WIRTINGER TYPE ABSOLUTE VALUE INTEGRAL INEQUALITIES FOR FUNCTIONS WITH OPERATOR VALUES IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. In this paper we show among others that, if $A(a)=A(b)=0$ and $A'\in L_{2,w_{a,b}}([a,b],\mathcal{B}(H))$, where

$$w_{a,b}(t;a,b)=(b-t)(t-a),$$

then

$$\int_a^b |A(t)|^2\,dt \le \int_a^b w_{a,b}(t;a,b)\left|A'(t)\right|^2 dt.$$

The inequality is sharp in the operator order of $\mathcal{B}(H)$. Applications related to the trapezoid and of Grüss' type inequalities are also provided.

1. Introduction

It is well known that, see for instance [4], or [8], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{C})$ is a function with complex values and $u(a)=u(b)=0$, then $\operatorname{Re}u(a)=\operatorname{Re}u(b)=0$ and $\operatorname{Im}u(a)=\operatorname{Im}u(b)=0$ and by writing (1.1) for $\operatorname{Re}u$ and $\operatorname{Im}u$ and adding the obtained inequalities, we get

$$\int_a^b |u(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |u'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K\in\mathbb{C}$.

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Similarly, if $u \in C^1([a,b],\mathbb{C})$ with $u(a)=0$, then by (1.2) we have

$$\int_a^b |u(t)|^2 dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b |u'(t)|^2 dt \tag{1.4}$$

and the equality holds if and only if

$$u(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [3], [4] and [7]-[11].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$.

In the recent paper [6] we obtained the following inequality of Wirtinger type in the operator order of $\mathcal{B}(H)$:

Theorem 1. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $A(a) = A(b) = 0$, then*

$$\int_a^b |A(t)|^2 dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b |A'(t)|^2 dt. \tag{1.5}$$

If only $A(a)=0$, then

$$\int_a^b |A(t)|^2 dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b |A'(t)|^2 dt. \tag{1.6}$$

Motivated by the above results, in this paper we show among others that, if $A(a) = A(b) = 0$ and $A' \in L_{2,w_{a,b}}([a,b],\mathcal{B}(H))$, where

$$w_{a,b}(t;a,b) = (b-t)(t-a),$$

then

$$\int_a^b |A(t)|^2 dt \leq \int_a^b w_{a,b}(t;a,b)|A'(t)|^2 dt.$$

The inequality is sharp in the operator order of $\mathcal{B}(H)$. Applications related to the trapezoid and of Grüss' type inequalities are also provided.

2. Main Results

We start with the following result:

Theorem 2. *Assume that the function $A : [a,b] \to \mathcal{B}(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b).*

(i) *If $A(a) = 0$ and $A' \in L_{2,w_a}([a,b],\mathcal{B}(H))$, where*

$$w_a(t;a,b) := \frac{1}{2}\left[(b-a)^2 - (t-a)^2\right] = (b-t)\left(\frac{b+t}{2}-a\right),$$

then

$$\int_a^b |A(t)|^2 dt \leq \int_a^b w_a(t;a,b)|A'(t)|^2 dt \tag{2.1}$$

in the operator order of $\mathcal{B}(H)$. The inequality is sharp.

(ii) *If* $A(b)=0$ *and* $A'\in L_{2,w_b}([a,b],\mathcal{B}(H))$, *where*

$$w_b(t;a,b):=\frac{1}{2}\left[(b-a)^2-(b-t)^2\right]=(t-a)\left(b-\frac{a+t}{2}\right),$$

then

$$\int_a^b |A(t)|^2\,dt\le\int_a^b w_b(t;a,b)\,|A'(t)|^2\,dt. \tag{2.2}$$

The inequality is sharp.

(iii) *If* $A(a)=A(b)=0$ *and* $A'\in L_{2,w_{a,b}}([a,b],\mathcal{B}(H))$, *where*

$$w_{a,b}(t;a,b)=(b-t)(t-a),$$

then

$$\int_a^b |A(t)|^2\,dt\le\int_a^b w_{a,b}(t;a,b)\,|A'(t)|^2\,dt. \tag{2.3}$$

The inequality is sharp.

