RIEMANN-STIELTJES INTEGRAL INEQUALITIES OF WIRTINGER TYPE FOR VECTOR AND OPERATOR VALUED FUNCTIONS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. In this paper we show among others that, if $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$, $f'\in L_2([a,b],H)$ and u is monotonic nondecreasing, then

$$\int_a^b \|f(t)\|^2\,du(t) \le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right]\int_a^b \|f'(s)\|^2\,ds.$$

Applications related to the trapezoid and to Grüss' type inequalities are also provided.

1. Introduction

It is well known that, see for instance [5], or [14], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{C})$ is a function with complex values and $u(a)=u(b)=0$, then $\operatorname{Re}u(a)=\operatorname{Re}u(b)=0$ and $\operatorname{Im}u(a)=\operatorname{Im}u(b)=0$ and by writing (1.1) for $\operatorname{Re}u$ and $\operatorname{Im}u$ and adding the obtained inequalities, we get

$$\int_a^b |u(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |u'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

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for some complex constant $K \in \mathbb{C}$.

Similarly, if $u \in C^1([a,b],\mathbb{C})$ with $u(a) = 0$, then by (1.2) we have

$$\int_a^b |u(t)|^2 dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b |u'(t)|^2 dt \tag{1.4}$$

and the equality holds if and only if

$$u(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [12]-[18].

In the recent paper [9] we obtained among others the following generalization of Wirtinger inequalities for vector valued functions in Hilbert spaces:

Theorem 1. *Assume that $f : [a,b] \to H$ is of class C^1 on $[a,b]$ and $f(a) = f(b) = 0$. Then*

$$\int_a^b \|f(t)\|^2 dt \le \frac{(b-a)^2}{\pi^2} \int_a^b \|f'(t)\|^2 dt. \tag{1.5}$$

If only $f(a) = 0$, then

$$\int_a^b \|f(t)\|^2 dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b \|f'(t)\|^2 dt. \tag{1.6}$$

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$.

We have the following inequality of Wirtinger type in the operator order of $\mathcal{B}(H)$, see [10]:

Theorem 2. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $A(a) = A(b) = 0$, then*

$$\int_a^b |A(t)|^2 dt \le \frac{(b-a)^2}{\pi^2} \int_a^b |A'(t)|^2 dt \tag{1.7}$$

in the operator order of $\mathcal{B}(H)$.

If only $A(a) = 0$, then

$$\int_a^b |A(t)|^2 dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b |A'(t)|^2 dt. \tag{1.8}$$

Motivated by the above results, in this paper we obtain sharp upper bounds for the Riemann-Stieltjes integrals

$$\int_a^b \|f(t)\|^2 du(t) \text{ and } \int_a^b |A(t)|^2 du(t)$$

in the case when $f : [a,b] \to H$ is absolutely continuous with $f(a) = f(b) = 0$, $A : [a,b] \to \mathcal{B}(H)$ is strongly differentiable on (a,b) with $A(a) = A(b) = 0$ and the integrator u is monotonic nondecreasing on $[a,b]$. Applications related to the trapezoid and of Grüss' type inequalities are also provided.

2. Main Results

We have the following result:

Theorem 3. *Assume that* $f:[a,b]\to H$ *is absolutely continuous with* $f(a)=f(b)=0$ *and* $f'\in L_2([a,b],H)$. *If* $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}(2.1)\quad &\int_a^b \|f(t)\|^2\,du(t)\\ &\le \frac{1}{2}\int_a^b\left[(t-a)\int_a^t\|f'(s)\|^2\,ds+(b-t)\int_t^b\|f'(s)\|^2\,ds\right]du(t)\\ &\le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right]\int_a^b\|f'(s)\|^2\,ds\\ &=\frac{1}{2}\left[(b-a)[u(b)-u(a)]-\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b\|f'(s)\|^2\,ds.\end{aligned}$$

The inequalities are sharp in (2.1).

Proof. Since $f(a)=f(b)=0$, hence $f(t)=\int_a^t f'(s)\,ds$ and $f(t)=-\int_t^b f'(s)\,ds$ for $t\in[a,b]$.

