STEKLOFF'S TYPE ABSOLUTE VALUE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR FUNCTIONS OF OPERATORS IN HILBERT SPACES

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Abstract. Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. In this paper we show among others that, if $A : [a, b] \to \mathcal{B}(H)$ is strongly differentiable on (a, b) with $\int_a^b A(t)\, dt = 0$ and $A' \in L_2([a, b], \mathcal{B}(H))$ and if $u : [a, b] \to \mathbb{R}$ is monotonic nondecreasing, then

$$\begin{aligned}&\int_a^b |A(t)|^2 du(t)\\ &\leq \left[\frac{1}{12}(b-a)[u(b)-u(a)] + \frac{1}{b-a}\int_a^b \left(t - \frac{a+b}{2}\right)^2 du(t)\right]\\ &\times \int_a^b |A'(s)|^2 ds\end{aligned}$$

in the operator order of $\mathcal{B}(H)$. Applications for Grüss' type inequalities are provided. Some examples for the operator exponential are also given.

1. Introduction

It is well known that, see for instance [7], or [15], if $u \in C^1([a, b], \mathbb{R})$, namely u is continuous on $[a, b]$ and has a derivative that is continuous on (a, b) and satisfies $u(a) = u(b) = 0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\, dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2 dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K \sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a, b], \mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\, dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2 dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [1], [3], [7] and [14]-[17].

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In 1901, W. Stekloff, [19], proved that, if $u \in C^1([a,b],\mathbb{R})$ and $\int_a^b u(t)\,dt = 0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.3}$$

In addition, if $u(a) = u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h:[a,b]\to\mathbb{C}$.

Theorem 1. *If $h \in C^1([a,b],\mathbb{C})$ and $\int_a^b h(t)\,dt = 0$, then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.5}$$

In addition, if $h(a) = h(b)$, then

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u = \operatorname{Re} h$ and $u = \operatorname{Im} h$ and by adding the corresponding inequalities.

In the recent paper we obtained the following weighted version of the above results:

Theorem 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{C})$. If $\frac{f'}{\sqrt{w}} \in L_2[a,b]$ and $\int_a^b f(t)\,w(t)\,dt = 0$, then*

$$\int_a^b |f(t)|^2\,w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.7}$$

In addition, if $f(a) = f(b)$, then we have the better inequality

$$\int_a^b |f(t)|^2\,w(t)\,dt \le \frac{1}{4\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.8}$$

We have the following inequality of Wirtinger type in the operator order of $\mathcal{B}(H)$, [12]:

Theorem 3. *Assume that $A:[a,b]\to\mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $\int_a^b A(t)\,dt = 0$, then*

$$\int_a^b |A(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |A'(t)|^2\,dt. \tag{1.9}$$

In addition, if $A(a) = A(b)$, then we have the better inequality

$$\int_a^b |A(t)|^2\,dt \le \frac{(b-a)^2}{4\pi^2}\int_a^b |A'(t)|^2\,dt. \tag{1.10}$$

Motivated by the above results, in this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |A(t)|^2\,du(t)$ in the case that $A:[a,b]\to\mathcal{B}(H)$ is strongly differentiable on (a,b) with $\int_a^b A(t)\,dt = 0$, $A' \in L_2([a,b],\mathcal{B}(H))$ and $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing. Applications for Grüss' type inequalities are provided. Some examples for the operator exponential are also given.

2. Main Results

We have for $\alpha : [a,b] \to \mathbb{C}$ and $A : [a,b] \to \mathcal{B}(H)$,

$$
\begin{aligned}
0 &\le \left|\overline{\alpha(t)}A(s) - \overline{\alpha(s)}A(t)\right|^2 \\
&= |\alpha(t)|\,|A(s)|^2 - \alpha(s)\overline{\alpha(t)}A^*(t)A(s) \\
&- \alpha(t)\overline{\alpha(s)}A^*(s)A(t) + |\alpha(s)|^2|A(t)|^2,
\end{aligned}
$$

which gives that

$$
\begin{aligned}
&|\alpha(t)|^2|A(s)|^2 + |\alpha(s)|^2|A(t)|^2 \\
&\ge \alpha(s)\overline{\alpha(t)}A^*(t)A(s) + \alpha(t)\overline{\alpha(s)}A^*(s)A(t)
\end{aligned}
$$

for all $s,t \in [a,b]$.

