NEW RIEMANN-STIELTJES INTEGRAL INEQUALITIES OF WIRTINGER TYPE FOR VECTOR AND OPERATOR VALUED FUNCTIONS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. In this paper we show among others that, if $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$, $f'\in L_2([a,b],H)$ and u is monotonic nondecreasing, then

$$\int_a^b \|f(t)\|^2 du(t) \le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2} du(t) \int_a^b \|f'(s)\|^2 ds$$
$$\le \frac{1}{4}(b-a)[u(b)-u(a)]\int_a^b \|f'(s)\|^2 ds.$$

Applications related to the trapezoid and to Grüss' type inequalities are also provided.

1. Introduction

It is well known that, see for instance [5], or [15], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2 dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2 dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{C})$ is a function with complex values and $u(a)=u(b)=0$, then $\operatorname{Re}u(a)=\operatorname{Re}u(b)=0$ and $\operatorname{Im}u(a)=\operatorname{Im}u(b)=0$ and by writing (1.1) for $\operatorname{Re}u$ and $\operatorname{Im}u$ and adding the obtained inequalities, we get

$$\int_a^b |u(t)|^2 dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |u'(t)|^2 dt \tag{1.3}$$

with the equality holding if and only if

$$u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K\in\mathbb{C}$.

1991 *Mathematics Subject Classification.* 26D15; 46C05.

Key words and phrases. Wirtinger's inequality, Trapezoid inequality, Fejér's inequality, Grüss' inequality, Functions with Values in Hilbert spaces.

RGMIA Res. Rep. Coll. 24 (2021), Art. 67, 16 pp. Received 25/05/21

Similarly, if $u \in C^1([a,b],\mathbb{C})$ with $u(a)=0$, then by (1.2) we have

$$\int_a^b |u(t)|^2 \, dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b |u'(t)|^2 \, dt \tag{1.4}$$

and the equality holds if and only if

$$u(t) = L \sin \left[\frac{\pi (t-a)}{2(b-a)} \right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [13]-[19].

In the recent paper [9] we obtained among others the following generalization of Wirtinger inequalities for vector valued functions in Hilbert spaces:

Theorem 1. *Assume that $f : [a,b] \to H$ is of class C^1 on $[a,b]$ and $f(a) = f(b) = 0$. Then*

$$\int_a^b \|f(t)\|^2 \, dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b \|f'(t)\|^2 \, dt. \tag{1.5}$$

If only $f(a) = 0$, then

$$\int_a^b \|f(t)\|^2 \, dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b \|f'(t)\|^2 \, dt. \tag{1.6}$$

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$.

We have the following inequality of Wirtinger type in the operator order of $\mathcal{B}(H)$, see [10]:

Theorem 2. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $A(a) = A(b) = 0$, then*

$$\int_a^b |A(t)|^2 \, dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |A'(t)|^2 \, dt \tag{1.7}$$

in the operator order of $\mathcal{B}(H)$.

If only $A(a) = 0$, then

$$\int_a^b |A(t)|^2 \, dt \leq \frac{4(b-a)^2}{\pi^2} \int_a^b |A'(t)|^2 \, dt. \tag{1.8}$$

In the previous paper [11] we obtained the following result for the Riemann-Stieltjes integral:

Theorem 3. *Assume that* $f : [a, b] \to H$ *is absolutely continuous with* $f(a) = f(b) = 0$ *and* $f' \in L_2([a, b], H)$. *If* $u : [a, b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$
\begin{aligned}
(1.9) \quad & \int_a^b \|f(t)\|^2 du(t) \\
& \leq \frac{1}{2} \int_a^b \left[(t-a) \int_a^t \|f'(s)\|^2 ds + (b-t) \int_t^b \|f'(s)\|^2 ds \right] du(t) \\
& \leq \frac{1}{2} \left[\frac{1}{2} (b-a) [u(b) - u(a)] + \int_a^b \left| t - \frac{a+b}{2} \right| du(t) \right] \int_a^b \|f'(s)\|^2 ds \\
& = \frac{1}{2} \left[(b-a) [u(b) - u(a)] - \int_a^b \operatorname{sgn} \left(t - \frac{a+b}{2} \right) u(t) \, dt \right] \int_a^b \|f'(s)\|^2 ds.
\end{aligned}
$$

The inequalities are sharp in (1.9).