We have by Cauchy-Bunyakowsky-Schwarz inequality for the Riemann-Stieltjes integral, that

$$\begin{aligned}&\int_a^b\|f(t)\|^2\,du(t)\\ &=\int_a^b\|f(t)\|\,\|f(t)\|\,du(t)=\int_a^b\left\|\int_a^t f'(s)\,ds\right\|\left\|\int_t^b f'(s)\,ds\right\|du(t)\\ &\le\int_a^b\left(\int_a^t\|f'(s)\|\,ds\right)\left(\int_t^b\|f'(s)\|\,ds\right)du(t)\\ &\le\left(\int_a^b\left(\int_a^t\|f'(s)\|\,ds\right)^2du(t)\right)^{1/2}\left(\int_a^b\left(\int_t^b\|f'(s)\|\,ds\right)^2du(t)\right).\end{aligned}$$

Using the arithmetic mean-geometric mean inequality

$$\sqrt{\alpha\beta}\le\frac{1}{2}(\alpha+\beta),\ \ \alpha,\beta\ge0,$$

we have

$$\begin{aligned}&\left(\int_a^b\left(\int_a^t\|f'(s)\|\,ds\right)^2du(t)\right)^{1/2}\left(\int_a^b\left(\int_t^b\|f'(s)\|\,ds\right)^2du(t)\right)\\ &\le\frac{1}{2}\left[\int_a^b\left(\int_a^t\|f'(s)\|\,ds\right)^2du(t)+\int_a^b\left(\int_t^b\|f'(s)\|\,ds\right)^2du(t)\right]\\ &=\frac{1}{2}\int_a^b\left[\left(\int_a^t\|f'(s)\|\,ds\right)^2+\left(\int_t^b\|f'(s)\|\,ds\right)^2\right]du(t)=:K.\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz inequality we also have

$$\left(\int_a^t \|f'(s)\|\,ds\right)^2 \le (t-a)\int_a^t \|f'(s)\|^2\,ds$$

and

$$\left(\int_t^b \|f'(s)\|\,ds\right)^2 \le (b-t)\int_t^b \|f'(s)\|^2\,ds,$$

and then

$$\begin{aligned}&\left(\int_a^t \|f'(s)\|\,ds\right)^2 + \left(\int_t^b \|f'(s)\|\,ds\right)^2\\ &\le (t-a)\int_a^t \|f'(s)\|^2\,ds + (b-t)\int_t^b \|f'(s)\|^2\,ds\\ &\le \max\{t-a,b-t\}\int_a^b \|f'(s)\|^2\,ds.\end{aligned}$$

Therefore,

$$\begin{aligned}K &\le \frac{1}{2}\int_a^b\left[(t-a)\int_a^t \|f'(s)\|^2\,ds + (b-t)\int_t^b \|f'(s)\|^2\,ds\right]du(t)\\ &\le \frac{1}{2}\int_a^b\left[\max\{t-a,b-t\}\int_a^b \|f'(s)\|^2\,ds\right]du(t)\\ &= \frac{1}{2}\int_a^b \|f'(s)\|^2\,ds\int_a^b[\max\{t-a,b-t\}]\,du(t)\\ &= \frac{1}{2}\int_a^b \|f'(s)\|^2\,ds\int_a^b\left(\frac{1}{2}(b-a)+\left|t-\frac{a+b}{2}\right|\right)du(t)\\ &= \frac{1}{2}\int_a^b \|f'(s)\|^2\,ds\left[\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right],\end{aligned}$$

which proves the desired result (2.1).

Using integration by parts for the Riemann-Stieltjes integral, we have further

$$\begin{aligned}&\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\\ &=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)du(t)+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)du(t)\\ &=\left(\frac{a+b}{2}-t\right)u(t)\Big|_a^{\frac{a+b}{2}}+\int_a^{\frac{a+b}{2}}u(t)\,dt\\ &+\left(t-\frac{a+b}{2}\right)u(t)\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b u(t)\,dt\\ &=-\frac{b-a}{2}u(a)+\int_a^{\frac{a+b}{2}}u(t)\,dt+\frac{b-a}{2}u(b)-\int_{\frac{a+b}{2}}^b u(t)\,dt.\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\\
&=\frac{1}{2}(b-a)[u(b)-u(a)]\\
&-\frac{b-a}{2}u(a)+\int_a^{\frac{a+b}{2}}u(t)\,dt+\frac{b-a}{2}u(b)-\int_{\frac{a+b}{2}}^b u(t)\,dt\\
&=(b-a)[u(b)-u(a)]+\int_a^{\frac{a+b}{2}}u(t)\,dt-\int_{\frac{a+b}{2}}^b u(t)\,dt\\
&=(b-a)[u(b)-u(a)]-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt,
\end{aligned}$$

and the equality is proved.