Consider also $u : [a,b] \to [0,\infty)$ monotonic nondecreasing and assume that $A : [a,b] \to \mathcal{B}(H)$ is continuous.

Integrating over $du(t)$ and $du(s)$ on $[a,b]$, then we get

$$
\begin{aligned}
&\int_a^b |\alpha(t)|^2 du(t)\int_a^b |A(s)|^2 du(s) + \int_a^b |\alpha(s)|^2 du(s)\int_a^b |A(t)|^2 du(t) \\
&\ge \int_a^b \overline{\alpha(t)}A^*(t)\,du(t)\int_a^b \alpha(s)A(s)\,du(s) \\
&+ \int_a^b \overline{\alpha(s)}A^*(s)\,du(s)\int_a^b \alpha(t)A(t)\,du(t) \\
&= 2\left|\int_a^b \alpha(s)A(s)\,du(s)\right|^2,
\end{aligned}
$$

which proves that

$$
\int_a^b |\alpha(t)|^2 du(t)\int_a^b |A(t)|^2 du(t) \ge \left|\int_a^b \alpha(t)A(t)\,du(t)\right|^2, \tag{2.1}
$$

provided that α and A are continuous.

In the case of Lebesgue and Bochner integrals, the inequality (2.1) also holds for $\alpha \in L_2[a,b]$ and $A \in L_2([a,b],\mathcal{B}(H))$.

We have the following Riemann-Stieltjes integral inequality:

Theorem 4. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is strongly differentiable on (a,b) with $\int_a^b A(t)\,dt = 0$ and $A' \in L_2([a,b],\mathcal{B}(H))$. If $u : [a,b] \to \mathbb{R}$ is monotonic* *nondecreasing, then*

$$\int_a^b |A(t)|^2 du(t) \tag{2.2}$$
$$\leq \left[\frac{1}{12}(b-a)[u(b)-u(a)] + \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right)^2 du(t)\right]$$
$$\times \int_a^b |A'(s)|^2 ds$$
$$= \left[\frac{1}{3}(b-a)[u(b)-u(a)] - \frac{2}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) u(t)\,dt\right]$$
$$\times \int_a^b |A'(s)|^2 ds.$$

The inequality is sharp.

Proof. We use Montgomery identity for funcyions with values in $\mathcal{B}(H)$,

$$A(t) = \frac{1}{b-a}\int_a^b A(s)\,ds + \frac{1}{b-a}\int_a^b k(t,s)\,A'(s)\,ds, \tag{2.3}$$

where

$$k(t,s) = \begin{cases} s-a, & s\in[a,t], \\ s-b, & s\in(t,b]. \end{cases}$$

Since $\int_a^b A(s)\,ds = 0$, hence by Cauchy-Bunyakowsky-Schwarz inequality (2.1) we have

$$|A(t)|^2 = \frac{1}{(b-a)^2}\left|\int_a^b k(t,s)\,A'(s)\,ds\right|^2 \tag{2.4}$$
$$\leq \frac{1}{(b-a)^2}\int_a^b |k(t,s)|^2 ds \int_a^b |A'(s)|^2 ds.$$

Observe that

$$\int_a^b |k(t,s)|^2 ds = \int_a^t (s-a)^2 ds + \int_t^b (s-b)^2 ds = \frac{1}{3}\left[(t-a)^3 + (b-t)^3\right],$$

then by (2.4) we get

$$|A(t)|^2 \leq \frac{1}{3(b-a)^2}\left[(t-a)^3 + (b-t)^3\right]\int_a^b |A'(s)|^2 ds$$

for $t\in[a,b]$.

Taking the Riemann-Stieltjes integral we get

$$\int_a^b |A(t)|^2 du(t) \leq \frac{1}{3(b-a)^2}\int_a^b \left[(t-a)^3 + (b-t)^3\right] du(t) \int_a^b |A'(s)|^2 ds. \tag{2.5}$$

Since

$$\frac{1}{3}\left[\left(\frac{t-a}{b-a}\right)^3+\left(\frac{b-t}{b-a}\right)^3\right]=\frac{1}{12}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2,$$

hence

$$\begin{aligned}&\frac{1}{3}\int_a^b\left[\left(\frac{t-a}{b-a}\right)^3+\left(\frac{b-t}{b-a}\right)^3\right]du(t)\\&=\frac{1}{12}\left[u(b)-u(a)\right]+\frac{1}{(b-a)^2}\int_a^b\left(t-\frac{a+b}{2}\right)^2du(t)\end{aligned}$$

which, by (2.5), proves the first part of (2.2).

Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_a^b\left[(t-a)^3+(b-t)^3\right]du(t)\\&=\left[(t-a)^3+(b-t)^3\right]u(t)\Big|_a^b-3\int_a^b\left[(t-a)^2-(b-t)^2\right]u(t)\,dt\\&=(b-a)^3u(b)-(b-a)^3u(a)-3(b-a)\int_a^b(2t-a-b)\,u(t)\,dt\\&=(b-a)^3\left[u(b)-u(a)\right]-6(b-a)\int_a^b\left(t-\frac{a+b}{2}\right)u(t)\,dt.\end{aligned}\tag{2.6}$$

If we use (2.5) and (2.6) we derive the desired result (2.2).

The sharpness of the inequality follows by Corollary 2 below. □

Corollary 1. *Assume that $A:[a,b]\to\mathcal{B}(H)$ is absolutely continuous with $\int_a^b A(t)\,dt=0$ and $A'\in L_2([a,b],\mathcal{B}(H))$. If $w:[a,b]\to(0,\infty)$ is integrable on $[a,b]$ with $\int_a^b w(s)\,ds=1$, then*

$$\begin{aligned}&\int_a^b|A(t)|^2w(t)\,dt\\&\le\left[\frac{1}{12}(b-a)+\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)^2w(t)\,dt\right]\int_a^b|A'(s)|^2ds.\end{aligned}\tag{2.7}$$

Remark 1. *Assume that $B:[a,b]\to\mathcal{B}(H)$ is absolutely continuous and $B'\in L_2([a,b],\mathcal{B}(H))$. By taking*

$$A=B-\frac{1}{b-a}\int_a^b B(s)\,ds$$

in (2.2) we obtain

$$\int_a^b \left| B(t) - \frac{1}{b-a}\int_a^b B(s)\,ds \right|^2 du(t) \tag{2.8}$$
$$\leq \left[\frac{1}{12}(b-a)[u(b)-u(a)] + \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right)^2 du(t) \right]$$
$$\times \int_a^b |B'(s)|^2\,ds$$
$$= \left[\frac{1}{3}(b-a)[u(b)-u(a)] - \frac{2}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) u(t)\,dt \right]$$
$$\times \int_a^b |B'(s)|^2\,ds,$$

provided that $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing.

Using Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann-Stieltjes integral with monotonic integrators (2.1), we have

$$\left| \int_a^b B(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b B(s)\,ds \right|^2$$
$$\leq [u(b)-u(a)] \int_a^b \left| B(t) - \frac{1}{b-a}\int_a^b B(s)\,ds \right|^2 du(t).$$

Therefore by (2.8) we derive

$$\left| \int_a^b B(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b B(s)\,ds \right|^2 \tag{2.9}$$
$$\leq \left[\frac{1}{12}(b-a)[u(b)-u(a)] + \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right)^2 du(t) \right]$$
$$\times \int_a^b |B'(s)|^2\,ds.$$

If $w:[a,b]\to(0,\infty)$ is integrable on $[a,b]$ with $\int_a^b w(s)\,ds = 1$, then we have the following result comparing the weighted integral mean with the integral mean,

$$\left| \frac{1}{b-a}\int_a^b B(t)\,w(t)\,dt - \frac{1}{b-a}\int_a^b B(s)\,ds \right| \tag{2.10}$$
$$\leq \frac{1}{b-a}\int_a^b \left| B(t) - \frac{1}{b-a}\int_a^b B(s)\,ds \right|^2 w(t)\,dt$$
$$\leq \left[\frac{1}{12} + \frac{1}{(b-a)^2}\int_a^b \left(t-\frac{a+b}{2}\right)^2 w(t)\,dt \right] \int_a^b |B'(s)|^2\,ds.$$

Corollary 2. *Assume that $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable on (a,b) and $B'\in L_2([a,b],\mathcal{B}(H))$, then*

$$\left|B\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2\leq\frac{1}{12}(b-a)\int_a^b\left|B'(s)\right|^2 ds. \tag{2.11}$$

The constant $\frac{1}{12}$ is best possible in (2.11).