Proof. (i). Let $x\in H$, $x\neq 0$. Since $A(a)=0$, hence $A(t)x=\int_a^t A'(s)x\,ds$ and we have by Schwarz inequality

$$\begin{aligned}\int_a^b\|A(t)x\|^2\,dt&=\int_a^b\left\|\int_a^t A'(s)x\,ds\right\|^2dt\\&\le\int_a^b(t-a)\left(\int_a^t\|A'(s)x\|^2\,ds\right)dt=:B.\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}B&=\int_a^b\left(\int_a^t\|A'(s)x\|^2\,ds\right)d\left(\frac{(t-a)^2}{2}\right)\\&=\left(\int_a^t\|A'(s)x\|^2\,ds\right)\frac{(t-a)^2}{2}\Bigg|_a^b-\int_a^b\frac{(t-a)^2}{2}\|A'(t)x\|^2\,dt\\&=\frac{(b-a)^2}{2}\left(\int_a^b\|A'(s)x\|^2\,ds\right)-\int_a^b\frac{(t-a)^2}{2}\|A'(t)x\|^2\,dt\\&=\int_a^b\left[\frac{(b-a)^2}{2}-\frac{(t-a)^2}{2}\right]\|A'(t)x\|^2\,dt=\int_a^b w_a(t;a,b)\,\|A'(t)x\|^2\,dt.\end{aligned}$$

Therefore,

$$\int_a^b\|A(t)x\|^2\,dt\le\int_a^b w_a(t;a,b)\,\|A'(t)x\|^2\,dt \tag{2.4}$$

for all $x\in H$.

Now, observe that

$$\begin{aligned}\int_a^b\|A(t)x\|^2\,dt&=\int_a^b\langle A(t)x,A(t)x\rangle\,dt=\int_a^b\left\langle(A(t))^*A(t)x,x\right\rangle dt\\&=\int_a^b\left\langle|A(t)|^2x,x\right\rangle dt=\left\langle\left(\int_a^b|A(t)|^2\,dt\right)x,x\right\rangle\end{aligned}$$

and

$$\int_a^b w_a(t;a,b)\langle A'(t)x, A'(t)x\rangle dt = \int_a^b w_a(t;a,b)\left\langle |A'(t)|^2 x, x\right\rangle dt$$
$$= \left\langle \left(\int_a^b w_a(t;a,b)|A'(t)|^2 dt\right)x, x\right\rangle$$

and by (2.4) we get

$$\left\langle \left(\int_a^b |A(t)|^2 dt\right)x, x\right\rangle \le \left\langle \left(\int_a^b w_a(t;a,b)|A'(t)|^2 dt\right)x, x\right\rangle$$

for all $x \in H$, which is equivalent, in the operator order with (2.3).

Now, if we consider the scalar case and take $f(t) = t - a$, then

$$\int_a^b |f(t)|^2 dt = \int_a^b (t-a)^2 dt = \frac{1}{3}(b-a)^3$$

and

$$\int_a^b w_a(t;a,b)|f'(t)|^2 dt$$
$$= \int_a^b w_a(t;a,b)\, dt = \frac{1}{2}\int_a^b \left[(b-a)^2 - (t-a)^2\right]$$
$$= \frac{1}{2}\left[(b-a)^3 - \frac{1}{3}(b-a)^3\right] = \frac{1}{3}(b-a)^3,$$

which shows that both terms in (2.1) are equal with $\frac{1}{3}(b-a)^3$.

(ii). Follows in a similar way by observing that $A(t)x = -\int_b^t A'(s)xds$, $x \in H$. The function $f(t) = b - t$ realizes the equality in the scalar case.

(iii). Let $x \in H$, $x \neq 0$. Since $A(a) = A(b) = 0$, hence

$$\int_a^b \|A(t)x\|^2 dt$$
$$= \int_a^b \left\|\int_a^t A'(s)xds\right\| \left\|\int_t^b A'(s)x\right\| dt$$
$$\le \int_a^b (t-a)^{1/2}\left(\int_a^t \|A'(s)x\|^2 ds\right)^{1/2}(b-t)^{1/2}\left(\int_t^b \|A'(s)x\|^2 ds\right)^{1/2} dt$$
$$=: C.$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we also have

$$C \le \left(\int_a^b \left[(t-a)^{1/2} \left(\int_a^t \left\| A'(s)\, x \right\|^2 ds \right)^{1/2} \right]^2 dt \right)^{1/2}$$
$$\times \left(\int_a^b \left[(b-t)^{1/2} \left(\int_t^b \left\| A'(s)\, x \right\|^2 ds \right)^{1/2} \right]^2 dt \right)^{1/2}$$
$$= \left(\int_a^b (t-a) \left(\int_a^t \left\| A'(s)\, x \right\|^2 ds \right) dt \right)^{1/2}$$
$$\times \left(\int_a^b (b-t) \left(\int_t^b \left\| A'(s)\, x \right\|^2 ds \right) dt \right)^{1/2}$$
$$= \left(\int_a^b w_a(t;a,b) \left\| A'(t)\, x \right\|^2 dt \right)^{1/2} \left(\int_a^b w_b(t;a,b) \left\| A'(t)\, x \right\|^2 dt \right)^{1/2}.$$