Now, for $H=\mathbb{R}$ and if f is real valued, then integrating by parts we have

$$\begin{aligned}
\int_a^b f^2(t)\,du(t)&=f^2(t)\,u(t)\big|_a^b-\int_a^b 2f(t)\,f'(t)\,u(t)\,dt\\
&=-2\int_a^b f(t)\,f'(t)\,u(t)\,dt.
\end{aligned}$$

Now, consider the functions

$$f(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right],\\ b-t, & t\in\left(\frac{a+b}{2},b\right]\end{cases}$$

and $u(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$. The function f is absolutely continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$. Also

$$f'(t)=\begin{cases} 1, & t\in\left(a,\frac{a+b}{2}\right),\\ -1, & t\in\left(\frac{a+b}{2},b\right),\end{cases}$$

which gives that $\int_a^b |f'(t)|^2\,dt=b-a$.

Therefore

$$\begin{aligned}
&\frac{1}{2}\left[(b-a)[u(b)-u(a)]-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b |f'(s)|^2\,ds\\
&=\frac{1}{2}\left[2(b-a)-\int_a^b\left(\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\right)^2dt\right](b-a)=\frac{1}{2}(b-a)^2.
\end{aligned}$$

Also,

$$\begin{aligned}
-2\int_a^b f(t)\,f'(t)\,u(t)\,dt&=2\int_a^{\frac{a+b}{2}}(t-a)\,dt+2\int_{\frac{a+b}{2}}^b(b-t)\,dt\\
&=\frac{(b-a)^2}{4}+\frac{(b-a)^2}{4}=\frac{1}{2}(b-a)^2.
\end{aligned}$$

This example gives in all sides of (2.1) the same quantity $\frac{1}{2}(b-a)^2$, which proves the sharpness of all inequalities in (2.1). □

Corollary 1. *Assume that $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_2([a,b],H)$, then*

$$\left\|f\left(\frac{a+b}{2}\right)\right\|^2\leq\frac{1}{4}(b-a)\int_a^b\|f'(s)\|^2\,ds. \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible.

Proof. If we take $u(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$, then

$$\begin{aligned}\int_a^b\|f(t)\|^2\,du(t) &= \|f(t)\|^2u(t)\Big|_a^b-\int_a^b u(t)\,d\left(\|f(t)\|^2\right)\\ &=\|f(b)\|^2u(b)-\|f(a)\|^2u(a)\\ &-\int_a^{\frac{a+b}{2}}u(t)\,d\left(\|f(t)\|^2\right)-\int_{\frac{a+b}{2}}^b u(t)\,d\left(\|f(t)\|^2\right)\\ &=\|f(b)\|^2+\|f(a)\|^2+\left\|f\left(\frac{a+b}{2}\right)\right\|^2\\ &-\|f(a)\|^2-\|f(b)\|^2+\left\|f\left(\frac{a+b}{2}\right)\right\|^2\\ &=2\left\|f\left(\frac{a+b}{2}\right)\right\|^2\end{aligned}$$

and

$$\begin{aligned}&(b-a)[u(b)-u(a)]-\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\\ &=2(b-a)-(b-a)=b-a.\end{aligned}$$

By (2.1) we get

$$2\left\|f\left(\frac{a+b}{2}\right)\right\|^2\leq\frac{1}{2}(b-a)\int_a^b\|f'(s)\|^2\,ds,$$

which is equivalent to (2.2).

For $H=\mathbb{R}$, we take

$$f(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right],\\ b-t, & t\in\left(\frac{a+b}{2},b\right].\end{cases}$$

Observe that f is absolutely continuous with $|f'(t)|=1$, $t\in(a,b)$. Then

$$f\left(\frac{a+b}{2}\right)=\frac{b-a}{2},\quad\int_a^b\|f'(s)\|^2\,ds=b-a,$$

which gives the same quantity $\frac{(b-a)^2}{4}$ in both sides of (2.2). □

Corollary 2. *Assume that $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_2([a,b],H)$. If $w:[a,b]\to(0,\infty)$ is integrable with $\int_a^b w(s)\,ds=$*

1, *then*

$$\int_a^b \|f(t)\|^2 w(t)\,dt \le \frac{1}{2}\int_a^b \left[(t-a)\int_a^t \|f'(s)\|^2 ds + (b-t)\int_t^b \|f'(s)\|^2 ds\right] w(t)\,dt \le \frac{1}{2}\left[\frac{1}{2}(b-a) + \int_a^b \left|t-\frac{a+b}{2}\right| w(t)\,dt\right]\int_a^b \|f'(s)\|^2 ds. \tag{2.3}$$

The proof follows by (2.1) on taking $u(t) = \int_a^t w(s)\,ds$, $t\in[a,b]$.

