SOME INEQUALITIES RELATED TO OPIAL'S RESULT FOR FUNCTIONS WITH VALUES IN HILBERT SPACES

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ABSTRACT. Assume that $(H;\langle\cdot,\cdot\rangle)$ is a complex Hilbert space. In this paper we provide some upper bounds for the integral

$$\left|\int_a^b \langle f(t), g(t)\rangle dt\right|$$

under various assumptions for the absolutely continuous functions $f, g : [a,b] \to H$ including the case when $f(a) = g(b) = 0$. In particular, we obtain the following Wirtinger type sharp inequality

$$\int_a^b \|f(t)\|^2 dt \le \int_a^b (b-t)(t-a) \|f'(t)\|^2 dt,$$

provided that $f(a) = f(b) = 0$. Applications for Trapezoid and Grüss' type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u' \in L_2[a,b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| dt \le \frac{1}{4}(b-a) \int_a^b |u'(t)|^2 dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \le t \le \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \le b, \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| dt \le \frac{1}{2}(b-a) \int_a^b |u'(t)|^2 dt, \tag{1.2}$$

with equality if and only if $u(t) = c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [8] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [9].

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Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [3], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [5]-[7] and [10].

In 1975, G. G. Vrânceanu extended Opial's inequality (1.2) for functions with values in Hilbert spaces $(H; \langle\cdot,\cdot\rangle)$ as follows:

Theorem 2. *Assume that the function $f : [a,b] \to H$ has a continuous derivative and $f(a) = 0$, then*

$$\int_a^b |\langle f(t), f'(t)\rangle| \, dt \le \frac{1}{2}(b-a) \int_a^b \|f'(t)\|^2 \, dt. \tag{1.3}$$

In the recent paper [4] we obtained the following better result:

Theorem 3. *Assume that $f : [a,b] \to H$ is absolutely continuous on $[a,b]$ and $f' \in L_2([a,b], H)$.*

(i) *If either $f(a) = 0$ or $f(b) = 0$, then*

$$\begin{aligned} &\int_a^b |\langle f'(t), f(t)\rangle| \, dt \\ &\le \left(\int_a^b (t-a)\|f'(t)\|^2 dt\right)^{1/2} \left(\int_a^b (b-t)\|f'(t)\|^2 dt\right)^{1/2} \\ &\le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2 dt. \end{aligned} \tag{1.4}$$

(ii) *If $f(a) = f(b) = 0$, then*

$$\begin{aligned} &\int_a^b |\langle f'(t), f(t)\rangle| \, dt \\ &\le \left[\int_a^b K(t)\|f'(t)\|^2 dt\right]^{1/2} \left[\int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 dt\right]^{1/2} \\ &\le \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2 dt, \end{aligned} \tag{1.5}$$

where

$$K(t) := \begin{cases} t-a & \text{if } a \le t \le \frac{a+b}{2}, \\ \\ b-t & \text{if } \frac{a+b}{2} < t \le b. \end{cases}$$

Motivated by the above results, in this paper we provide some upper bounds for the integral

$$\left|\int_a^b \langle f(t), g(t)\rangle \, dt\right|$$

under various assumptions for the absolutely continuous functions f, $g : [a,b] \to H$ including the case when $f(a) = g(b) = 0$. In particular, we obtain the following Wirtinger type sharp inequality

$$\int_a^b \|f(t)\|^2 dt \le \int_a^b (b-t)(t-a)\|f'(t)\|^2 dt,$$

provided that $f(a) = f(b) = 0$. Applications for Trapezoid and Grüss' type inequalities are also given.

2. Main Results

Assume that $(H;\langle\cdot,\cdot\rangle)$ is a complex Hilbert space. We consider the positive weight

$$w_a(t;a,b):=\frac{1}{2}\left[(b-a)^2-(t-a)^2\right]=(b-t)\left(\frac{b+t}{2}-a\right),$$

for $t\in[a,b]$.

Theorem 4. *Assume that $f,\ g:[a,b]\to H$ are absolutely continuous with $f(a)=g(a)=0$ and $f',\ g'\in L_{2,w_a}([a,b],H)$, then*

$$\begin{aligned}(2.1)\qquad &\left|\int_a^b\langle f(t),g(t)\rangle dt\right|\\ &\le\int_a^b|\langle f(t),g(t)\rangle|\,dt\\ &\le\left(\int_a^b w_a(t;a,b)\,\|f'(t)\|^2dt\right)^{1/2}\left(\int_a^b w_a(t;a,b)\,\|g'(t)\|^2dt\right)^{1/2}\\ &\le\frac{1}{2}\int_a^b w_a(t;a,b)\left[\|f'(t)\|^2+\|g'(t)\|^2\right]dt.\end{aligned}$$

The inequalities in (2.1) are sharp.

Proof. Since $f(a)=g(a)=0$, hence $f(t)=\int_a^t f'(s)\,ds$ and $g(t)=\int_a^t g'(s)\,ds$ and we have by Schwarz inequality in H,

$$\begin{aligned}(2.2)\qquad &\int_a^b|\langle f(t),g(t)\rangle|\,dt\le\int_a^b\|f(t)\|\,\|g(t)\|\,dt\\ &=\int_a^b\left\|\int_a^t f'(s)\,ds\right\|\left\|\int_a^t g'(s)\,ds\right\|dt\\ &=\int_a^b(t-a)(t-a)^{-1/2}\left\|\int_a^t f'(s)\,ds\right\|(t-a)^{-1/2}\left\|\int_a^t g'(s)\,ds\right\|dt\\ &=:A.\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(t-a)^{-1/2}\left\|\int_a^t f'(s)\,ds\right\|\le\left(\int_a^t\|f'(s)\|^2ds\right)^{1/2}$$

and

$$(t-a)^{-1/2}\left\|\int_a^t g'(s)\,ds\right\|\le\left(\int_a^t\|g'(s)\|^2ds\right)^{1/2}$$

for all $t\in[a,b]$.