Using the arithmetic mean-geometric mean inequality

$$\sqrt{\alpha\beta} \le \frac{1}{2}(\alpha+\beta), \ \alpha, \ \beta \ge 0$$

we get

$$\left(\int_a^b w_a(t;a,b) \left\| A'(t)\, x \right\|^2 dt \right)^{1/2} \left(\int_a^b w_b(t;a,b) \left\| A'(t)\, x \right\|^2 dt \right)^{1/2}$$
$$\le \frac{1}{2} \left[\int_a^b w_a(t;a,b) \left\| A'(t)\, x \right\|^2 dt + \int_a^b w_b(t;a,b) \left\| A'(t)\, x \right\|^2 dt \right]$$
$$= \frac{1}{2} \left[\int_a^b \frac{1}{2} \left[(b-a)^2 - (t-a)^2 \right] \left\| A'(t)\, x \right\|^2 dt \right.$$
$$\left. + \int_a^b \frac{1}{2} \left[(b-a)^2 - (b-t)^2 \right] \left\| A'(t)\, x \right\|^2 dt \right]$$

$$= \frac{1}{2} \left[\int_a^b \frac{1}{2} \left[(b-a)^2 - (t-a)^2 \right] \left\| A'(t)\, x \right\|^2 dt \right.$$
$$\left. + \int_a^b \frac{1}{2} \left[(b-a)^2 - (b-t)^2 \right] \left\| A'(t)\, x \right\|^2 dt \right]$$
$$= \frac{1}{4}(b-a)^2 \int_a^b \left[\left\| A'(t)\, x \right\|^2 + \left\| A'(t)\, x \right\|^2 \right] dt$$
$$- \frac{1}{4} \int_a^b \left[(t-a)^2 \left\| A'(t)\, x \right\|^2 + (b-t)^2 \left\| A'(t)\, x \right\|^2 \right] dt$$
$$= \frac{1}{2}(b-a)^2 \int_a^b \left\| A'(t)\, x \right\|^2 dt - \frac{1}{2} \int_a^b \left[(t-a)^2 + (b-t)^2 \right] \left\| A'(t)\, x \right\|^2 dt.$$

Since

$$\frac{(t-a)^2+(b-t)^2}{2}=\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2,$$

hence

$$\begin{aligned}&\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\|A'(t)x\|^2dt\\&=\int_a^b\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\|A'(t)x\|^2dt\\&=\frac{1}{4}(b-a)^2\int_a^b\|A'(t)x\|^2dt+\int_a^b\left(t-\frac{a+b}{2}\right)^2\|A'(t)x\|^2dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}(b-a)^2\int_a^b\|A'(t)x\|^2dt-\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\|A'(t)x\|^2dt\\&=\frac{1}{2}(b-a)^2\int_a^b\|A'(t)x\|^2dt-\frac{1}{4}(b-a)^2\int_a^b\|A'(t)x\|^2dt\\&-\int_a^b\left(t-\frac{a+b}{2}\right)^2\|A'(t)x\|^2dt\\&=\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\|A'(t)x\|^2dt,\end{aligned}$$

which shows that

$$\int_a^b\|A(t)x\|^2dt\le\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\|A'(t)x\|^2dt.$$

Finally, since

$$\int_a^b\|A(t)x\|^2dt=\left\langle\left(\int_a^b|A(t)|^2dt\right)x,x\right\rangle$$

and

$$\begin{aligned}&\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\|A'(t)x\|^2dt\\&=\left\langle\left(\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]|A'(t)|^2dt\right)x,x\right\rangle\\&=\left\langle\left(\int_a^b(b-t)(t-a)|A'(t)|^2dt\right)x,x\right\rangle\end{aligned}$$

for all $x\in H$, which is equivalent to

In the case when $H=\mathbb{R}$, consider the function

$$f(t)=(t-a)(b-t),\ t\in[a,b].$$

Then $f'(t)=a+b-2t$, $t\in(a,b)$,

$$\int_a^b|f(t)|^2dt=\int_a^b(t-a)^2(b-t)^2dt=\frac{1}{30}(b-a)^5$$

and

$$\begin{aligned}&\int_a^b (b-t)(t-a)\left|f'(t)\right|^2 dt\\&=\int_a^b (b-t)(t-a)\left|(a+b)t-2t\right|^2 dt\\&=4\int_a^b (b-t)(t-a)\left(t-\frac{a+b}{2}\right)^2 dt=\frac{1}{30}(b-a)^5,\end{aligned}$$

which show that the inequalities (2.3) is sharp. □

We also have:

Theorem 3. *Assume that the function $A:[a,b]\to\mathcal{B}(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b) with $A(a)=A(b)=0$ and $A\in L_{2,p_{a,b}}([a,b],\mathcal{B}(H))$, where*

$$p_{a,b}(t;a,b):=\frac{1}{4}(b-a)^2-\begin{cases}(t-a)^2, & t\in\left[a,\frac{a+b}{2}\right],\\ \\ (b-t)^2, & t\in\left(\frac{a+b}{2},b\right],\end{cases}\tag{2.5}$$

then

$$\int_a^b \left|A(t)\right|^2 dt\le\frac{1}{2}\int_a^b p_{a,b}(t;a,b)\left|A'(t)\right|^2 dt.\tag{2.6}$$

The constant $\frac{1}{2}$ is best possible.