Theorem 4. *Assume that* $A:[a,b]\to\mathcal{B}(H)$ *is strongly differentiable on* (a,b) *with* $A(a)=A(b)=0$ *and* $A'\in L_2([a,b],\mathcal{B}(H))$. *Then*

$$\int_a^b |A(t)|^2 du(t) \le \frac{1}{2}\int_a^b \left[(t-a)\int_a^t |A'(s)|^2 ds + (b-t)\int_t^b |A'(s)|^2 ds\right] du(t) \le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t)\right]\int_a^b |A'(s)|^2 ds = \frac{1}{2}\left[(b-a)[u(b)-u(a)] - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b |A'(s)|^2 ds \tag{2.4}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $x\in H$, $x\neq 0$ and take $f(t)=A(t)x$, $t\in[a,b]$ in (2.1) to get

$$\int_a^b \|A(t)x\|^2 du(t) \le \frac{1}{2}\int_a^b \left[(t-a)\int_a^t \|A'(s)x\|^2 ds + (b-t)\int_t^b \|A'(s)x\|^2 ds\right] du(t) \le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t)\right]\int_a^b \|A'(s)x\|^2 ds = \frac{1}{2}\left[(b-a)[u(b)-u(a)] - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right] \times \int_a^b \|A'(s)x\|^2 ds. \tag{2.5}$$

Observe that

$$\int_a^b \|A(t)x\|^2 du(t) = \int_a^b \langle A(t)x, A(t)x\rangle du(t) = \int_a^b \left\langle (A(t))^* A(t)x, x\right\rangle du(t) = \int_a^b \left\langle |A(t)|^2 x, x\right\rangle du(t) = \left\langle \left(\int_a^b |A(t)|^2 du(t)\right)x, x\right\rangle,$$

$$\int_a^b \left[(t-a)\int_a^t \|A'(s)x\|^2 ds + (b-t)\int_t^b \|A'(s)x\|^2 ds\right] du(t)$$
$$= \int_a^b \left[(t-a)\left\langle\left(\int_a^t |A'(s)|^2 ds\right)x, x\right\rangle + (b-t)\left\langle\left(\int_t^b |A'(s)|^2 ds\right)x, x\right\rangle\right] du(t)$$
$$= \left\langle\left(\int_a^b \left[(t-a)\left(\int_a^t |A'(s)|^2 ds\right) + (b-t)\left(\int_t^b |A'(s)|^2 ds\right)\right] du(t)\right)x, x\right\rangle$$

and

$$\int_a^b \|A'(s)x\|^2 ds = \left\langle\left(\int_a^b |A'(s)|^2 ds\right)x, x\right\rangle.$$

Then by (2.5) we get

$$\left\langle\left(\int_a^b |A(t)|^2 du(t)\right)x, x\right\rangle$$
$$\leq \left\langle\left(\int_a^b \left[(t-a)\left(\int_a^t |A'(s)|^2 ds\right) + (b-t)\left(\int_t^b |A'(s)|^2 ds\right)\right] du(t)\right)x, x\right\rangle$$
$$\leq \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t - \frac{a+b}{2}\right| du(t)\right]$$
$$\times \left\langle\left(\int_a^b |A'(s)|^2 ds\right)x, x\right\rangle$$
$$= \frac{1}{2}\left[(b-a)[u(b)-u(a)] - \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right)u(t)\,dt\right]$$
$$\times \left\langle\left(\int_a^b |A'(s)|^2 ds\right)x, x\right\rangle,$$

for all $x \in H$, which is equivalent, in the operator order, to (2.4). □

Corollary 3. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is strongly differentiable on (a,b) with $A(a) = A(b) = 0$ and $A' \in L_2([a,b], \mathcal{B}(H))$. Then*

$$\left|A\left(\frac{a+b}{2}\right)\right|^2 \leq \frac{1}{4}(b-a)\int_a^b |f'(s)|^2 ds. \tag{2.6}$$

The proof is like the one for the Corollary 1.