Motivated by the above results, in this paper we obtain new sharp upper bounds for the Riemann-Stieltjes integrals

$$
\int_a^b \|f(t)\|^2 du(t) \text{ and } \int_a^b |A(t)|^2 du(t)
$$

in the case when $f : [a, b] \to H$ is absolutely continuous with $f(a) = f(b) = 0$, $A : [a, b] \to \mathcal{B}(H)$ is strongly differentiable on (a, b) with $A(a) = A(b) = 0$ and the integrator u is monotonic nondecreasing on $[a, b]$. Applications related to the trapezoid and of Grüss' type inequalities are also provided.

2. Main Results

In the case of vector valued functions with values in Hilbert spaces, we have the following result:

Theorem 4. *Assume that* $f : [a, b] \to H$ *is absolutely continuous with* $f(a) = f(b) = 0$ *and* $f' \in L_2([a, b], H)$. *If* $u : [a, b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$
\begin{aligned}
(2.1) \quad & \int_a^b \|f(t)\|^2 du(t) \\
& \leq \frac{1}{2} \int_a^b (t-a)^{1/2} (b-t)^{1/2} du(t) \int_a^b \|f'(s)\|^2 ds \\
& = \frac{1}{2} \int_a^b (t-a)^{-1/2} (b-t)^{-1/2} \left(t - \frac{a+b}{2} \right) u(t) \, dt \int_a^b \|f'(s)\|^2 ds \\
& \leq \frac{1}{4} (b-a) [u(b) - u(a)] \int_a^b \|f'(s)\|^2 ds.
\end{aligned}
$$

The inequalities are sharp in (2.1).

Proof. Since $f(a)=f(b)=0$, hence $f(t)=\int_a^t f'(s)\,ds=-\int_t^b f'(s)\,ds$, $t\in[a,b]$. Then we have

$$\begin{aligned}
&\int_a^b \|f(t)\|^2\,du(t)\\
&=\int_a^b \|f(t)\|\,\|f(t)\|\,du(t)=\int_a^b \left\|\int_a^t f'(s)\,ds\right\|\left\|\int_t^b f'(s)\,ds\right\|du(t)\\
&\leq \int_a^b \left(\int_a^t \|f'(s)\|\,ds\right)\left(\int_t^b \|f'(s)\|\,ds\right)du(t)\\
&=\int_a^b (t-a)^{1/2}(t-a)^{-1/2}\left(\int_a^t \|f'(s)\|\,ds\right)\\
&\times(b-t)^{1/2}(b-t)^{-1/2}\left(\int_t^b \|f'(s)\|\,ds\right)du(t)\\
&=:B.
\end{aligned}$$

Using Cauchy-Bunyakowsky-Schwarz inequality, we have

$$(t-a)^{-1/2}\left(\int_a^t \|f'(s)\|\,ds\right)\leq\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}$$

and

$$(b-t)^{-1/2}\left(\int_t^b \|f'(s)\|\,ds\right)\leq\left(\int_t^b \|f'(s)\|^2\,ds\right)^{1/2}$$

for all $t\in[a,b]$.

These imply that

$$\begin{aligned}
B&\leq\int_a^b (t-a)^{1/2}(b-t)^{1/2}\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}\left(\int_t^b \|f'(s)\|^2\,ds\right)^{1/2}du(t)\\
&=:C.
\end{aligned}$$

By the arithmetic mean-geometric mean inequality

$$\sqrt{\alpha\beta}\leq\frac{1}{2}(\alpha+\beta),\ \ \alpha,\beta\geq 0,$$

we have

$$\begin{aligned}
&\left(\int_a^t \|f'(s)\|^2\,ds\right)^{1/2}\left(\int_t^b \|f'(s)\|^2\,ds\right)^{1/2}\\
&\leq\frac{1}{2}\left[\int_a^t \|f'(s)\|^2\,ds+\int_t^b \|f'(s)\|^2\,ds\right]=\frac{1}{2}\int_a^b \|f'(s)\|^2\,ds.
\end{aligned}$$

Therefore

$$C\leq\frac{1}{2}\int_a^b \|f'(s)\|^2\,ds\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t),$$

which proves the first inequality in (2.1).