Proof. From (2.1) we have

$$\begin{aligned}\int_a^{\frac{a+b}{2}}\left|A(t)\right|^2 dt&\le\int_a^{\frac{a+b}{2}} w_a\left(t;a,\frac{a+b}{2}\right)\left|A'(t)\right|^2 dt.\\&=\frac{1}{2}\int_a^{\frac{a+b}{2}}\left[\left(\frac{a+b}{2}-a\right)^2-(t-a)^2\right]\left|A'(t)\right|^2 dt,\end{aligned}\tag{2.7}$$

while from (2.2) we have

$$\begin{aligned}\int_{\frac{a+b}{2}}^b\left|A(t)\right|^2 dt&\le\int_{\frac{a+b}{2}}^b w_b\left(t;\frac{a+b}{2},b\right)\left|A'(t)\right|^2 dt\\&=\frac{1}{2}\int_{\frac{a+b}{2}}^b\left[\left(b-\frac{a+b}{2}\right)^2-(b-t)^2\right]\left|A'(t)\right|^2 dt.\end{aligned}\tag{2.8}$$

If we add these two inequalities, then we get

$$\begin{aligned}
\int_a^b |A(t)|^2\,dt &\le \frac{1}{2}\int_a^{\frac{a+b}{2}} \left[\frac{1}{4}(b-a)^2-(t-a)^2\right]|A'(t)|^2\,dt \\
&+\frac{1}{2}\int_{\frac{a+b}{2}}^b \left[\frac{1}{4}(b-a)^2-(b-t)^2\right]|A'(t)|^2\,dt \\
&=\frac{1}{8}(b-a)^2\int_a^{\frac{a+b}{2}}|A'(t)|^2\,dt+\frac{1}{8}(b-a)^2\int_{\frac{a+b}{2}}^b|A'(t)|^2\,dt \\
&-\frac{1}{2}\int_a^{\frac{a+b}{2}}(t-a)^2|A'(t)|^2\,dt-\frac{1}{2}\int_{\frac{a+b}{2}}^b(b-t)^2|A'(t)|^2\,dt \\
&=\frac{1}{8}(b-a)^2\int_a^b|A'(t)|^2\,dt-\frac{1}{2}\int_a^{\frac{a+b}{2}}(t-a)^2|A'(t)|^2\,dt \\
&-\frac{1}{2}\int_{\frac{a+b}{2}}^b(b-t)^2|A'(t)|^2\,dt \\
&=\frac{1}{2}\int_a^b p_{a,b}(t;a,b)\,|A'(t)|^2\,dt,
\end{aligned}$$

which proves the desired inequality.

For the case $H=\mathbb{R}$ we consider the function

$$f(t):=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right], \\ b-t, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then f is absolutely continuous, $f(a)=f(b)=0$, $|f'(t)|=1$, $t\in(a,b)$,

$$\begin{aligned}
\int_a^b f^2(t)\,dt &= \int_a^{\frac{a+b}{2}}(t-a)^2\,dt+\int_{\frac{a+b}{2}}^b(t-b)^2\,dt \\
&=\frac{1}{24}(b-a)^3+\frac{1}{24}(b-a)^3=\frac{1}{12}(b-a)^3
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_a^b p_{a,b}(t;a,b)\,[f'(t)]^2\,dt \\
&=\frac{1}{2}\int_a^b p_{a,b}(t;a,b)\,dt \\
&=\frac{1}{2}\int_a^b\frac{1}{4}(b-a)^2\,dt-\frac{1}{2}\left[\int_a^{\frac{a+b}{2}}(t-a)^2\,dt+\int_{\frac{a+b}{2}}^b(t-b)^2\,dt\right] \\
&=\frac{1}{8}(b-a)^3-\frac{1}{24}(b-a)^3=\frac{1}{12}(b-a)^3,
\end{aligned}$$

which shows that in both sides of (2.6) we get the same quantity $\frac{1}{12}(b-a)^3$.

This proves the sharpness of the constant $\frac{1}{2}$. □

3. Applications

For $A:[a,b]\rightarrow B(H)$ strongly measurable and Bochner squared integrable, namely $A\in L_2([a,b],\mathcal{B}(H))$, we have

$$\frac{1}{b-a}\int_a^b |A(t)|^2\,dt \geq \left|\frac{1}{b-a}\int_a^b A(t)\,dt\right|^2 \tag{3.1}$$

in the operator order of $\mathcal{B}(H)$.