Corollary 4. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is strongly differentiable on (a,b) with $A(a) = A(b) = 0$ and $A' \in L_2([a,b], \mathcal{B}(H))$. If $w : [a,b] \to (0,\infty)$ is integrable* *with* $\int_a^b w(s)\,ds = 1$, *then*

$$\begin{aligned}(2.7)\qquad & \int_a^b |A(t)|^2 w(t)\,dt \\ & \le \frac{1}{2}\int_a^b \left[(t-a)\int_a^t |A'(s)|^2 ds + (b-t)\int_t^b |A'(s)|^2 ds\right] w(t)\,dt \\ & \le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t-\frac{a+b}{2}\right| w(t)\,dt\right]\int_a^b |A'(s)|^2 ds.\end{aligned}$$

3. Applications for Trapezoid and Grüss' Type Inequalities

We have:

Proposition 1. *Assume that* $h:[a,b]\to H$ *is absolutely continuous and* $h'\in L_2([a,b],H)$. *If* $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}(3.1)\qquad & \left\|\int_a^b h(t)\,du(t)\right. \\ & \left. -\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right]h(b) - \left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right]h(a)\right\|^2 \\ & \le \frac{1}{2}[u(b)-u(a)]\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t)\right] \\ & \times \int_a^b \left\|h'(t)-\frac{h(b)-h(a)}{b-a}\right\|^2 ds.\end{aligned}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}& \int_a^b \left[h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\right]du(t) \\ & = \int_a^b h(t)\,du(t) - \int_a^b \frac{(b-th(a))+(t-ah(b))}{b-a}du(t)\end{aligned}$$

$$\begin{aligned}& = \int_a^b h(t)\,du(t) - u(t)\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\Big|_a^b \\ & + \frac{h(b)-h(a)}{b-a}\int_a^b u(t)\,dt \\ & = \int_a^b h(t)\,du(t) - u(b)h(b) + u(a)h(a) + \left(\int_a^b u(t)\,dt\right)\frac{h(b)-h(a)}{b-a} \\ & = \int_a^b h(t)\,du(t) \\ & - \left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,dt\right]h(b) - \left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right]h(a).\end{aligned}$$

If we choose

$$f(t)=h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a},\ t\in[a,b],$$

then we have $f(a)=f(b)=0$,

$$f'(t)=h'(t)-\frac{h(b)-h(a)}{b-a},\ t\in(a,b)$$

and by (2.1) we get

$$\begin{aligned}(3.2)\qquad &\int_a^b\left\|h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\right\|^2du(t)\\ &\le\frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right]\\ &\times\int_a^b\left\|h'(t)-\frac{h(b)-h(a)}{b-a}\right\|^2ds.\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators, we have

$$\begin{aligned}(3.3)\qquad &[u(b)-u(a)]\int_a^b\left\|h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\right\|^2du(t)\\ &\ge\left\|\int_a^b\left(h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\right)du(t)\right\|^2.\end{aligned}$$

By making use of (3.2) and (3.3) we derive the desired result (3.1). □

Corollary 5. *Assume that $h:[a,b]\to H$ is absolutely continuous and $h'\in L_2([a,b],H)$. If $w:[a,b]\to(0,\infty)$ is integrable with $\int_a^b w(s)\,ds=1$, then*

$$\begin{aligned}(3.4)\qquad &\left\|\int_a^b w(t)h(t)\,dt-\frac{(b-E(w,[a,b]))f(a)+(E(w,[a,b])-a)f(b)}{b-a}\right\|^2\\ &\le\frac{1}{2}\left[\frac{1}{2}(b-a)+\int_a^b\left|t-\frac{a+b}{2}\right|w(t)\,dt\right]\\ &\times\int_a^b\left\|h'(t)-\frac{h(b)-h(a)}{b-a}\right\|^2dt,\end{aligned}$$

where $E(w,[a,b]):=\int_a^b tw(t)\,dt$.

Proposition 2. *Assume that $h:[a,b]\to H$ is absolutely continuous and $h'\in L_2([a,b],H)$. Then*

$$(3.5)\qquad \left\|h\left(\frac{a+b}{2}\right)-\frac{h(a)+h(b)}{2}\right\|^2\le\frac{1}{8}(b-a)\int_a^b\left\|h'(t)-h'(a+b-t)\right\|^2ds$$

and

$$(3.6)\qquad \left\|h\left(\frac{a+b}{2}\right)-\frac{h(a)+h(b)}{2}\right\|^2\le\frac{1}{4}(b-a)\int_a^b\left\|h'(s)-\frac{h(b)-h(a)}{b-a}\right\|^2ds.$$

Proof. We use (2.2) for the function

$$f(t) := \frac{h(t) + h(a+b-t)}{2} - \frac{h(a) + h(b)}{2},$$

which is absolutely continuous with $f(a) = f(b) = 0$ and

$$f'(t) := \frac{h'(t) - h'(a+b-t)}{2}$$

to get (3.5).