Therefore

$$A\le\int_a^b(t-a)\left(\int_a^t\|f'(s)\|^2ds\right)^{1/2}\left(\int_a^t\|g'(s)\|^2ds\right)^{1/2}dt.$$

By utilising Cauchy-Bunyakowsky-Schwarz weighted integral inequality, we have

$$\begin{aligned}
&\int_a^b (t-a)\left(\int_a^t \|f'(s)\|^2 ds\right)^{1/2}\left(\int_a^t \|g'(s)\|^2 ds\right)^{1/2} dt \qquad (2.3)\\
&\leq \left[\int_a^b (t-a)\left(\left(\int_a^t \|f'(s)\|^2 ds\right)^{1/2}\right)^2 dt\right]^{1/2}\\
&\times \left[\int_a^b (t-a)\left(\left(\int_a^t \|g'(s)\|^2 ds\right)^{1/2}\right)^2 dt\right]^{1/2}\\
&= \left[\int_a^b (t-a)\left(\int_a^t \|f'(s)\|^2 ds\right) dt\right]^{1/2}\\
&\times \left[\int_a^b (t-a)\left(\int_a^t \|g'(s)\|^2 ds\right) dt\right]^{1/2}\\
&=: B.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_a^b (t-a)\left(\int_a^t \|f'(s)\|^2 ds\right) dt\\
&= \int_a^b \left(\int_a^t \|f'(s)\|^2 ds\right) d\left(\frac{(t-a)^2}{2}\right)\\
&= \left(\int_a^t \|f'(s)\|^2 ds\right)\left(\frac{(t-a)^2}{2}\right)\Bigg|_a^b - \int_a^b \frac{(t-a)^2}{2}\|f'(t)\|^2 dt\\
&= \frac{(b-a)^2}{2}\left(\int_a^b \|f'(s)\|^2 ds\right) - \int_a^b \frac{(t-a)^2}{2}\|f'(t)\|^2 dt\\
&= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right]\|f'(t)\|^2 dt
\end{aligned}$$

and

$$\int_a^b (t-a)\left(\int_a^t \|g'(s)\|^2 ds\right) dt = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right]\|g'(t)\|^2 dt.$$

Therefore

$$\begin{aligned}
B &\leq \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right]\|f'(t)\|^2 dt\right)^{1/2} \qquad (2.4)\\
&\times \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2}\right]\|g'(t)\|^2 dt\right)^{1/2}\\
&= \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2 dt\right)^{1/2}\left(\int_a^b w_a(t;a,b)\|g'(t)\|^2 dt\right)^{1/2}.
\end{aligned}$$

By making use of (2.2)-(2.4) we derive the first part in (2.1). The second part follows by the arithmetic mean-geometric mean inequality,

$$\sqrt{\alpha\beta} \leq \frac{\alpha+\beta}{2},\ \ \alpha,\beta \geq 0.$$

Now, consider in the case that $H=\mathbb{R}$, $f(t)=g(t)=t-a$. Then

$$\int_a^b |f(t)\,g(t)|\,dt = \int_a^b (t-a)^2\,dt = \frac{1}{3}(b-a)^3$$

and

$$\begin{aligned}
&\frac{1}{2}\int_a^b w_a(t;a,b)\left[|f'(t)|^2+|g'(t)|^2\right]dt \\
&= \int_a^b w_a(t;a,b)\,dt = \frac{1}{2}\int_a^b \left[(b-a)^2-(t-a)^2\right] \\
&= \frac{1}{2}\left[(b-a)^3-\frac{1}{3}(b-a)^3\right] = \frac{1}{3}(b-a)^3,
\end{aligned}$$

which shows that all terms in (2.1) are equal with $\frac{1}{3}(b-a)^3$. □

Remark 1. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a)=f'(a)=0$ and $f',\ f'' \in L_{2,w_a}([a,b],H)$ then*

$$\begin{aligned}
(2.5)\qquad &\int_a^b |\langle f(t), f'(t)\rangle|\,dt \\
&\leq \left(\int_a^b w_a(t;a,b)\,\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_a(t;a,b)\,\|f''(t)\|^2\,dt\right)^{1/2} \\
&\leq \frac{1}{2}\int_a^b w_a(t;a,b)\left[\|f'(t)\|^2+\|f''(t)\|^2\right]dt.
\end{aligned}$$

The inequality follows by (2.17) for $g=f'$.

Corollary 1. *Assume that $f:[a,b]\to H$ is absolutely continuous with $f(a)=0$ and $f' \in L_{2,w_a}([a,b],H)$, then*

$$\int_a^b \|f(t)\|^2\,dt \leq \int_a^b w_a(t;a,b)\,\|f'(t)\|^2\,dt. \tag{2.6}$$

The inequality in (2.6) is sharp.

Now consider the dual weight

$$w_b(t;a,b) := \frac{1}{2}\left[(b-a)^2-(b-t)^2\right] = (t-a)\left(b-\frac{a+t}{2}\right),$$

for $t\in[a,b]$.

Theorem 5. *Assume that* $f,\ g:[a,b]\rightarrow H$ *are absolutely continuous with* $f(b)=g(b)=0$ *and* $f',\ g'\in L_{2,w_b}[a,b]$, *then*

$$\left|\int_a^b \langle f(t),g(t)\rangle dt\right| \tag{2.7}$$
$$\leq \int_a^b |\langle f(t),g(t)\rangle| dt$$
$$\leq \left(\int_a^b w_b(t;a,b)\|f'(t)\|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|g'(t)\|^2 dt\right)^{1/2}$$
$$\leq \frac{1}{2}\int_a^b w_b(t;a,b)\left[\|f'(t)\|^2+\|g'(t)\|^2\right]dt.$$

The inequalities in (2.7) are sharp.