Indeed, since

$$\begin{aligned} 0 &\leq |A(t)-A(s)|^2 = (A(t)-A(s))^*(A(t)-A(s)) \\ &= |A(t)|^2 - A^*(s)A(t) - A^*(t)A(s) + |A(s)|^2, \end{aligned}$$

hence

$$|A(t)|^2 + |A(s)|^2 \geq A^*(s)A(t) + A^*(t)A(s)$$

for all $t,s\in[a,b]$.

Integrating over $s,t\in[a,b]$, we get

$$\int_a^b\int_a^b \left[|A(t)|^2+|A(s)|^2\right]dtds \geq \int_a^b\int_a^b \left[A^*(s)A(t)+A^*(t)A(s)\right]dtds \tag{3.2}$$

in the operator order of $\mathcal{B}(H)$.

Observe that

$$\begin{aligned} \int_a^b\int_a^b \left[|A(t)|^2+|A(s)|^2\right]dtds &= (b-a)\int_a^b |A(t)|^2\,dt + (b-a)\int_a^b |A(s)|^2\,ds \\ &= 2(b-a)\int_a^b |A(t)|^2\,dt \end{aligned}$$

and

$$\begin{aligned} &\int_a^b\int_a^b \left[A^*(s)A(t)+A^*(t)A(s)\right]dtds \\ &= \int_a^b A^*(s)\,ds\int_a^b A(t)\,dt + \int_a^b A^*(t)\,dt\int_a^b A(s)\,ds \\ &= 2\left(\int_a^b A(t)\,dt\right)^*\int_a^b A(t)\,dt, \end{aligned}$$

and by (3.2) we derive (3.1).

In a similar way, if $w:[a,b]\rightarrow[0,\infty)$ with $\int_a^b w(t)\,dt=1$, then

$$\int_a^b w(t)|A(t)|^2\,dt \geq \left|\int_a^b w(t)A(t)\,dt\right|^2, \tag{3.3}$$

provided that $A\in L_{2,w}([a,b],\mathcal{B}(H)) := \left\{A:[a,b]\rightarrow B(H),\ \int_a^b w(t)\|A(t)\|^2\,dt<\infty\right\}$.

Assume that $B:[a,b]\rightarrow\mathcal{B}(H)$ is absolutely continuous with $B(a)=B(b)=0$ and $f'\in L_{2,w_a}([a,b],\mathcal{B}(H))\cap L_{2,w_b}([a,b],\mathcal{B}(H))$ then

$$\int_a^b |B(t)|^2\,dt \leq \int_a^b (b-t)(t-a)|B'(t)|^2\,dt \tag{3.4}$$

and

$$\int_a^b |B(t)|^2 \, dt \leq \frac{1}{2} \int_a^b L(t; a, b) |B'(t)|^2 \, dt. \tag{3.5}$$

We have the following trapezoid type inequalities:

Proposition 1. *Let $A \in C^1([a, b], \mathcal{B}(H))$. Then*

$$\left| \frac{A(a) + A(b)}{2} - \frac{1}{b-a} \int_a^b A(t) \, dt \right|^2 \leq \frac{1}{4} \frac{1}{b-a} \int_a^b (b-t)(t-a) |A'(t) - A'(a+b-t)|^2 \, dt \tag{3.6}$$

and

$$\left| \frac{A(a) + A(b)}{2} - \frac{1}{b-a} \int_a^b A(t) \, dt \right|^2 \leq \frac{1}{8} \frac{1}{b-a} \int_a^b L(t; a, b) |A'(t) - A'(a+b-t)|^2 \, dt. \tag{3.7}$$

Proof. If $A \in C^1([a, b], \mathcal{B}(H))$, then by taking

$$B(t) := \frac{A(t) + A(a+b-t)}{2} - \frac{A(a) + A(b)}{2}, \ t \in [a, b]$$

we have $B(a) = B(b) = 0$,

$$B'(t) = \frac{A'(t) - A'(a+b-t)}{2}$$

and by (3.4) we derive

$$\int_a^b \left| \frac{A(t) + A(a+b-t)}{2} - \frac{A(a) + A(b)}{2} \right|^2 dt \leq \frac{1}{4} \int_a^b (b-t)(t-a) |A'(t) - A'(a+b-t)|^2 \, dt. \tag{3.8}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality (3.3) we have

$$\begin{aligned} &(b-a) \int_a^b \left| \frac{A(t) + A(a+b-t)}{2} - \frac{A(a) + A(b)}{2} \right|^2 dt \\ &\geq \left| \int_a^b \left[\frac{A(t) + A(a+b-t)}{2} - \frac{A(a) + A(b)}{2} \right] dt \right|^2 \\ &= \left| \int_a^b A(t) \, dt - \frac{A(a) + A(b)}{2} (b-a) \right|^2, \end{aligned}$$

which implies that

$$\left| \frac{A(a) + A(b)}{2} - \frac{1}{b-a} \int_a^b A(t) \, dt \right|^2 \leq \frac{1}{b-a} \int_a^b \left| \frac{A(t) + A(a+b-t)}{2} - \frac{A(a) + A(b)}{2} \right|^2 dt. \tag{3.9}$$

By utilising (3.8) and (3.9) we derive the desired result (3.6).