Also if we use (2.2) for the function

$$f(t) = h(t) - \frac{(b-t)h(a) + (t-a)h(b)}{b-a},$$

which is absolutely continuous with $f(a) = f(b) = 0$ and

$$f'(t) = h'(t) - \frac{h(b) - h(a)}{b-a}$$

to get (3.6). □

We have for $\alpha : [a,b] \to \mathbb{C}$ and $A : [a,b] \to \mathcal{B}(H)$,

$$\begin{aligned} 0 &\le \left| \overline{\alpha(t)} A(s) - \overline{\alpha(s)} A(t) \right|^2 \\ &= |\alpha(t)| \, |A(s)|^2 - \alpha(s) \overline{\alpha(t)} A^*(t) A(s) \\ &\quad - \alpha(t) \overline{\alpha(s)} A^*(s) A(t) + |\alpha(s)|^2 |A(t)|^2, \end{aligned}$$

which gives that

$$\begin{aligned} &|\alpha(t)|^2 |A(s)|^2 + |\alpha(s)|^2 |A(t)|^2 \\ &\ge \alpha(s) \overline{\alpha(t)} A^*(t) A(s) + \alpha(t) \overline{\alpha(s)} A^*(s) A(t) \end{aligned}$$

for all $s, t \in [a,b]$.

Consider also $u : [a,b] \to [0,\infty)$ monotonic nondecreasing and assume that $A : [a,b] \to \mathcal{B}(H)$ is continuous.

Integrating over $du(t)$ and $du(s)$ on $[a,b]$, then we get

$$\begin{aligned} &\int_a^b |\alpha(t)|^2 du(t) \int_a^b |A(s)|^2 du(s) + \int_a^b |\alpha(s)|^2 du(s) \int_a^b |A(t)|^2 du(t) \\ &\ge \int_a^b \overline{\alpha(t)} A^*(t) du(t) \int_a^b \alpha(s) A(s) du(s) \\ &+ \int_a^b \overline{\alpha(s)} A^*(s) du(s) \int_a^b \alpha(t) A(t) du(t) \\ &= 2 \left| \int_a^b \alpha(s) A(s) du(s) \right|^2, \end{aligned}$$

which proves that

$$\int_a^b |\alpha(t)|^2 du(t) \int_a^b |A(t)|^2 du(t) \ge \left| \int_a^b \alpha(t) A(t) du(t) \right|^2, \tag{3.7}$$

provided that α and A are continuous.

For $\alpha(t) = 1$, we then get

$$[u(b) - u(a)] \int_a^b |A(t)|^2 du(t) \geq \left| \int_a^b A(t) du(t) \right|^2 . \tag{3.8}$$

In the case of Lebesgue and Bochner integrals, the inequality (3.7) also holds for $\alpha \in L_2[a,b]$ and $A \in L_2([a,b], \mathcal{B}(H))$.

We have:

Proposition 3. *Assume that $B : [a,b] \to H$ is strongly differentiable and $B' \in L_2([a,b], H)$. If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then*

$$\begin{aligned} &\left| \int_a^b B(t) du(t) \right. \\ &\left. - \left[u(b) - \frac{1}{b-a} \int_a^b u(t) dt \right] B(b) - \left[\frac{1}{b-a} \int_a^b u(t) dt - u(a) \right] B(a) \right|^2 \\ &\leq \frac{1}{2} [u(b) - u(a)] \left[\frac{1}{2} (b-a) [u(b) - u(a)] + \int_a^b \left| t - \frac{a+b}{2} \right| du(t) \right] \\ &\times \int_a^b \left| B'(t) - \frac{B(b) - B(a)}{b-a} \right|^2 ds. \end{aligned} \tag{3.9}$$

Proof. As in the proof of Proposition 1, we have

$$\begin{aligned} &\int_a^b \left[B(t) - \frac{(b-t) B(a) + (t-a) B(b)}{b-a} \right] du(t) \\ &= \int_a^b B(t) du(t) \\ &- \left[u(b) - \frac{1}{b-a} \int_a^b u(t) dt \right] B(b) - \left[\frac{1}{b-a} \int_a^b u(t) dt - u(a) \right] B(a). \end{aligned}$$