Proof. Since $f(b)=g(b)=0$, hence $f(t)=-\int_t^b f'(s)\,ds$ and $g(t)=-\int_t^b g'(s)\,ds$ and we have

$$\int_a^b |\langle f(t),g(t)\rangle| dt \tag{2.8}$$
$$\leq \int_a^b \|f(t)\|\,\|g(t)\|\,dt$$
$$=\int_a^b \left\|\int_t^b f'(s)\,ds\right\|\left\|\int_t^b g'(s)\,ds\right\|dt$$
$$=\int_a^b (b-t)(b-t)^{-1/2}\left\|\int_t^b f'(s)\,ds\right\|(b-t)^{-1/2}\left\|\int_t^b g'(s)\,ds\right\|dt$$
$$=: C.$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(b-t)^{-1/2}\left\|\int_t^b f'(s)\,ds\right\|\leq\left(\int_t^b \|f'(s)\|^2 ds\right)^{1/2}$$

and

$$(b-t)^{-1/2}\left\|\left\|\int_t^b g'(s)\,ds\right\|\right\|\leq\left(\int_t^b \|g'(s)\|^2 ds\right)^{1/2}$$

for all $t\in[a,b]$.

Therefore

$$C\leq\int_a^b (b-t)\left(\int_t^b \|f'(s)\|^2 ds\right)^{1/2}\left(\int_t^b \|g'(s)\|^2 ds\right)^{1/2}dt.$$

By utilising Cauchy-Bunyakowsky-Schwarz weighted integral inequality, we have

$$\int_a^b (b-t)\left(\int_t^b \|f'(s)\|^2 ds\right)^{1/2}\left(\int_t^b \|g'(s)\|^2 ds\right)^{1/2}dt \tag{2.9}$$

$$\leq \left[\int_a^b (b-t)\left(\left(\int_t^b \|f'(s)\|^2 ds\right)^{1/2}\right)^2 dt\right]^{1/2}$$
$$\times \left[\int_a^b (b-t)\left(\left(\int_t^b \|g'(s)\|^2 ds\right)^{1/2}\right)^2 dt\right]^{1/2}$$
$$= \left[\int_a^b (b-t)\left(\int_t^b \|f'(s)\|^2 ds\right) dt\right]^{1/2}\left[\int_a^b (b-t)\left(\int_t^b \|g'(s)\|^2 ds\right) dt\right]^{1/2}$$
$$=: D.$$

Using integration by parts, we have

$$\int_a^b (b-t)\left(\int_t^b \|f'(s)\|^2 ds\right) dt$$
$$= -\int_a^b \left(\int_t^b \|f'(s)\|^2 ds\right) d\left(\frac{(b-t)^2}{2}\right)$$
$$= -\left(\int_t^b \|f'(s)\|^2 ds\right)\left(\frac{(b-t)^2}{2}\right)\Bigg|_a^b - \int_a^b \frac{(b-t)^2}{2}\|f'(t)\|^2 dt$$
$$= \frac{(b-a)^2}{2}\left(\int_a^b \|f'(s)\|^2 ds\right) - \int_a^b \frac{(b-t)^2}{2}\|f'(t)\|^2 dt$$
$$= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]\|f'(t)\|^2 dt$$

and

$$\int_a^b (b-t)\left(\int_t^b \|g'(s)\|^2 ds\right) dt = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]\|g'(t)\|^2 dt.$$

Therefore

$$D \leq \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]\|f'(t)\|^2 dt\right)^{1/2} \tag{2.10}$$
$$\times \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]\|g'(t)\|^2 dt\right)^{1/2}$$
$$= \left(\int_a^b w_b(t;a,b)\|f'(t)\|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|g'(t)\|^2 dt\right)^{1/2}.$$

By making use of (2.8)-(2.10) we derive the first part in (2.7). The second part follows by the arithmetic mean-geometric mean inequality,

$$\sqrt{\alpha\beta} \leq \frac{\alpha+\beta}{2},\ \alpha,\beta \geq 0. \tag{2.11}$$

The sharpness follows by taking $f(t) = b-t$, $t \in [a,b]$. □

Remark 2. *Assume that f' is absolutely continuous on $[a,b]$. If with $f(b) = f'(b) = 0$ and $f',\ f'' \in L_{2,w_b}[a,b]$, then*

$$\int_a^b |\langle f(t), f'(t)\rangle|\,dt \le \left(\int_a^b w_b(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|f''(t)\|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b w_b(t;a,b)\left[\|f'(t)\|^2+\|f''(t)\|^2\right]dt. \tag{2.12}$$

Corollary 2. *Assume that $f:[a,b]\to H$ is absolutely continuous with $f(b)=0$ and $f'\in(L_{2,w_b}[a,b],H)$, then*

$$\int_a^b \|f(t)\|^2\,dt \le \int_a^b w_b(t;a,b)\|f'(t)\|^2\,dt. \tag{2.13}$$

The inequality in (2.13) is sharp.

We also have:

Theorem 6. *Assume that $f,\ g:[a,b]\to H$ are absolutely continuous with $f(a) = g(b) = 0$ and $f'\in L_{2,w_a}([a,b],H)$, $g'\in L_{2,w_b}([a,b],H)$ then*

$$\begin{aligned}&\left|\int_a^b \langle f(t), g(t)\rangle\,dt\right| \\ &\le \int_a^b |\langle f(t), g(t)\rangle|\,dt \\ &\le \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|g'(t)\|^2\,dt\right)^{1/2} \\ &\le \frac{1}{4}(b-a)^2\int_a^b\left[\|f'(t)\|^2+\|g'(t)\|^2\right]dt \\ &-\frac{1}{4}\int_a^b\left[(t-a)^2\|f'(t)\|^2+(b-t)^2\|g'(t)\|^2\right]dt.\end{aligned} \tag{2.14}$$

The inequalities in (2.7) are sharp.