The inequality (3.7) follows from (3.5). □

Proposition 2. *Let $A \in C^1([a,b], \mathcal{B}(H))$. Then*

$$\left| \frac{1}{b-a} \int_a^b A(t)\,dt - \frac{A(a)+A(b)}{2} \right|^2 \leq \frac{1}{b-a} \int_a^b (b-t)(t-a) \left| A'(t) - \frac{A(b)-A(a)}{b-a} \right|^2 dt, \tag{3.10}$$

and

$$\left| \frac{1}{b-a} \int_a^b A(t)\,dt - \frac{A(a)+A(b)}{2} \right|^2 \leq \frac{1}{2} \frac{1}{b-a} \int_a^b L(t;a,b) \left| A'(t) - \frac{A(b)-A(a)}{b-a} \right|^2 dt. \tag{3.11}$$

Proof. If $A \in C^1([a,b], \mathcal{B}(H))$, then by taking

$$B(t) := A(t) - \frac{A(a)(b-t) + A(b)(t-a)}{b-a}, \ t \in [a,b]$$

we have $B(a) = B(b) = 0$ and by (3.4) we have

$$\int_a^b \left| A(t) - \frac{A(a)(b-t) + A(b)(t-a)}{b-a} \right|^2 dt \leq \int_a^b (b-t)(t-a) \left| A'(t) - \frac{A(b)-A(a)}{b-a} \right|^2 dt \tag{3.12}$$

By (3.3) we have

$$\begin{aligned} \frac{1}{b-a} \int_a^b \left| A(t) - \frac{A(a)(b-t) + A(b)(t-a)}{b-a} \right|^2 dt &\geq \left| \frac{1}{b-a} \int_a^b \left(A(t) - \frac{A(a)(b-t) + A(b)(t-a)}{b-a} \right) dt \right|^2 \\ &= \left| \frac{1}{b-a} \int_a^b A(t)\,dt - \frac{A(a)+A(b)}{2} \right|^2 . \end{aligned} \tag{3.13}$$

By utilising (3.12) and (3.13) we derive

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b A(t)\,dt - \frac{A(a)+A(b)}{2} \right|^2 \\ &\leq \frac{1}{b-a} \int_a^b \left| A(t) - \frac{A(a)(b-t) + A(b)(t-a)}{b-a} \right|^2 dt \\ &\leq \frac{1}{b-a} \int_a^b (b-t)(t-a) \left| A'(t) - \frac{A(b)-A(a)}{b-a} \right|^2 dt, \end{aligned}$$

which proves (3.10).

The inequality (3.11) follows from (3.5). □

We also have:

Proposition 3. *Let $A \in C([a,b], \mathcal{B}(H))$. Then*

$$\left| \frac{b+a}{2} \int_a^b A(s)\,ds - \int_a^b tA(t)\,dt \right|^2 \leq (b-a) \int_a^b (b-t)(t-a) \left| A(t) - \frac{1}{b-a} \int_a^b A(s)\,ds \right|^2 dt. \tag{3.14}$$

and

$$\left| \frac{b+a}{2} \int_a^b A(s)\,ds - \int_a^b tA(t)\,dt \right|^2 \leq \frac{1}{2}(b-a) \int_a^b L(t;a,b) \left| A(t) - \frac{1}{b-a} \int_a^b A(s)\,ds \right|^2 dt. \tag{3.15}$$

Proof. Assume that $A : [a,b] \to \mathcal{B}(H)$ is continuous, then by taking

$$B(t) := \int_a^t A(s)\,ds - \frac{t-a}{b-a} \int_a^b A(s)\,ds, \; t \in [a,b]$$

we have $B(a) = B(b) = 0$, and by (3.4)

$$\int_a^b \left| \int_a^t A(s)\,ds - \frac{t-a}{b-a} \int_a^b A(s)\,ds \right|^2 dt \leq \int_a^b (b-t)(t-a) \left| A(t) - \frac{1}{b-a} \int_a^b A(s)\,ds \right|^2 dt. \tag{3.16}$$