Also

$$(t-a)^{1/2}(b-t)^{1/2}\leq\frac{1}{2}(t-a+b-t)=\frac{1}{2}(b-a),$$

and the last part of (2.1) is also proved.

Now, for the equality part, by using the integration by part formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t) \\
&= (t-a)^{1/2}(b-t)^{1/2}u(t)\Big|_a^b \\
&-\frac{1}{2}\int_a^b \left[(t-a)^{-1/2}(b-t)^{1/2}-(t-a)^{1/2}(b-t)^{-1/2}\right]u(t)\,dt \\
&= -\frac{1}{2}\int_a^b \left[(t-a)^{-1/2}(b-t)^{1/2}-(t-a)^{1/2}(b-t)^{-1/2}\right]u(t)\,dt \\
&= -\frac{1}{2}\int_a^b (t-a)^{-1/2}(b-t)^{-1/2}\left[b-t-(t-a)\right]u(t)\,dt \\
&= \int_a^b (t-a)^{-1/2}(b-t)^{-1/2}\left(t-\frac{a+b}{2}\right)u(t)\,dt,
\end{aligned}$$

which proves the identity.

Further, consider the functions

$$f(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right], \\ b-t, & t\in\left(\frac{a+b}{2},b\right] \end{cases}$$

and $u(t)=\operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$. The function f is absolutely continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$. Also

$$f'(t)=\begin{cases} 1, & t\in\left(a,\frac{a+b}{2}\right) \\ -1, & t\in\left(\frac{a+b}{2},b\right), \end{cases}$$

which gives that $\int_a^b |f'(t)|^2\,dt=b-a.$

Therefore

$$\frac{1}{4}(b-a)\left[u(b)-u(a)\right]\int_a^b |f'(s)|^2\,ds=\frac{1}{2}(b-a)^2,$$

Also,

$$\begin{aligned}
-2\int_a^b f(t)f'(t)u(t)\,dt &= 2\int_a^{\frac{a+b}{2}}(t-a)\,dt+2\int_{\frac{a+b}{2}}^b (b-t)\,dt \\
&= \frac{(b-a)^2}{4}+\frac{(b-a)^2}{4}=\frac{1}{2}(b-a)^2.
\end{aligned}$$

This example gives in all sides of (2.1) the same quantity $\frac{1}{2}(b-a)^2$, which proves the sharpness of all inequalities in (2.1). □

Corollary 1. *Assume that $f:[a,b]\to H$ is absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_2([a,b],H)$, then*

$$\left\|f\left(\frac{a+b}{2}\right)\right\|^2\le\frac{1}{4}(b-a)\int_a^b \|f'(s)\|^2\,ds. \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible.

Proof. If we take $u(t) = \operatorname{sgn}\left(t - \frac{a+b}{2}\right)$, $t \in [a,b]$, then

$$\begin{aligned}\int_a^b \|f(t)\|^2 du(t) &= \|f(t)\|^2 u(t)\Big|_a^b - \int_a^b u(t)\, d\left(\|f(t)\|^2\right) \\ &= \|f(b)\|^2 u(b) - \|f(a)\|^2 u(a) \\ &\quad - \int_a^{\frac{a+b}{2}} u(t)\, d\left(\|f(t)\|^2\right) - \int_{\frac{a+b}{2}}^b u(t)\, d\left(\|f(t)\|^2\right) \\ &= \|f(b)\|^2 + \|f(a)\|^2 + \left\|f\left(\frac{a+b}{2}\right)\right\|^2 \\ &\quad - \|f(a)\|^2 - \|f(b)\|^2 + \left\|f\left(\frac{a+b}{2}\right)\right\|^2 \\ &= 2\left\|f\left(\frac{a+b}{2}\right)\right\|^2\end{aligned}$$

and

$$\frac{1}{4}(b-a)\left[u(b) - u(a)\right] = \frac{1}{2}(b-a).$$

By (2.1) we get

$$2\left\|f\left(\frac{a+b}{2}\right)\right\|^2 \le \frac{1}{2}(b-a)\int_a^b \|f'(s)\|^2 ds,$$

which is equivalent to (2.2).