Proof. Since $f(a)=g(b)=0$, hence $f(t)=\int_a^t f'(s)\,ds$ and $g(t)=-\int_t^b g'(s)\,ds$. Therefore

$$\begin{aligned}&\int_a^b |\langle f(t), g(t)\rangle|\,dt \\ &\le \int_a^b \|f(t)\|\|g(t)\|\,dt = \int_a^b\left|\int_a^t f'(s)\,ds\right|\left|\int_t^b g'(s)\,ds\right|dt \\ &= \int_a^b (t-a)^{1/2}(b-t)^{1/2}(t-a)^{-1/2} \\ &\times\left\|\int_a^t f'(s)\,ds\right\|(b-t)^{-1/2}\left\|\int_t^b g'(s)\,ds\right\|dt \\ &=: E.\end{aligned} \tag{2.15}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(t-a)^{-1/2}\left\|\int_a^t f'(s)\,ds\right\| \le \left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}$$

and

$$(b-t)^{-1/2}\left\|\int_t^b g'(s)\,ds\right\| \le \left(\int_t^b \|g'(s)\|^2\,ds\right)^{1/2},$$

which imply that

$$\begin{aligned}E &\le \int_a^b (t-a)^{1/2}(b-t)^{1/2}\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}\left(\int_t^b \|g'(s)\|^2\,ds\right)^{1/2}\\ &= \int_a^b (t-a)^{1/2}\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}(b-t)^{1/2}\left(\int_t^b \|g'(s)\|^2\,ds\right)^{1/2}.\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we also have

$$\begin{aligned}&\int_a^b (t-a)^{1/2}\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}(b-t)^{1/2}\left(\int_t^b \|g'(s)\|^2\,ds\right)^{1/2}\\ &\le \left(\int_a^b \left[(t-a)^{1/2}\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}\right]^2 dt\right)^{1/2}\\ &\times \left(\int_a^b \left[(b-t)^{1/2}\left(\int_t^b \|g'(s)\|^2\,ds\right)^{1/2}\right]^2 dt\right)^{1/2}\\ &= \left(\int_a^b (t-a)\left(\int_a^t \|f'(s)\|^2\,ds\right)\right)^{1/2}\left(\int_a^b (b-t)\left(\int_t^b \|g'(s)\|^2\,ds\right)dt\right)^{1/2}\\ &= \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|g'(t)\|^2\,dt\right)^{1/2},\end{aligned}$$

which proves the first inequality in (2.14).

By (2.11) inequality, we have that

$$\begin{aligned}&\left(\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|g'(t)\|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}\left[\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt + \int_a^b w_b(t;a,b)\|g'(t)\|^2\,dt\right]\\ &= \frac{1}{2}\left[\int_a^b \frac{1}{2}\left[(b-a)^2-(t-a)^2\right]\|f'(t)\|^2\,dt\right.\\ &\left.+\int_a^b \frac{1}{2}\left[(b-a)^2-(b-t)^2\right]\|g'(t)\|^2\,dt\right]\end{aligned}$$

$$= \frac{1}{2}\left[\int_a^b \frac{1}{2}\left[(b-a)^2-(t-a)^2\right]\|f'(t)\|^2\,dt\right.$$
$$\left.+\int_a^b \frac{1}{2}\left[(b-a)^2-(b-t)^2\right]\|g'(t)\|^2\,dt\right]$$
$$= \frac{1}{4}(b-a)^2\int_a^b\left[\|f'(t)\|^2+\|g'(t)\|^2\right]dt$$
$$-\frac{1}{4}\int_a^b\left[(t-a)^2\|f'(t)\|^2+(b-t)^2\|g'(t)\|^2\right]dt$$

and the second part of inequality (2.14) is proved. □

Remark 3. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a)=f'(b)=0$ and $f'\in L_{2,w_a}([a,b],H)$, $f''\in L_{2,w_b}([a,b],H)$ then*

$$\int_a^b |\langle f(t), f'(t)\rangle|\,dt \tag{2.16}$$
$$\le \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|f''(t)\|^2\,dt\right)^{1/2}$$
$$\le \frac{1}{4}(b-a)^2\int_a^b\left[\|f'(t)\|^2+\|f''(t)\|^2\right]dt$$
$$-\frac{1}{4}\int_a^b\left[(t-a)^2\|f'(t)\|^2+(b-t)^2\|f''(t)\|^2\right]dt.$$

Corollary 3. *Assume that $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_{2,w_a}([a,b],H)\cap L_{2,w_b}([a,b],H)$ then*

$$\int_a^b \|f(t)\|^2\,dt \tag{2.17}$$
$$\le \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}$$
$$\le \int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\|f'(t)\|^2\,dt$$
$$=\int_a^b (b-t)(t-a)\|f'(t)\|^2\,dt.$$

The inequalities (2.17) are sharp.

Proof. From (2.14) we get for $g=f$ that

$$\int_a^b \|f(t)\|^2\,dt$$
$$\le \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|f'(t)\|^2\,dt\right)^{1/2}$$
$$\le \frac{1}{2}(b-a)^2\int_a^b\|f'(t)\|^2\,dt-\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\|f'(t)\|^2\,dt.$$

Since

$$\frac{(t-a)^2+(b-t)^2}{2}=\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2,$$

hence

$$\begin{aligned}
&\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\left\|f'(t)\right\|^2dt\\
&=\int_a^b\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\left\|f'(t)\right\|^2dt\\
&=\frac{1}{4}(b-a)^2\int_a^b\left\|f'(t)\right\|^2dt+\int_a^b\left(t-\frac{a+b}{2}\right)^2\left\|f'(t)\right\|^2dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}(b-a)^2\int_a^b\left\|f'(t)\right\|^2dt-\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\left\|f'(t)\right\|^2dt\\
&=\frac{1}{2}(b-a)^2\int_a^b\left\|f'(t)\right\|^2dt-\frac{1}{4}(b-a)^2\int_a^b\left\|f'(t)\right\|^2dt\\
&-\int_a^b\left(t-\frac{a+b}{2}\right)^2\left\|f'(t)\right\|^2dt\\
&=\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\left\|f'(t)\right\|^2dt,
\end{aligned}$$

which proves the second part of the inequality (2.17).