Proof. We use the inequality (2.2) for the nondecreasing function $u(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$. Integrating by parts in the Riemann-Stieltjes integral,

$$\begin{aligned}
&\int_a^b |A(t)|^2\,du(t)\\
&=|A(t)|^2u(t)\Big|_a^b-\int_a^b u(t)\,d\left(|A(t)|^2\right)\\
&=|A(b)|^2u(b)-|A(a)|^2u(a)-\left(-\int_a^{\frac{a+b}{2}}d\left(|A(t)|^2\right)+\int_{\frac{a+b}{2}}^b d\left(|A(t)|^2\right)\right)\\
&=|A(b)|^2+|A(a)|^2+\int_a^{\frac{a+b}{2}}d\left(|A(t)|^2\right)-\int_{\frac{a+b}{2}}^b d\left(|A(t)|^2\right).
\end{aligned}$$

Since

$$\int_a^{\frac{a+b}{2}}d\left(|A(t)|^2\right)=\left|A\left(\frac{a+b}{2}\right)\right|^2-|A(a)|^2$$

and

$$\int_{\frac{a+b}{2}}^b d\left(|A(t)|^2\right)=|A(b)|^2-\left|A\left(\frac{a+b}{2}\right)\right|^2,$$

hence

$$\begin{aligned}
\int_a^b |A(t)|^2\,du(t)&=|A(b)|^2+|A(a)|^2\\
&\quad+\left|A\left(\frac{a+b}{2}\right)\right|^2-|A(a)|^2-|A(b)|^2+\left|A\left(\frac{a+b}{2}\right)\right|^2\\
&=2\left|A\left(\frac{a+b}{2}\right)\right|^2.
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{1}{3}(b-a)[u(b)-u(a)]-\frac{2}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)u(t)\,dt\\
&=\frac{2}{3}(b-a)-\frac{2}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{2}{3}(b-a)-\frac{1}{2}(b-a)\\
&=\frac{1}{6}(b-a).
\end{aligned}$$

Then by (2.2) we get

$$\left|A\left(\frac{a+b}{2}\right)\right|^2\leq\frac{1}{12}(b-a)\int_a^b\left|A'(s)\right|^2 ds.$$

Finally, if we take $A=B-\frac{1}{b-a}\int_a^b B(s)\,ds$, then we get the desired result (2.11).

In the scalar case, consider the function

$$A_0(t) := \begin{cases} \frac{1}{2}(t-a)^2, & t \in \left[a, \frac{a+b}{2}\right], \\ \frac{1}{2}(t-b)^2, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then

$$A_0\left(\frac{a+b}{2}\right) = \frac{1}{8}(b-a)^2$$

and

$$A_0'(t) := \begin{cases} t-a, & t \in \left(a, \frac{a+b}{2}\right), \\ t-b, & t \in \left(\frac{a+b}{2}, b\right). \end{cases}$$

We have

$$\int_a^b A_0(t)\,dt = \frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)^2\,dt + \frac{1}{2}\int_{\frac{a+b}{2}}^b (t-b)^2\,dt = \frac{(b-a)^3}{24},$$

$$\int_a^b |A'(t)|^2\,dt = \int_a^{\frac{a+b}{2}} (t-a)^2\,dt + \int_{\frac{a+b}{2}}^b (t-b)^2\,dt = \frac{(b-a)^3}{12}$$

and

$$(b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t)\,dt = \frac{1}{8}(b-a)^3 - \frac{(b-a)^3}{24} = \frac{(b-a)^3}{12}.$$

Therefore

$$\left|(b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t)\,dt\right|^2 = \frac{(b-a)^6}{144}$$

and

$$\frac{1}{12}(b-a)^3\int_a^b |A'(t)|^2\,dt = \frac{(b-a)^6}{144},$$

which proves the sharpness of the constant $\frac{1}{12}$. □

3. Some Examples

Consider the function $A(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $A'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. By making use of (2.11) we get

$$\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT)\,dt\right|^2 \le \frac{1}{12}(b-a)^3\int_a^b |T\exp(tT)|^2\,dt. \tag{3.1}$$