For $H = \mathbb{R}$, we take

$$f(t) = \begin{cases} t - a, & t \in \left[a, \frac{a+b}{2}\right], \\ b - t, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Observe that f is absolutely continuous with $|f'(t)| = 1$, $t \in (a,b)$. Then

$$f\left(\frac{a+b}{2}\right) = \frac{b-a}{2}, \quad \int_a^b \|f'(s)\|^2 ds = b - a,$$

which gives the same quantity $\frac{(b-a)^2}{4}$ in both sides of (2.2). □

Corollary 2. *Assume that $f : [a,b] \to H$ is absolutely continuous with $f(a) = f(b) = 0$ and $f' \in L_2([a,b], H)$. If $w : [a,b] \to (0,\infty)$ is integrable with $\int_a^b w(s)\, ds = 1$, then*

$$\begin{aligned}\int_a^b \|f(t)\|^2 w(t)\, dt &\le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2} w(t)\, dt \int_a^b \|f'(s)\|^2 ds \\ &\le \frac{1}{4}(b-a)\int_a^b \|f'(s)\|^2 ds.\end{aligned} \tag{2.3}$$

We have the following operator inequalities:

Theorem 5. *Assume that* $A:[a,b]\to\mathcal{B}(H)$ *is strongly differentiable on* (a,b) *with* $A(a)=A(b)=0$ *and* $A'\in L_2([a,b],\mathcal{B}(H))$. *Then*

$$\begin{aligned}
&\int_a^b |A(t)|^2\,du(t) \\
&\le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t)\int_a^b |A'(s)|^2\,ds \\
&= \frac{1}{2}\int_a^b (t-a)^{-1/2}(b-t)^{-1/2}\left(t-\frac{a+b}{2}\right)u(t)\,dt\int_a^b |A'(s)|^2\,ds \\
&\le \frac{1}{4}(b-a)[u(b)-u(a)]\int_a^b |A'(s)|^2\,ds
\end{aligned}\tag{2.4}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $x\in H$, $x\neq 0$ and take $f(t)=A(t)x$ in (2.1) to get

$$\begin{aligned}
&\int_a^b \|A(t)x\|^2\,du(t) \\
&\le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t)\int_a^b \|A'(s)x\|^2\,ds \\
&= \frac{1}{2}\int_a^b (t-a)^{-1/2}(b-t)^{-1/2}\left(t-\frac{a+b}{2}\right)u(t)\,dt\int_a^b \|A'(s)x\|^2\,ds \\
&\le \frac{1}{4}(b-a)[u(b)-u(a)]\int_a^b \|A'(s)x\|^2\,ds.
\end{aligned}\tag{2.5}$$

Observe that

$$\begin{aligned}
\int_a^b \|A(t)x\|^2\,du(t) &= \int_a^b \langle A(t)x, A(t)x\rangle\,du(t) = \int_a^b \left\langle (A(t))^* A(t)x, x\right\rangle du(t) \\
&= \int_a^b \left\langle |A(t)|^2 x, x\right\rangle du(t) = \left\langle \left(\int_a^b |A(t)|^2\,du(t)\right)x, x\right\rangle
\end{aligned}$$

and

$$\int_a^b \|A'(s)x\|^2\,ds = \left\langle \left(\int_a^b |A'(s)|^2\,ds\right)x, x\right\rangle$$

for $x\in H$, $x\neq 0$.

Then by (2.5) we get

$$\begin{aligned}
&\left\langle \left(\int_a^b |A(t)|^2\,du(t)\right)x, x\right\rangle \\
&\le \left\langle \left(\frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t)\int_a^b |A'(s)|^2\,ds\right)x, x\right\rangle \\
&= \left\langle \left(\frac{1}{2}\int_a^b (t-a)^{-1/2}(b-t)^{-1/2}\left(t-\frac{a+b}{2}\right)u(t)\,dt\int_a^b |A'(s)|^2\,ds\right)x, x\right\rangle \\
&\le \left\langle \left(\frac{1}{4}(b-a)[u(b)-u(a)]\int_a^b |A'(s)|^2\,ds\right)x, x\right\rangle,
\end{aligned}$$