In the case when $H=\mathbb{R}$, consider the function

$$f(t)=(t-a)(b-t),\ t\in[a,b].$$

Then $f'(t)=a+b-2t$, $t\in(a,b)$,

$$\int_a^b|f(t)|^2dt=\int_a^b(t-a)^2(b-t)^2dt=\frac{1}{30}(b-a)^5$$

and

$$\begin{aligned}
&\int_a^b(b-t)(t-a)\left|f'(t)\right|^2dt\\
&=\int_a^b(b-t)(t-a)\left|(a+b)t-2t\right|^2dt\\
&=4\int_a^b(b-t)(t-a)\left(t-\frac{a+b}{2}\right)^2dt=\frac{1}{30}(b-a)^5,
\end{aligned}$$

which show that the inequalities (2.17) are sharp. □

Theorem 7. *Assume that $f,\ g:[a,b]\to H$ are absolutely continuous with $f(a)=g(a)=0$ and $f(b)=g(b)=0$ with $f',\ g'\in L_{2,w_a}([a,b],H)\cap L_{2,w_b}([a,b],H)$ then*

$$\int_a^b\left|\langle f(t),g(t)\rangle\right|dt\leq\frac{1}{4}\int_a^bL(t;a,b)\left[\left\|f'(t)\right\|^2+\left\|g'(t)\right\|^2\right]dt, \tag{2.18a}$$

where

$$L(t;a,b) := \frac{1}{4}(b-a)^2 - \begin{cases} (t-a)^2, \ t \in \left[0, \frac{a+b}{2}\right], \\ (b-t)^2, \ t \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{2.19}$$

Proof. From (2.1) we have

$$\begin{aligned} &\int_a^{\frac{a+b}{2}} |\langle f(t), g(t)\rangle| \, dt \\ &\leq \frac{1}{2}\int_a^{\frac{a+b}{2}} w_a\left(t; a, \frac{a+b}{2}\right)\left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt. \\ &= \frac{1}{4}\int_a^{\frac{a+b}{2}} \left[\left(\frac{a+b}{2} - a\right)^2 - (t-a)^2\right]\left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt, \end{aligned} \tag{2.20}$$

while from (2.7) we have

$$\begin{aligned} &\int_{\frac{a+b}{2}}^b |\langle f(t), g(t)\rangle| \, dt \\ &\leq \frac{1}{2}\int_{\frac{a+b}{2}}^b w_b\left(t; \frac{a+b}{2}, b\right)\left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \\ &= \frac{1}{4}\int_{\frac{a+b}{2}}^b \left[\left(b - \frac{a+b}{2}\right)^2 - (b-t)^2\right]\left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt. \end{aligned} \tag{2.21}$$

If we add these two inequalities, then we get

$$\begin{aligned} \int_a^b |\langle f(t), g(t)\rangle| \, dt &\leq \frac{1}{4}\int_a^{\frac{a+b}{2}} \left[\frac{1}{4}(b-a)^2 - (t-a)^2\right]\left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \\ &+ \frac{1}{4}\int_{\frac{a+b}{2}}^b \left[\frac{1}{4}(b-a)^2 - (b-t)^2\right]\left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \\ &= \frac{1}{16}(b-a)^2 \int_a^{\frac{a+b}{2}} \left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \\ &+ \frac{1}{16}(b-a)^2 \int_{\frac{a+b}{2}}^b \left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \\ &- \frac{1}{4}\int_a^{\frac{a+b}{2}} (t-a)^2 \left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \\ &- \frac{1}{4}\int_{\frac{a+b}{2}}^b (b-t)^2 \left[\|f'(t)\|^2 + \|g'(t)\|^2\right] dt \end{aligned}$$

$$= \frac{1}{16}(b-a)^2 \int_a^b \left[\|f'(t)\|^2 + \|g'(t)\|^2 \right] dt$$
$$- \frac{1}{4} \int_a^{\frac{a+b}{2}} (t-a)^2 \left[\|f'(t)\|^2 + \|g'(t)\|^2 \right] dt$$
$$- \frac{1}{4} \int_{\frac{a+b}{2}}^b (b-t)^2 \left[\|f'(t)\|^2 + \|g'(t)\|^2 \right] dt$$
$$= \frac{1}{4} \int_a^b L(t;a,b) \left[\|f'(t)\|^2 + \|g'(t)\|^2 \right] dt,$$

which proves the desired result. □

Corollary 4. *Assume that $f : [a,b] \to H$ is absolutely continuous with $f(a) = f(b) = 0$ with $f' \in L_{2,w_a}([a,b], H) \cap L_{2,w_b}([a,b], H)$, then*

$$\int_a^b \|f(t)\|^2 dt \le \frac{1}{2} \int_a^b L(t;a,b) \|f'(t)\|^2 dt, \tag{2.22}$$

where $L(t;a,b)$ is defined by (2.19). The constant $\frac{1}{2}$ is best possible.

Proof. For the case $H = \mathbb{R}$ we consider the function

$$f(t) := \begin{cases} t-a, & t \in \left[a, \frac{a+b}{2}\right], \\ b-t, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then f is absolutely continuous, $|f'(t)| = 1$, $t \in (a,b)$,

$$\int_a^b f^2(t)\, dt = \int_a^{\frac{a+b}{2}} (t-a)^2 dt + \int_{\frac{a+b}{2}}^b (t-b)^2 dt$$
$$= \frac{1}{24}(b-a)^3 + \frac{1}{24}(b-a)^3 = \frac{1}{12}(b-a)^3$$

and

$$\frac{1}{2} \int_a^b L(t;a,b) [f'(t)]^2 dt$$
$$= \frac{1}{2} \int_a^b L(t;a,b)\, dt$$
$$= \frac{1}{2} \int_a^b \frac{1}{4}(b-a)^2 dt - \frac{1}{2} \left[\int_a^{\frac{a+b}{2}} (t-a)^2 dt + \int_{\frac{a+b}{2}}^b (t-b)^2 dt \right]$$
$$= \frac{1}{8}(b-a)^3 - \frac{1}{24}(b-a)^3 = \frac{1}{12}(b-a)^3,$$

which shows that in both sides of (2.22) we get the same quantity $\frac{1}{12}(b-a)^3$.

