frac{1}{2} \bigvee_a^b (v) + \left| \bigvee_a^t (v) - \frac{1}{2} \bigvee_a^b (v) \right| \right] |g'(t)|\,dt. \end{cases}
\end{aligned}
$$

Proof. By Lemma 1 and (2.5) we get

$$
\begin{aligned}
& \left| \int_a^b g(t)\,dv(t) - \frac{v(b)-v(a)}{b-a} \int_a^b g(s)\,ds \right| \\
& = \left| \int_a^b \left(g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right) dv(t) \right| \\
& \le \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right| dV(t) \\
& \le \frac{1}{b-a} \int_a^b \{ (t-a)[V(b) - V(t)] + (b-t)[V(t) - V(a)] \} |g'(t)|\,dt \\
& = \frac{1}{b-a} \int_a^b \left\{ (t-a) \bigvee_t^b (v) + (b-t) \bigvee_a^t (v) \right\} |g'(t)|\,dt \\
& \le \begin{cases} \frac{1}{b-a} \int_a^b \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] |g'(t)|\,dt \bigvee_a^b (v) \\[2ex] \int_a^b \left[\frac{1}{2} \bigvee_a^b (v) + \left| \bigvee_a^t (v) - \frac{1}{2} \bigvee_a^b (v) \right| \right] |g'(t)|\,dt, \end{cases}
\end{aligned}
$$

which proves the desired result (2.10). □

Remark 3. *Assume that $g:[a,b]\to\mathbb{C}$ is absolutely continuous. If $h:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$, then*

$$\left|\int_a^b g(t)\,h(t)\,dt-\frac{1}{b-a}\int_a^b g(s)\,ds\int_a^b h(s)\,ds\right| \tag{2.11}$$
$$\le\frac{1}{b-a}\int_a^b\left\{(t-a)\int_t^b|h(s)|\,ds+(b-t)\int_a^t|h(s)|\,ds\right\}|g'(t)|\,dt$$
$$\le\begin{cases}\frac{1}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|g'(t)|\,dt\int_a^b|h(s)|\,ds,\\[2ex] \int_a^b\left[\frac{1}{2}\int_a^b|h(s)|\,ds+\left|\int_a^t|h(s)|\,ds-\frac{1}{2}\int_a^b|h(s)|\,ds\right|\right]|g'(t)|\,dt.\end{cases}$$

3. Applications for Selfadjoint and Unitary Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, & \text{for } -\infty<s\le\lambda,\\[1ex] 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [11, p. 256]:

Theorem 4 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\min\operatorname{Sp}(A)$ and $b=\max\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\max\operatorname{Sp}(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) *$E_\lambda\le E_{\lambda'}$ for $\lambda\le\lambda'$;*
b) *$E_{a-0}=0, E_b=1_H$ and $E_{\lambda+0}=E_\lambda$ for all $\lambda\in\mathbb{R}$;*
c) *We have the representation*

$$A=\int_{a-0}^b\lambda\,dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A)\in\mathcal{B}(H)$ such that for every $\varepsilon>0$ there exists a $\delta>0$ satisfying the inequality

$$\left\|\varphi(A)-\sum_{k=1}^n\varphi(\lambda_k')\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\le\varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 4. *With the assumptions of Theorem 4 for A, E_λ and φ we have the representations*

$$\varphi(A)x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\ y \in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We have the following result:

Theorem 5. *Assume that $f : I \to \mathbb{C}$ is locally absolutely continuous with $[a, b] \subset \mathring{I}$ (the interior of I) and $f' \in L_2[a, b]$. Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in \operatorname{Sp}(A)\} =: \min \operatorname{Sp}(A)$ and $b = \max\{\lambda \,|\lambda \in \operatorname{Sp}(A)\} =: \max \operatorname{Sp}(A)$. Then*

$$\begin{aligned} &\left| f(A) - \frac{1}{b-a}\left(\int_a^b f(s)\, ds\right) 1_H \right| \\ &\leq \frac{1}{b-a} \int_{a-0}^{b} |g'(t)| \left[(t-a)(1_H - E_t) + (b-t) E_t\right] dt \\ &\leq \frac{1}{b-a}\left(\int_a^b \left[\frac{1}{2}(b-a) + \left|t - \frac{a+b}{2}\right|\right] |g'(t)|\, dt\right) 1_H \end{aligned} \tag{3.4}$$

in the operator order of $\mathcal{B}(H)$, where $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the spectral family of A.