If T is invertible, then [4]

$$\int_a^b \exp(tT)\,dt = T^{-1}[\exp(bT) - \exp(aT)]. \tag{3.2}$$

From (3.1) we derive

$$\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - T^{-1}[\exp(bT) - \exp(aT)]\right|^2 \le \frac{1}{12}(b-a)^3\int_a^b |T\exp(tT)|^2\,dt. \tag{3.3}$$

For T invertible, if we consider $B(t) = T\exp(tT)$, then $B'(t) = T^2\exp(tT)$ and

$$\int_a^b B(t)\,dt = \exp(bT) - \exp(aT).$$

By (2.11) we derive

$$\left|(b-a)\,T\exp\left(\frac{a+b}{2}T\right) - \exp(bT) + \exp(aT)\right|^2 \leq \frac{1}{12}(b-a)^3\int_a^b \left|T^2\exp(tT)\right|^2 dt. \tag{3.4}$$

Since for any operator $V \in \mathcal{B}(H)$ we have $|V|^2 \leq \|V\|^2$ and $\|\exp(tT)\| \leq \exp(|t|\,\|T\|)$, $t\in\mathbb{R}$, $T \in \mathcal{B}(H)$, then by (3.1) we get

$$\begin{aligned}\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT)\,dt\right|^2 &\leq \frac{1}{12}(b-a)^3\int_a^b \|T\exp(tT)\|^2\,dt \leq \frac{1}{12}\|T\|^2(b-a)^3\int_a^b \|\exp(tT)\|^2\,dt \\ &\leq \frac{1}{12}\|T\|^2(b-a)^3\int_a^b \exp(2\|T\|\,|t|)\,dt.\end{aligned} \tag{3.5}$$

If $0 \leq a \leq b$, then

$$\int_a^b \exp(2\|T\|\,|t|)\,dt = \int_a^b \exp(2\|T\|\,t)\,dt = \frac{\exp(2\|T\|\,b) - \exp(2\|T\|\,a)}{2\|T\|}$$

and by (3.5) we get

$$\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT)\,dt\right|^2 \leq \frac{1}{24}(b-a)^3\|T\|\left[\exp(2\|T\|\,b) - \exp(2\|T\|\,a)\right] \tag{3.6}$$

for any $T \in \mathcal{B}(H)$.

Moreover, if T is invertible, then we also have the exponential inequality

$$\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - T^{-1}\left[\exp(bT) - \exp(aT)\right]\right|^2 \leq \frac{1}{24}(b-a)^3\|T\|\left[\exp(2\|T\|\,b) - \exp(2\|T\|\,a)\right]. \tag{3.7}$$

Consider the function $A(t)=\exp[(1-t)A](B-A)\exp(tB)$, $t\in[0,1]$. Then, integrating by parts

$$\begin{aligned}
&\int_0^1 f(t)\,dt \\
&=\int_0^1 (\exp[(1-t)A]B\exp(tB)-\exp[(1-t)A]A\exp(tB))\,dt \\
&=\int_0^1 \exp[(1-t)A](\exp(tB))'\,dt+\int_0^1 (\exp[(1-t)A])'\exp(tB)\,dt \\
&= \exp[(1-t)A]\exp(tB)|_0^1 + A\int_0^1 \exp[(1-t)A]\exp(tB)\,dt \\
&+ \exp[(1-t)A]\exp(tB)|_0^1 - \int_0^1 (\exp[(1-t)A])B\exp(tB)\,dt \\
&=2(\exp B-\exp A)-\int_0^1 \exp[(1-t)A](B-A)\exp(tB) \\
&=2(\exp B-\exp A)-\int_0^1 f(t)\,dt,
\end{aligned}$$

which gives the following identity of interest [5]

$$\int_0^1 \exp[(1-t)A](B-A)\exp(tB)\,dt=\exp B-\exp A$$

for all $A,\ B\in\mathcal{B}(H)$.