By (3.3) we have

$$\frac{1}{b-a} \int_a^b \left| \int_a^t A(s)\,ds - \frac{t-a}{b-a} \int_a^b A(s)\,ds \right|^2 dt \geq \left| \frac{1}{b-a} \int_a^b \left(\int_a^t A(s)\,ds - \frac{t-a}{b-a} \int_a^b A(s)\,ds \right) dt \right|^2,$$

namely

$$\left| \int_a^b \left(\int_a^t A(s)\,ds - \frac{t-a}{b-a} \int_a^b A(s)\,ds \right) dt \right|^2 \leq (b-a) \int_a^b \left| \int_a^t A(s)\,ds - \frac{t-a}{b-a} \int_a^b A(s)\,ds \right|^2 dt \tag{3.17}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^t A(s)\, ds - \frac{t-a}{b-a} \int_a^b A(s)\, ds \right) dt \\
&= \int_a^b \left(\int_a^t A(s)\, ds \right) dt - \frac{b-a}{2} \int_a^b A(s)\, ds \\
&= b \int_a^b A(s)\, ds - \int_a^b tA(t)\, dt - \frac{b-a}{2} \int_a^b A(s)\, ds \\
&= \frac{b+a}{2} \int_a^b A(s)\, ds - \int_a^b tA(t)\, dt.
\end{aligned} \tag{3.18}$$

Therefore by (3.16)-(3.18) we derive

$$\begin{aligned}
&\left| \frac{b+a}{2} \int_a^b A(s)\, ds - \int_a^b tA(t)\, dt \right|^2 \\
&\leq (b-a) \int_a^b (b-t)(t-a) \left| A(t) - \frac{1}{b-a} \int_a^b A(s)\, ds \right|^2 dt,
\end{aligned} \tag{3.19}$$

which proves (3.14).

The inequality (3.15) follows from (3.5). □

Consider now the *weighted Čebyšev functional*

$$C_w(\alpha, A) := \int_a^b w(t)\, \alpha(t)\, A(t)\, dt - \int_a^b w(t)\, \alpha(t)\, dt \int_a^b w(t)\, A(t)\, dt$$

where $w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$, $\alpha : [a,b] \to \mathbb{C}$ and $A : [a,b] \to \mathcal{B}(H)$ are functions such that the involved integrals exist and $\int_a^b w(t)\, dt = 1$.

We have for $\alpha : [a,b] \to \mathbb{C}$ and $A : [a,b] \to \mathcal{B}(H)$,

$$\begin{aligned}
0 &\leq \left| \overline{\alpha(t)} A(s) - \overline{\alpha(s)} A(t) \right|^2 \\
&= |\alpha(t)|\, |A(s)|^2 - \alpha(s)\, \overline{\alpha(t)} A^*(t)\, A(s) \\
&\quad - \alpha(t)\, \overline{\alpha(s)} A^*(s)\, A(t) + |\alpha(s)|^2\, |A(t)|^2,
\end{aligned}$$

which gives that

$$\begin{aligned}
&|\alpha(t)|^2\, |A(s)|^2 + |\alpha(s)|^2\, |A(t)|^2 \\
&\geq \alpha(s)\, \overline{\alpha(t)} A^*(t)\, A(s) + \alpha(t)\, \overline{\alpha(s)} A^*(s)\, A(t)
\end{aligned}$$

for all $s, t \in [a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned}
&\int_a^b |\alpha(t)|^2\, dt \int_a^b |A(s)|^2\, ds + \int_a^b |\alpha(s)|^2\, ds \int_a^b |A(t)|^2\, dt \\
&\geq \int_a^b \overline{\alpha(t)} A^*(t)\, dt \int_a^b \alpha(s)\, A(s)\, ds + \int_a^b \overline{\alpha(s)} A^*(s)\, ds \int_a^b \alpha(t)\, A(t)\, dt \\
&= 2 \left| \int_a^b \alpha(s)\, A(s)\, ds \right|^2,
\end{aligned}$$

which proves that

$$\int_a^b |\alpha(t)|^2 dt \int_a^b |A(t)|^2 dt \geq \left| \int_a^b \alpha(t) A(t) dt \right|^2, \tag{3.20}$$

provided that $\alpha \in L_2([a,b], \mathbb{C})$ and $A \in L_2([a,b], \mathcal{B}(H))$.