If we choose

$$A(t) = B(t) - \frac{(b-t) B(a) + (t-a) B(b)}{b-a}, \ t \in [a,b],$$

then we have $f(a) = f(b) = 0$,

$$A'(t) = B'(t) - \frac{B(b) - B(a)}{b-a}, \ t \in (a,b)$$

and by (2.6) we get

$$\begin{aligned} &\int_a^b \left| B(t) - \frac{(b-t) B(a) + (t-a) B(b)}{b-a} \right|^2 du(t) \\ &\leq \frac{1}{2} \left[\frac{1}{2} (b-a) [u(b) - u(a)] + \int_a^b \left| t - \frac{a+b}{2} \right| du(t) \right] \\ &\times \int_a^b \left| B'(t) - \frac{B(b) - B(a)}{b-a} \right|^2 ds. \end{aligned} \tag{3.10}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators (3.8), we have

$$[u(b)-u(a)]\int_a^b \left|h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\right|^2 du(t) \geq \left|\int_a^b \left(h(t)-\frac{(b-t)h(a)+(t-a)h(b)}{b-a}\right) du(t)\right|^2. \tag{3.11}$$

By making use of (3.10) and (3.11) we derive the desired result (3.9). □

Corollary 6. *Assume that* $B:[a,b]\to H$ *is strongly differentiable and* $B'\in L_2([a,b],H)$. *If* $w:[a,b]\to(0,\infty)$ *is integrable with* $\int_a^b w(s)\,ds=1$, *then*

$$\left|\int_a^b w(t)B(t)\,dt-\frac{(b-E(w,[a,b]))B(a)+(E(w,[a,b])-a)B(b)}{b-a}\right|^2 \leq \frac{1}{2}\left[\frac{1}{2}(b-a)+\int_a^b\left|t-\frac{a+b}{2}\right|w(t)\,dt\right] \times\int_a^b\left|B'(t)-\frac{B(b)-B(a)}{b-a}\right|^2 dt. \tag{3.12}$$

We have:

Proposition 4. *Assume that* $B:[a,b]\to H$ *is strongly differentiable and* $B'\in L_2([a,b],H)$. *Then*

$$\left|B\left(\frac{a+b}{2}\right)-\frac{B(a)+B(b)}{2}\right|^2\leq\frac{1}{8}(b-a)\int_a^b\left|B'(t)-B'(a+b-t)\right|^2 ds \tag{3.13}$$

and

$$\left|B\left(\frac{a+b}{2}\right)-\frac{B(a)+B(b)}{2}\right|^2\leq\frac{1}{4}(b-a)\int_a^b\left|B'(t)-\frac{B(b)-B(a)}{b-a}\right|^2 ds. \tag{3.14}$$

The proof is similar to the one from Proposition 2.

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{3.15}$$

In 1935, Grüss [13] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n), \tag{3.16}$$

provided that there exists the real numbers m, M, n, N such that

$$m\leq f(t)\leq M \quad\text{and}\quad n\leq g(t)\leq N \quad\text{for a.e. } t\in[a,b]. \tag{3.17}$$

The constant $\frac{1}{4}$ is best possible in (3.15) in the sense that it cannot be replaced by a smaller quantity.

Theorem 5. *Assume that* $h:[a,b]\to H$ *is absolutely continuous and* $h'\in L_2([a,b],H)$. *If* $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing, then*

$$\|C(h,u)\|^2 \leq \frac{1}{2}[u(b)-u(a)]\left[\frac{1}{2}[u(b)-u(a)]+\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right] \times\left(\frac{1}{b-a}\int_a^b\|h(t)\|^2dt-\left\|\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2\right), \tag{3.18}$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_a^b\left(\int_a^t h(s)\,ds-\frac{t-a}{b-a}\int_a^b h(s)\,ds\right)du(t)\\&=\left(\int_a^t h(s)\,ds-\frac{t-a}{b-a}\int_a^b h(s)\,ds\right)u(t)\Big|_a^b\\&-\int_a^b u(t)\,d\left(\int_a^t h(s)\,ds-\frac{t-a}{b-a}\int_a^b h(s)\,ds\right)\\&=-\int_a^b u(t)h(t)\,dt+\frac{1}{b-a}\int_a^b h(s)\,ds\int_a^b u(t)\,dt,\end{aligned}$$

which gives that

$$C(h,u)=\frac{1}{b-a}\int_a^b\left(\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right)du(t). \tag{3.19}$$

Consider

$$g(t):=\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds,\ t\in[a,b],$$

then g is absolutely continuous, $g(a)=g(b)=0$,

$$g'(t):=\frac{1}{b-a}\int_a^b h(s)\,ds-h(t),\ t\in[a,b]$$

and by (2.1) we get

$$\int_a^b\left\|\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right\|^2du(t) \leq\frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right] \times\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2dt. \tag{3.20}$$