Also

$$\begin{aligned}
A'(t)&=-A\exp[(1-t)A](B-A)\exp(tB) \\
&\quad+\exp[(1-t)A](B-A)B\exp(tB)\,dt \\
&=\exp[(1-t)A](B-A)B\exp(tB)\,d \\
&\quad-\exp[(1-t)A]A(B-A)\exp(tB) \\
&=\exp[(1-t)A][(B-A)B-A(B-A)]\exp(tB) \\
&=\exp[(1-t)A]\left(B^2-2AB+A^2\right)\exp(tB).
\end{aligned}$$

By utilising (2.6) we get

$$\begin{aligned}
&\left|\exp\left(\frac{1}{2}A\right)(B-A)\exp\left(\frac{1}{2}B\right)-\exp B+\exp A\right|^2 \\
&\le\frac{1}{12}\int_a^b \left|\exp[(1-t)A]\left(B^2-2AB+A^2\right)\exp(tB)\right|^2 dt,
\end{aligned}\tag{3.8}$$

for all $A,\ B\in\mathcal{B}(H)$.

Since

$$\begin{aligned}&\left|\exp\left[(1-t)A\right]\left(B^2-2AB+A^2\right)\exp(tB)\right|^2\\&\leq\left\|\exp\left[(1-t)A\right]\left(B^2-2AB+A^2\right)\exp(tB)\right\|^2\\&\leq\left\|\exp\left[(1-t)A\right]\right\|^2\left\|B^2-2AB+A^2\right\|^2\left\|\exp(tB)\right\|^2\\&\leq\exp\left[2(1-t)\|A\|\right]\left\|B^2-2AB+A^2\right\|^2\exp(2t\|B\|)\\&=\left\|B^2-2AB+A^2\right\|^2\exp\left[2\left[(1-t)\|A\|+t\|B\|\right]\right]\\&=\left\|B^2-2AB+A^2\right\|^2\exp\left\{2\left[(1-t)\|A\|+t\|B\|\right]\right\},\end{aligned}$$

hence by (3.8) we get

$$\begin{aligned}&\left|\exp\left(\frac{1}{2}A\right)(B-A)\exp\left(\frac{1}{2}B\right)-\exp B+\exp A\right|^2\\&\leq\frac{1}{12}\left\|B^2-2AB+A^2\right\|^2\int_a^b\exp\left\{2\left[(1-t)\|A\|+t\|B\|\right]\right\}dt\\&=\frac{1}{12}\left\|B^2-2AB+A^2\right\|^2\begin{cases}\exp(2\|A\|) \text{ if } \|B\|=\|A\|,\\ \frac{\exp(2\|B\|)-\exp(2\|A\|)}{2(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|.\end{cases}\end{aligned}\tag{3.9}$$

Further, let $A,\ B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $A,\ B>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $A(t):=((1-t)A+tB)^{-1}$, $t\in[0,1]$ and observe that

$$A'(t)=-((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1},\ t\in[0,1].$$

By utilising (2.11) we then get

$$\begin{aligned}&\left|\left(\frac{A+B}{2}\right)^{-1}-\int_0^1((1-t)A+tB)^{-1}dt\right|^2\\&\leq\frac{1}{12}\int_0^1\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2dt\end{aligned}\tag{3.10}$$

for $A,\ B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$.

Since

$$\begin{aligned}&\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|\\&\leq\left\|((1-t)A+tB)^{-1}\right\|^2\|B-A\|,\end{aligned}$$

hence by (3.10) we derive

$$\begin{aligned}&\left|\left(\frac{A+B}{2}\right)^{-1}-\int_0^1((1-t)A+tB)^{-1}dt\right|^2\\&\leq\frac{1}{12}\|B-A\|^2\int_0^1\left\|((1-t)A+tB)^{-1}\right\|^4dt.\end{aligned}\tag{3.11}$$

Now, if $A \geq m > 0$ and $B \geq m > 0$, then $((1-t)A+tB)^{-1} \leq m^{-1}$ for $t \in [0,1]$ and by (3.11) we obtain the simpler inequality

$$\left| \left(\frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t)A+tB)^{-1} dt \right|^2 \leq \frac{1}{12} \frac{\|B-A\|^2}{m^4}.$$

Since $f(u) = u^{-1}$ is operator convex on $(0,\infty)$, then by taking the square root and using the Hermite-Hadamard operator inequality [8], we derive

$$0 \leq \int_0^1 ((1-t)A+tB)^{-1} dt - \left(\frac{A+B}{2} \right)^{-1} \leq \frac{\sqrt{3}}{6} \frac{\|B-A\|}{m^2},$$

provided that $A \geq m > 0$ and $B \geq m > 0$.

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