This proves the sharpness of the constant $\frac{1}{2}$. □

Remark 4. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a) = f'(a) = 0$ and $f(b) = f'(b) = 0$ with $f', f'' \in L_{2,w_a}[a,b] \cap L_{2,w_b}[a,b]$, then*

$$\int_a^b |f(t) f'(t)|\, dt \le \frac{1}{4} \int_a^b L(t;a,b) \left[|f'(t)|^2 + |f''(t)|^2 \right] dt. \tag{2.23}$$

3. Applications

Assume that $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_{2,w_a}([a,b],H)\cap L_{2,w_b}([a,b],H)$ then

$$\begin{aligned}(3.1)\qquad &\int_a^b \|f(t)\|^2 dt\\ &\le \left(\int_a^b w_a(t;a,b)\|f'(t)\|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\|f'(t)\|^2 dt\right)^{1/2}\\ &\le \int_a^b (b-t)(t-a)\|f'(t)\|^2 dt\end{aligned}$$

and

$$(3.2)\qquad \int_a^b \|f(t)\|^2 dt \le \frac{1}{2}\int_a^b L(t;a,b)\|f'(t)\|^2 dt.$$

We have the following trapezoid type inequalities:

Proposition 1. *Let $g\in C^1([a,b],H)$. Then*

$$\begin{aligned}(3.3)\qquad &\left\|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|^2\\ &\le \frac{1}{4}\left(\frac{1}{b-a}\int_a^b w_a(t;a,b)\|g'(t)-g'(a+b-t)\|^2 dt\right)^{1/2}\\ &\times\left(\frac{1}{b-a}\int_a^b w_b(t;a,b)\|g'(t)-g'(a+b-t)\|^2 dt\right)^{1/2}\\ &\le \frac{1}{4}\frac{1}{b-a}\int_a^b (b-t)(t-a)\|g'(t)-g'(a+b-t)\|^2 dt\end{aligned}$$

and

$$\begin{aligned}(3.4)\qquad &\left\|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right\|^2\\ &\le \frac{1}{8}\int_a^b L(t;a,b)\|g'(t)-g'(a+b-t)\|^2 dt.\end{aligned}$$

Proof. If $g\in C^1([a,b],H)$, then by taking

$$f(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b]$$

we have $f(a)=f(b)=0$,

$$f'(t)=\frac{g'(t)-g'(a+b-t)}{2}$$

and by (3.1) we derive

$$\begin{aligned}
&\int_a^b \left\| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right\|^2 dt \\
&\le \frac{1}{4}\left(\int_a^b w_a(t;a,b) \left\| g'(t) - g'(a+b-t) \right\|^2 dt \right)^{1/2} \\
&\times \left(\int_a^b w_b(t;a,b) \left\| g'(t) - g'(a+b-t) \right\|^2 dt \right)^{1/2} \\
&\le \frac{1}{4} \int_a^b (b-t)(t-a) \left\| g'(t) - g'(a+b-t) \right\|^2 dt.
\end{aligned} \tag{3.5}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned}
&(b-a)\int_a^b \left\| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right\|^2 dt \\
&\ge \left\| \int_a^b \left[\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right] dt \right\|^2 \\
&= \left\| \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}(b-a) \right\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left\| \frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt \right\|^2 \\
&\le \frac{1}{b-a}\int_a^b \left\| \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2} \right\|^2 dt.
\end{aligned} \tag{3.6}$$

By utilising (3.5) and (3.6) we derive the desired result (3.3).
The inequality (3.4) follows from (3.2). □

Proposition 2. *Let $g \in C^1([a,b],H)$. Then*

$$\begin{aligned}
&\left\| \frac{1}{b-a}\int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2} \right\|^2 \\
&\le \left(\frac{1}{b-a}\int_a^b w_a(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt \right)^{1/2} \\
&\times \left(\frac{1}{b-a}\int_a^b w_b(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt \right)^{1/2} \\
&\le \frac{1}{b-a}\int_a^b (b-t)(t-a) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt,
\end{aligned} \tag{3.7}$$

and

$$\left\| \frac{1}{b-a} \int_a^b g(t)\, dt - \frac{g(a)+g(b)}{2} \right\|^2 \tag{3.8}$$
$$\le \frac{1}{2} \int_a^b L(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt.$$

Proof. If $g \in C^1([a,b],H)$, then by taking

$$f(t) := g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a},\ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (3.1) we have

$$\int_a^b \left\| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right\|^2 dt \tag{3.9}$$
$$\le \left(\int_a^b w_a(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt \right)^{1/2}$$
$$\times \left(\int_a^b w_b(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt \right)^{1/2}$$
$$\le \int_a^b (b-t)(t-a) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt$$

By Jensen's inequality for $\|\cdot\|^2$ we have

$$\frac{1}{b-a} \int_a^b \left\| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right\|^2 dt \tag{3.10}$$
$$\ge \left\| \frac{1}{b-a} \int_a^b \left(g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right) dt \right\|^2$$
$$= \left\| \frac{1}{b-a} \int_a^b g(t)\, dt - \frac{g(a)+g(b)}{2} \right\|^2.$$

By utilising (3.9) and (3.10) we derive

$$\left\| \frac{1}{b-a} \int_a^b g(t)\, dt - \frac{g(a)+g(b)}{2} \right\|^2$$
$$\le \frac{1}{b-a} \int_a^b \left\| g(t) - \frac{g(a)(b-t)+g(b)(t-a)}{b-a} \right\|^2 dt$$
$$\le \left(\frac{1}{b-a} \int_a^b w_a(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt \right)^{1/2}$$
$$\times \left(\frac{1}{b-a} \int_a^b w_b(t;a,b) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt \right)^{1/2}$$
$$\le \frac{1}{b-a} \int_a^b (b-t)(t-a) \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 dt,$$

which proves (3.7).