Proof. Let $x \in H$ and small $\varepsilon > 0$. Consider the function

$$f_\varepsilon(t) := \begin{cases} f(a),\ t \in [a-\varepsilon, a) \\ f(t),\ t \in [a, b]. \end{cases},$$

Then by (2.5) on the interval $[a-\varepsilon, b]$ we get

$$\int_{a-\varepsilon}^{b}\left|f_{\varepsilon}(t)-\frac{1}{b-a+\varepsilon}\int_{a-\varepsilon}^{b}f_{\varepsilon}(s)\,ds\right|d\langle E_t x,x\rangle \tag{3.5}$$
$$\leq\frac{1}{b-a+\varepsilon}$$
$$\times\int_{a-\varepsilon}^{b}\{(t-a)[\langle E_b x,x\rangle-\langle E_t x,x\rangle]+(b-t)[\langle E_t x,x\rangle-\langle E_{a-\varepsilon}x,x\rangle]\}$$
$$\times|g'(t)|\,dt$$
$$\leq\frac{[\langle E_b x,x\rangle-\langle E_{a-\varepsilon}x,x\rangle]}{b-a+\varepsilon}$$
$$\times\int_{a-\varepsilon}^{b}\left[\frac{1}{2}(b-a+\varepsilon)+\left|t-\frac{a-\varepsilon+b}{2}\right|\right]|g'(t)|\,dt.$$

By taking the limit over $\varepsilon\to 0+$ in (3.5) and using Corollary 4, then we get

$$\left\langle\left|f(A)-\frac{1}{b-a}\left(\int_a^b f(s)\,ds\right)1_H\right|x,x\right\rangle$$
$$\leq\frac{1}{b-a}\int_{a-0}^{b}\{(t-a)[\langle x,x\rangle-\langle E_t x,x\rangle]+(b-t)\langle E_t x,x\rangle\}|g'(t)|\,dt$$
$$\leq\frac{\langle x,x\rangle}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|g'(t)|\,dt$$

namely

$$\left\langle\left|f(A)-\frac{1}{b-a}\left(\int_a^b f(s)\,ds\right)1_H\right|x,x\right\rangle$$
$$\leq\left\langle\left(\frac{1}{b-a}\int_{a-0}^{b}|g'(t)|[(t-a)(1_H-E_t)+(b-t)E_t]\,dt\right)x,x\right\rangle$$
$$\leq\left\langle\left(\frac{1}{b-a}\int_a^b\left[\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right]|g'(t)|\,dt\right)x,x\right\rangle$$

for all $x\in H$, which is equivalent, in the operator order, to (3.4). □

We say that the bounded linear operator $U:H\to H$ on the Hilbert space H is *unitary* iff $U^*=U^{-1}$.

It is well known that (see for instance [11, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda\leq E_\mu$ for $0\leq\lambda\leq\mu\leq 2\pi$;
b) $E_0=0$ and $E_{2\pi}=1_H$ (the *identity operator* on H);
c) $E_{\lambda+0}=E_\lambda$ for $0\leq\lambda<2\pi$;
d) $U=\int_0^{2\pi}e^{i\lambda}dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda=E_\lambda$ for all $\lambda\in[0,2\pi]$.

Also, for every continuous complex valued function $g:\mathcal{C}(0,1)\rightarrow\mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$g(U)=\int_0^{2\pi} g\left(e^{i\lambda}\right)dE_\lambda \tag{3.6}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle g(U)x,y\rangle=\int_0^{2\pi} g\left(e^{i\lambda}\right)d\langle E_\lambda x,y\rangle \tag{3.7}$$

and

$$\|g(U)x\|^2=\int_0^{2\pi}\left|g\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2=\int_0^{2\pi}\left|g\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x,x\rangle, \tag{3.8}$$

for any x, $y\in H$.

Theorem 6. *Assume that $g:\mathcal{C}(0,1)\rightarrow\mathbb{C}$ is of class C^1 on the circle $\mathcal{C}(0,1)$. If $U:H\rightarrow H$ is unitary, then*

$$\left|g(U)-\frac{1}{2\pi}\left(\int_0^{2\pi} g\left(e^{is}\right)ds\right)1_H\right|\leq\frac{1}{2\pi}\left(\int_0^{2\pi}(\pi+|t-\pi|)\left|g'\left(e^{it}\right)\right|dt\right)1_H, \tag{3.9}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ be the *spectral family* of U and $x\in H$. Consider the function $f(t)=g\left(e^{it}\right)$, $t\in[0,2\pi]$. By (2.5) we then have

$$\begin{aligned}&\int_0^{2\pi}\left|g\left(e^{it}\right)-\frac{1}{2\pi}\int_0^{2\pi} g\left(e^{is}\right)ds\right|d\langle E_t x,x\rangle\\&\leq\frac{\langle E_{2\pi}x,x\rangle-\langle E_0 x,x\rangle}{2\pi}\int_0^{2\pi}[\pi+|t-\pi|]\left|g'\left(e^{it}\right)\right|dt\end{aligned}$$

namely

$$\begin{aligned}&\left\langle\left(\int_0^{2\pi}\left|g\left(e^{it}\right)-\frac{1}{2\pi}\int_0^{2\pi} g\left(e^{is}\right)ds\right|dE_t\right)x,x\right\rangle\\&\leq\left\langle\left(\frac{1}{2\pi}\int_0^{2\pi}[\pi+|t-\pi|]\left|g'\left(e^{it}\right)\right|dt\right)x,x\right\rangle\end{aligned}$$

for all $x\in H$, which is, in the operator order, the desired inequality (3.9). □

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