for $x \in H$, $x \neq 0$, which is equivalent to (2.4). □

Corollary 3. *Assume that* $A : [a,b] \to \mathcal{B}(H)$ *is strongly differentiable on* (a,b) *with* $A(a) = A(b) = 0$ *and* $A' \in L_2([a,b], \mathcal{B}(H))$. *Then*

$$\left| A\left(\frac{a+b}{2}\right)\right|^2 \leq \frac{1}{4}(b-a)\int_a^b |f'(s)|^2 \, ds. \tag{2.6}$$

The proof is like the one for the Corollary 1.

Corollary 4. *Assume that* $A : [a,b] \to \mathcal{B}(H)$ *is strongly differentiable on* (a,b) *with* $A(a) = A(b) = 0$ *and* $A' \in L_2([a,b], \mathcal{B}(H))$. *If* $w : [a,b] \to (0,\infty)$ *is integrable with* $\int_a^b w(s)\,ds = 1$, *then*

$$\begin{aligned}\int_a^b |A(t)|^2 w(t)\,dt &\leq \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2} w(t)\,dt \int_a^b |A'(s)|^2 \, ds \\ &\leq \frac{1}{4}(b-a)\int_a^b |A'(s)|^2 \, ds.\end{aligned} \tag{2.7}$$

3. Applications for Trapezoid and Grüss' Type Inequalities

We have the following generalized trapezoid type norm inequality:

Proposition 1. *Assume that* $h : [a,b] \to H$ *is absolutely continuous and* $h' \in L_2([a,b], H)$. *If* $u : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}&\Bigg\| \int_a^b h(t)\,du(t) \\ &- \left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] h(b) - \left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] h(a)\Bigg\|^2 \\ &\leq \frac{1}{2}[u(b)-u(a)]\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t) \\ &\times \int_a^b \left\| h'(t) - \frac{h(b)-h(a)}{b-a}\right\|^2 ds \\ &\leq \frac{1}{4}(b-a)[u(b)-u(a)]^2 \int_a^b \left\| h'(t) - \frac{h(b)-h(a)}{b-a}\right\|^2 ds.\end{aligned} \tag{3.1}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_a^b \left[h(t) - \frac{(b-t)h(a) + (t-a)h(b)}{b-a}\right] du(t) \\ &= \int_a^b h(t)\,du(t) - \int_a^b \frac{(b-th(a)) + (t-ah(b))}{b-a} du(t)\end{aligned}$$

$$= \int_a^b h(t)\,du(t) - u(t)\,\frac{(b-t)\,h(a) + (t-a)\,h(b)}{b-a}\bigg|_a^b$$
$$+ \frac{h(b)-h(a)}{b-a}\int_a^b u(t)\,dt$$
$$= \int_a^b h(t)\,du(t) - u(b)\,h(b) + u(a)\,h(a) + \left(\int_a^b u(t)\,dt\right)\frac{h(b)-h(a)}{b-a}$$
$$= \int_a^b h(t)\,du(t)$$
$$- \left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,dt\right] h(b) - \left[\frac{1}{b-a}\int_a^b u(t)\,dt - u(a)\right] h(a).$$

If we choose

$$f(t) = h(t) - \frac{(b-t)\,h(a) + (t-a)\,h(b)}{b-a},\ t \in [a,b]$$

then we have $f(a) = f(b) = 0$,

$$f'(t) = h'(t) - \frac{h(b)-h(a)}{b-a},\ t \in (a,b)$$

and by (2.1) we get

$$\int_a^b \left\| h(t) - \frac{(b-t)\,h(a) + (t-a)\,h(b)}{b-a}\right\|^2 du(t) \tag{3.2}$$
$$\leq \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t)\int_a^b \left\| h'(t) - \frac{h(b)-h(a)}{b-a}\right\|^2 ds$$
$$\leq \frac{1}{4}(b-a)\,[u(b)-u(a)]\int_a^b \left\| h'(t) - \frac{h(b)-h(a)}{b-a}\right\|^2 ds.$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators, we have

$$[u(b)-u(a)]\int_a^b \left\| h(t) - \frac{(b-t)\,h(a) + (t-a)\,h(b)}{b-a}\right\|^2 du(t) \tag{3.3}$$
$$\geq \left\| \int_a^b \left(h(t) - \frac{(b-t)\,h(a) + (t-a)\,h(b)}{b-a}\right) du(t)\right\|^2.$$