The inequality (3.8) follows from (3.2). □

We also have:

Proposition 3. *Let $g \in C([a,b], H)$. Then*

$$\left\| \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b tg(t)\,dt \right\|^2 \le \left((b-a) \int_a^b w_a(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \right)^{1/2} \times \left((b-a) \int_a^b w_b(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \right)^{1/2} \le (b-a) \int_a^b (b-t)(t-a) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt. \tag{3.11}$$

and

$$\left\| \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b tg(t)\,dt \right\|^2 \le \frac{1}{2} \int_a^b L(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt. \tag{3.12}$$

Proof. Assume that $g : [a,b] \to H$ is continuous, then by taking

$$f(t) := \int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds, \ t \in [a,b]$$

we have $f(a) = f(b) = 0$, and by (3.1)

$$\int_a^b \left\| \int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \le \left(\int_a^b w_a(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \right)^{1/2} \times \left(\int_a^b w_b(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \right)^{1/2} \le \int_a^b (b-t)(t-a) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt. \tag{3.13}$$

By Jensen's inequality we also have

$$\frac{1}{b-a}\int_a^b \left\|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right\|^2 dt$$
$$\geq \left\|\frac{1}{b-a}\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt\right\|^2,$$

namely

$$\left\|\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt\right\|^2 \tag{3.14}$$
$$\leq (b-a)\int_a^b \left\|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right\|^2 dt$$

Observe that, integrating by parts, we have

$$\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt \tag{3.15}$$
$$= \int_a^b \left(\int_a^t g(s)\,ds\right) dt - \frac{b-a}{2}\int_a^b g(s)\,ds$$
$$= b\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt - \frac{b-a}{2}\int_a^b g(s)\,ds$$
$$= \frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt.$$

Therefore by (3.13)-(3.15) we derive

$$\left\|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right\|^2 \tag{3.16}$$
$$\leq \left((b-a)\int_a^b w_a(t;a,b)\left\|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right\|^2 dt\right)^{1/2}$$
$$\times \left((b-a)\int_a^b w_b(t;a,b)\left\|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right\|^2 dt\right)^{1/2}$$
$$\leq (b-a)\int_a^b (b-t)(t-a)\left\|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right\|^2 dt,$$

which proves (3.11).

The inequality (3.12) follows from (3.2). □

Consider now the *weighted Čebyšev functional*

$$C_w(\alpha, g) := \int_a^b w(t)\,\alpha(t)\,g(t)\,dt - \int_a^b w(t)\,\alpha(t)\,dt \int_a^b w(t)\,g(t)\,dt$$

where $w:[a,b]\to\mathbb{R}$ and $w(t)\geq 0$ for a.e. $t\in[a,b]$, $\alpha:[a,b]\to\mathbb{C}$ and $g:[a,b]\to H$ are functions such that the involved integrals exist and $\int_a^b w(t)\,dt=1$.

Theorem 8. *Assume that* $w:[a,b]\to(0,\infty)$ *is a probability density function on* $[a,b]$, α *is absolutely continuous on* $[a,b]$ *and* $g\in C([a,b],H)$, *then*

$$\begin{aligned}(3.17)\quad &\left\|C_w(\alpha,g)\right\|^2\\ &\leq\int_a^b|\alpha'(t)|^2dt\left(\int_a^b w^2(t)\,w_a(t;a,b)\left\|g(t)-\int_a^b w(s)\,g(s)\,ds\right\|^2dt\right)^{1/2}\\ &\times\left(\int_a^b w^2(t)\,w_b(t;a,b)\left\|g(t)-\int_a^b w(s)\,g(s)\,ds\right\|^2dt\right)^{1/2}\\ &\leq\int_a^b|\alpha'(t)|^2dt\int_a^b w^2(t)(b-t)(t-a)\left\|g(t)-\int_a^b w(s)\,g(s)\,ds\right\|^2dt\end{aligned}$$

and

$$\begin{aligned}(3.18)\qquad &\left\|C_w(\alpha,g)\right\|^2\\ &\leq\frac{1}{2}\int_a^b|\alpha'(t)|^2dt\int_a^b w^2(t)\,L(t;a,b)\left\|g(t)-\int_a^b w(s)\,g(s)\,ds\right\|^2dt.\end{aligned}$$

Proof. Integrating by parts, we have

$$\begin{aligned}&\int_a^b\alpha'(t)\left(\int_a^t w(s)\,g(s)\,ds-\int_a^t w(s)\,ds\int_a^b g(s)\,w(s)\,ds\right)dt\\ &=\left[\alpha(t)\left(\int_a^t w(s)\,g(s)\,ds-\int_a^t w(s)\,ds\int_a^b g(s)\,w(s)\,ds\right)\right|_a^b\\ &-\int_a^b\alpha(t)\left(w(t)\,g(t)-w(t)\int_a^b g(s)\,w(s)\,ds\right)dt\\ &=-\int_a^b w(t)\,\alpha(t)\,g(t)\,dt+\int_a^b\alpha(t)\,w(t)\,dt\int_a^b w(s)\,g(s)\,ds,\end{aligned}$$

which gives that

$$(3.19)\quad C_w(\alpha,g)=\int_a^b\alpha'(t)\left(\int_a^t w(s)\,ds\int_a^b w(s)\,g(s)\,ds-\int_a^t w(s)\,g(s)\,ds\right)dt.$$

Using (CBS) integral inequality we have

$$\begin{aligned}(3.20)\quad \left\|C_w(\alpha,g)\right\|^2&\leq\int_a^b|\alpha'(t)|^2dt\\ &\times\int_a^b\left\|\int_a^t w(s)\,ds\int_a^b w(s)\,g(s)\,ds-\int_a^t w(s)\,g(s)\,ds\right\|^2dt.\end{aligned}$$

If we take

$$h(t) := \int_a^t w(s)\,ds \int_a^b w(s)\,g(s)\,ds - \int_a^t w(s)\,g(s)\,ds$$

we observe that $h(a) = h(b) = 0$ and $h \in C^1([a,b], H)$.