Using (CBS) inequality, we have

$$\begin{aligned}
&[u(b)-u(a)]\int_a^b\left\|\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right\|^2 du(t) \\
&\geq\left\|\int_a^b\left(\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right)du(t)\right\|^2 \\
&=(b-a)^2\|C(h,u)\|^2.
\end{aligned}\tag{3.21}$$

By (3.20) and (3.21) we get

$$\begin{aligned}
&(b-a)^2\|C(h,u)\|^2 \\
&\leq\frac{1}{2}[u(b)-u(a)]\left[\frac{1}{2}(b-a)[u(b)-u(a)]+\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right] \\
&\times\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt,
\end{aligned}$$

which gives

$$\begin{aligned}
&\|C(h,u)\|^2 \\
&\leq\frac{1}{2}[u(b)-u(a)]\left[\frac{1}{2}[u(b)-u(a)]+\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right] \\
&\times\frac{1}{b-a}\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt.
\end{aligned}$$

Since, by the properties of inner product

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt \\
&=\frac{1}{b-a}\int_a^b\|h(t)\|^2 dt-\left\|\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2,
\end{aligned}$$

the theorem is thus proved. □

Finally, we can prove in a similar way the operator inequality:

Theorem 6. *Assume that* $A:[a,b]\to H$ *belongs to* $L_2([a,b],H)$. *If* $u:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}
&|C(A,u)|^2 \\
&\leq\frac{1}{2}[u(b)-u(a)]\left[\frac{1}{2}[u(b)-u(a)]+\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|du(t)\right] \\
&\times\left(\frac{1}{b-a}\int_a^b|A(t)|^2 dt-\left|\frac{1}{b-a}\int_a^b A(s)\,ds\right|^2\right).
\end{aligned}\tag{3.22}$$

References

[1] M. W. Alomari, On Beesack-Wirtinger Inequality, *Results Math.*, Online First 2017 Springer International Publishing DOI 10.1007/s00025-016-0644-6.

[2] T. M. Apostol, *Mathematical Analysis,* Addison-Wesley Publishing Company, Second Edition, 1981.

[3] P. R. Beesack, Extensions of Wirtinger's inequality. *Trans. R. Soc. Can.* **53**, 21–30 (1959)

[4] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [Online: http://rgmia.vu.edu.au/v5n2.html].

[5] J. B. Diaz and F. T. Metcalf, Variations on Wirtinger's inequality, in: *Inequalities* Academic Press, New York, 1967, pp. 79–103.

[6] S. S. Dragomir, Accurate approximations for the Riemann-Stieltjes integral via theory of inequalities. *J. Math. Inequal.* **3** (2009), no. 4, 663–681

[7] S. S. Dragomir, Approximating the Riemann-Stieltjes integral via a Chebyshev type functional. *Acta Comment. Univ. Tartu. Math.* **18** (2014), No. 2, 239–259. Preprint *RGMIA Res. Rep. Coll.* **10** (2007), Suplement Art. 18 [Online http://rgmia.org/papers/v10e/ARSICTF.pdf].

[8] S. S. Dragomir, Integral inequalities related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 59, 16 pp. [Online https://rgmia.org/papers/v21/v21a59.pdf].

[9] S. S. Dragomir, Integral inequalities for functions with values in Hilbert spaces related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art. 45, 19 pp. [Online https://rgmia.org/papers/v24/v24a45.pdf].

[10] S. S. Dragomir, Absolute value integral inequalities for functions with operator values in Hilbert spaces related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art. 47, 15 pp. [Online https://rgmia.org/papers/v24/v24a47.pdf].

[11] L. Fejér, Über die Fourierreihen, II, (In Hungarian) *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[12] R. Giova, An estimate for the best constant in the L_p-Wirtinger inequality with weights, *J. Func. Spaces Appl.*,Volume **6**, Number 1 (2008), 1-16.

[13] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b g(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b g(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[14] J. Jaroš, On an integral inequality of the Wirtinger type, *Appl. Math. Letters,* **24** (2011) 1389–1392.

[15] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[16] C. F. Lee, C. C. Yeh, C. H. Hong and R. P. Agarwal, Lyapunov and Wirtinger inequalities, *Appl. Math. Lett.* **17** (2004) 847–853.

[17] C. A. Swanson, Wirtinger's inequality, *SIAM J. Math. Anal.* **9** (1978) 484–491.

[18] T. Ricciardi, A sharp weighted Wirtinger inequality, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8), **8** (1) (2005), 259–267.

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