By making use of (3.2) and (3.3) we derive the desired result (3.1). □

Corollary 5. *Assume that* $h : [a,b] \to H$ *is absolutely continuous and* $h' \in L_2([a,b], H)$. *If* $w : [a,b] \to (0,\infty)$ *is integrable with* $\int_a^b w(s)\,ds = 1$, *then*

$$\left\| \int_a^b w(t)\,h(t)\,dt - \frac{(b - E(w,[a,b]))\,f(a) + (E(w,[a,b]) - a)\,f(b)}{b-a}\right\|^2 \tag{3.4}$$
$$\leq \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,w(t)\,dt\int_a^b \left\| h'(t) - \frac{h(b)-h(a)}{b-a}\right\|^2 dt$$
$$\leq \frac{1}{4}(b-a)\int_a^b \left\| h'(t) - \frac{h(b)-h(a)}{b-a}\right\|^2 ds,$$

where $E(w,[a,b]) := \int_a^b tw(t)\,dt$.

Proposition 2. *Assume that* $h:[a,b]\to H$ *is absolutely continuous and* $h' \in L_2([a,b],H)$. *Then*

$$\left\| h\left(\frac{a+b}{2}\right) - \frac{h(a)+h(b)}{2}\right\|^2 \le \frac{1}{8}(b-a)\int_a^b \|h'(t)-h'(a+b-t)\|^2\,ds \tag{3.5}$$

and

$$\left\| h\left(\frac{a+b}{2}\right) - \frac{h(a)+h(b)}{2}\right\|^2 \le \frac{1}{4}(b-a)\int_a^b \left\| h'(s) - \frac{h(b)-h(a)}{b-a}\right\|^2 ds. \tag{3.6}$$

Proof. We use (2.2) for the function

$$f(t) := \frac{h(t)+h(a+b-t)}{2} - \frac{h(a)+h(b)}{2},$$

which is absolutely continuous with $f(a)=f(b)=0$ and

$$f'(t) := \frac{h'(t)-h'(a+b-t)}{2}$$

to get (3.5).

Also if we use (2.2) for the function

$$f(t) = h(t) - \frac{(b-t)h(a)+(t-a)h(b)}{b-a},$$

which is absolutely continuous with $f(a)=f(b)=0$ and

$$f'(t) = h'(t) - \frac{h(b)-h(a)}{b-a}$$

to get (3.6). □

We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$\begin{aligned} 0 &\le \left|\overline{\alpha(t)}A(s) - \overline{\alpha(s)}A(t)\right|^2 \\ &= |\alpha(t)|\,|A(s)|^2 - \alpha(s)\overline{\alpha(t)}A^*(t)A(s) \\ &\quad - \alpha(t)\overline{\alpha(s)}A^*(s)A(t) + |\alpha(s)|^2|A(t)|^2, \end{aligned}$$

which gives that

$$\begin{aligned} &|\alpha(t)|^2|A(s)|^2 + |\alpha(s)|^2|A(t)|^2 \\ &\ge \alpha(s)\overline{\alpha(t)}A^*(t)A(s) + \alpha(t)\overline{\alpha(s)}A^*(s)A(t) \end{aligned}$$

for all $s,\ t\in[a,b]$.

Consider also $u:[a,b]\to[0,\infty)$ monotonic nondecreasing and assume that $A:[a,b]\to\mathcal{B}(H)$ is continuous.