By (3.1) we then get

$$\begin{aligned}
(3.21)\qquad & \int_a^b \left\| \int_a^t w(s)\,ds \int_a^b w(s)\,g(s)\,ds - \int_a^t w(s)\,g(s)\,ds \right\|^2 dt \\
&\leq \left(\int_a^b w^2(t)\,w_a(t;a,b) \left\| g(t) - \int_a^b w(s)\,g(s)\,ds \right\|^2 dt \right)^{1/2} \\
&\times \left(\int_a^b w^2(t)\,w_b(t;a,b) \left\| g(t) - \int_a^b w(s)\,g(s)\,ds \right\|^2 dt \right)^{1/2} \\
&\leq \int_a^b w^2(t)\,(b-t)\,(t-a) \left\| g(t) - \int_a^b w(s)\,g(s)\,ds \right\|^2 dt
\end{aligned}$$

and by (3.20) and (3.21) we derive (3.17). □

Remark 5. *If we take $w \equiv 1/(b-a)$ in (3.17), then we get the unweighted Grüss' type inequality*

$$\begin{aligned}
(3.22)\qquad & \|C(\alpha, g)\|^2 \\
&\leq \frac{1}{(b-a)^2} \int_a^b |\alpha'(t)|^2\,dt \\
&\times \left(\int_a^b w_a(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \right)^{1/2} \\
&\times \left(\int_a^b w_b(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \right)^{1/2} \\
&\leq \frac{1}{(b-a)^2} \int_a^b |\alpha'(t)|^2\,dt \\
&\times \int_a^b (b-t)\,(t-a) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt,
\end{aligned}$$

while from (3.18),

$$\begin{aligned}
(3.23)\qquad & \|C(\alpha, g)\|^2 \\
&\leq \frac{1}{2(b-a)^2} \int_a^b |\alpha'(t)|^2\,dt \int_a^b L(t;a,b) \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt.
\end{aligned}$$

Remark 6. *Observe also that*

$$\int_a^b w^2(t)(b-t)(t-a)\left\|g(t)-\int_a^b w(s)g(s)\,ds\right\|^2 dt$$
$$\leq \sup_{t\in[a,b]}[w(t)(b-t)(t-a)]\int_a^b w(t)\left\|g(t)-\int_a^b w(s)g(s)\,ds\right\|^2 dt$$
$$=\sup_{t\in[a,b]}[w(t)(b-t)(t-a)]\left[\int_a^b w(t)\|g(t)\|^2-\left\|\int_a^b w(s)g(s)\,ds\right\|^2\right],$$

then by (3.17) we get

$$\|C_w(\alpha,g)\|^2 \leq \sup_{t\in[a,b]}[w(t)(b-t)(t-a)] \times\int_a^b |\alpha'(t)|^2 dt\left[\int_a^b w(t)\|g(t)\|^2-\left\|\int_a^b w(s)g(s)\,ds\right\|^2\right] \tag{3.24}$$

and, similarly

$$\|C_w(\alpha,g)\|^2 \leq \frac{1}{2}\sup_{t\in[a,b]}[w(t)L(t;a,b)] \times\int_a^b |\alpha'(t)|^2 dt\left[\int_a^b w(t)\|g(t)\|^2-\left\|\int_a^b w(s)g(s)\,ds\right\|^2\right]. \tag{3.25}$$

References

[1] G. A. Anastassiou, Complex Opial type inequalities. *Rom. J. Math. Comput. Sci.* **9** (2019), no. 2, 93–97

[2] G.A. Anastassiou, Integer and fractional self adjoint operator Opial type inequalities. *J. Comput. Anal. Appl.* **23** (2017), no. 8, 1398–1411.

[3] P. R. Beesack, On an integral inequality of Z. Opial. *Trans. Am. Math. Soc.* **104** (1962), 470–475

[4] S. S. Dragomir, Some inequalities related to Vranceanu's extension of Opial's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art. 52, 9. pp. [Online https://rgmia.org/papers/v24/v24a52.pdf]..

[5] L.-G. Hua, On an inequality of Opial. *Sci. Sinica* 14 (1965), 789–790.

[6] N. Levinson, On an inequality of Opial and Beesack. *Proc. Amer. Math. Soc.* **15** (1964), 565–566.

[7] C. L. Mallows, An even simpler proof of Opial's inequality. *Proc. Amer. Math. Soc.* **16** (1965), 173.

[8] C. Olech, A simple proof of a certain result of Z. Opial. *Ann. Polon. Math.* **8** (1960), 61–63.

[9] Z. Opial, Sur une inégalité. *Ann. Polon. Math.* **8** (1960), 29–32.

[10] R. N. Pederson, On an inequality of Opial, Beesack and Levinson. *Proc. Amer. Math. Soc.* **16** (1965), 174.

[11] S. H. Saker, D. M. Abdou and I. Kubiaczyk, Opial and Pólya type inequalities via convexity. *Fasc. Math.* No. **60** (2018), 145–159.

[12] M. Z. Sarıkaya, On the generalization of Opial type inequality for convex function. *Konuralp J. Math.* **7** (2019), no. 2, 456–461.

[13] G. G. Vrânceanu, On an inequality of Opial. *Bull. Math. Soc. Sci. Math. R.S. Roumanie* (N.S.) 17(65) (1973), 315-316 (1975).

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