Theorem 4. *Assume that* $w : [a,b] \to (0, \infty)$ *is a probability density function on* $[a,b]$, α *is absolutely continuous on* $[a,b]$ *and* $A \in C([a,b], \mathcal{B}(H))$, *then*

$$\begin{aligned} &|C_w(\alpha, A)|^2 \\ &\leq \int_a^b |\alpha'(t)|^2 dt \int_a^b w^2(t)(b-t)(t-a) \left| A(t) - \int_a^b w(s) A(s) ds \right|^2 dt \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &|C_w(\alpha, A)|^2 \\ &\leq \frac{1}{2} \int_a^b |\alpha'(t)|^2 dt \int_a^b w^2(t) L(t; a, b) \left| A(t) - \int_a^b w(s) A(s) ds \right|^2 dt. \end{aligned} \tag{3.22}$$

Proof. Integrating by parts, we have

$$\begin{aligned} &\int_a^b \alpha'(t) \left(\int_a^t w(s) A(s) ds - \int_a^t w(s) ds \int_a^b A(s) w(s) ds \right) dt \\ &= \left[\alpha(t) \left(\int_a^t w(s) A(s) ds - \int_a^t w(s) ds \int_a^b A(s) w(s) ds \right) \right|_a^b \\ &- \int_a^b \alpha(t) \left(w(t) A(t) - w(t) \int_a^b A(s) w(s) ds \right) dt \\ &= -\int_a^b w(t) \alpha(t) A(t) dt + \int_a^b \alpha(t) w(t) dt \int_a^b w(s) A(s) ds, \end{aligned}$$

which gives that

$$C_w(\alpha, A) = \int_a^b \alpha'(t) \left(\int_a^t w(s) ds \int_a^b w(s) A(s) ds - \int_a^t w(s) A(s) ds \right) dt. \tag{3.23}$$

Using (CBS) integral inequality (3.20) we have

$$\begin{aligned} |C_w(\alpha, A)|^2 &\leq \int_a^b |\alpha'(t)|^2 dt \\ &\times \int_a^b \left| \int_a^t w(s) ds \int_a^b w(s) A(s) ds - \int_a^t w(s) A(s) ds \right|^2 dt. \end{aligned} \tag{3.24}$$

If we take

$$B(t) := \int_a^t w(s) ds \int_a^b w(s) A(s) ds - \int_a^t w(s) A(s) ds$$

we observe that $B(a) = B(b) = 0$ and $B \in C^1([a,b], H)$.

By (3.4) we then get

$$\int_a^b \left| \int_a^t w(s)\,ds \int_a^b w(s)\,A(s)\,ds - \int_a^t w(s)\,A(s)\,ds \right|^2 dt \leq \int_a^b w^2(t)\,(b-t)\,(t-a) \left| A(t) - \int_a^b w(s)\,A(s)\,ds \right|^2 dt \tag{3.25}$$

and by (3.24) and (3.25) we derive (3.21). □

Remark 1. *If we take $w \equiv 1/(b-a)$ in (3.21), then we get the unweighted Grüss' type inequality*

$$|C(\alpha, A)|^2 \leq \frac{1}{(b-a)^2} \int_a^b |\alpha'(t)|^2\,dt \times \int_a^b (b-t)\,(t-a) \left| A(t) - \frac{1}{b-a} \int_a^b A(s)\,ds \right|^2 dt, \tag{3.26}$$

while from (3.22),

$$|C(\alpha, A)|^2 \leq \frac{1}{2(b-a)^2} \int_a^b |\alpha'(t)|^2\,dt \int_a^b L(t;a,b) \left| A(t) - \frac{1}{b-a} \int_a^b A(s)\,ds \right|^2 dt. \tag{3.27}$$

Remark 2. *Observe also that*

$$\begin{aligned} &\int_a^b w^2(t)\,(b-t)\,(t-a) \left| A(t) - \int_a^b w(s)\,A(s)\,ds \right|^2 dt \\ &\leq \sup_{t\in[a,b]} [w(t)\,(b-t)\,(t-a)] \int_a^b w(t) \left| A(t) - \int_a^b w(s)\,A(s)\,ds \right|^2 dt \\ &= \sup_{t\in[a,b]} [w(t)\,(b-t)\,(t-a)] \left[\int_a^b w(t)\,|A(t)|^2 - \left| \int_a^b w(s)\,A(s)\,ds \right|^2 \right], \end{aligned}$$

then by (3.21) we get

$$|C(\alpha, A)|^2 \leq \sup_{t\in[a,b]} [w(t)\,(b-t)\,(t-a)] \times \int_a^b |\alpha'(t)|^2\,dt \left[\int_a^b w(t)\,|A(t)|^2 - \left| \int_a^b w(s)\,A(s)\,ds \right|^2 \right] \tag{3.28}$$

and, similarly

$$|C(\alpha, A)|^2 \leq \frac{1}{2} \sup_{t\in[a,b]} [w(t)\,L(t;a,b)] \times \int_a^b |\alpha'(t)|^2\,dt \left[\int_a^b w(t)\,|A(t)|^2 - \left| \int_a^b w(s)\,A(s)\,ds \right|^2 \right]. \tag{3.29}$$

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