Integrating over $du(t)$ and $du(s)$ on $[a,b]$, then we get

$$\begin{aligned}&\int_a^b |\alpha(t)|^2 du(t) \int_a^b |A(s)|^2 du(s) + \int_a^b |\alpha(s)|^2 du(s) \int_a^b |A(t)|^2 du(t) \\ &\geq \int_a^b \overline{\alpha(t)} A^*(t) du(t) \int_a^b \alpha(s) A(s) du(s) \\ &+ \int_a^b \overline{\alpha(s)} A^*(s) du(s) \int_a^b \alpha(t) A(t) du(t) \\ &= 2\left|\int_a^b \alpha(s) A(s) du(s)\right|^2,\end{aligned}$$

which proves that

$$\int_a^b |\alpha(t)|^2 du(t) \int_a^b |A(t)|^2 du(t) \geq \left|\int_a^b \alpha(t) A(t) du(t)\right|^2, \tag{3.7}$$

provided that α and A are continuous.

For $\alpha(t) = 1$, we then get

$$[u(b) - u(a)] \int_a^b |A(t)|^2 du(t) \geq \left|\int_a^b A(t) du(t)\right|^2. \tag{3.8}$$

In the case of Lebesgue and Bochner integrals, the inequality (3.7) also holds for $\alpha \in L_2[a,b]$ and $A \in L_2([a,b], \mathcal{B}(H))$.

We have:

Proposition 3. *Assume that $B : [a,b] \to H$ is strongly differentiable and $B' \in L_2([a,b], H)$. If $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, then*

$$\begin{aligned}&\Bigg|\int_a^b B(t) du(t) \\ &- \left[u(b) - \frac{1}{b-a}\int_a^b u(t) dt\right] B(b) - \left[\frac{1}{b-a}\int_a^b u(t) dt - u(a)\right] B(a)\Bigg|^2 \\ &\leq \frac{1}{2}[u(b) - u(a)] \int_a^b (t-a)^{1/2} (b-t)^{1/2} du(t) \\ &\times \int_a^b \left|B'(t) - \frac{B(b) - B(a)}{b-a}\right|^2 ds \\ &\leq \frac{1}{4}(b-a)[u(b) - u(a)]^2 \int_a^b \left|B'(t) - \frac{B(b) - B(a)}{b-a}\right|^2 ds.\end{aligned} \tag{3.9}$$

Proof. As in the proof of Proposition 1, we have

$$\begin{aligned}
&\int_a^b \left[B(t) - \frac{(b-t)B(a) + (t-a)B(b)}{b-a} \right] du(t) \\
&= \int_a^b B(t)\, du(t) \\
&- \left[u(b) - \frac{1}{b-a}\int_a^b u(t)\, dt \right] B(b) - \left[\frac{1}{b-a}\int_a^b u(t)\, dt - u(a) \right] B(a).
\end{aligned}$$

If we choose

$$A(t) = B(t) - \frac{(b-t)B(a) + (t-a)B(b)}{b-a},\ t \in [a,b],$$

then we have $f(a) = f(b) = 0$,

$$A'(t) = B'(t) - \frac{B(b) - B(a)}{b-a},\ t \in (a,b)$$

and by (2.3) we get

$$\begin{aligned}
&\int_a^b \left| B(t) - \frac{(b-t)B(a) + (t-a)B(b)}{b-a} \right|^2 du(t) \\
&\le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2} du(t) \int_a^b \left| B'(t) - \frac{B(b) - B(a)}{b-a} \right|^2 ds \\
&\le \frac{1}{4}(b-a)[u(b) - u(a)] \int_a^b \left| B'(t) - \frac{B(b) - B(a)}{b-a} \right|^2 ds.
\end{aligned} \tag{3.10}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality for the Riemann-Stieltjes integral with monotonic nondecreasing integrators (3.8), we have

$$\begin{aligned}
&[u(b) - u(a)] \int_a^b \left| B(t) - \frac{(b-t)B(a) + (t-a)B(b)}{b-a} \right|^2 du(t) \\
&\ge \left| \int_a^b \left(B(t) - \frac{(b-t)B(a) + (t-a)B(b)}{b-a} \right) du(t) \right|^2.
\end{aligned} \tag{3.11}$$

By making use of (3.10) and (3.11) we derive the desired result (3.9). □

Corollary 6. *Assume that $B : [a,b] \to H$ is strongly differentiable and $B' \in L_2([a,b], H)$. If $w : [a,b] \to (0,\infty)$ is integrable with $\int_a^b w(s)\, ds = 1$, then*

$$\begin{aligned}
&\left| \int_a^b w(t) B(t)\, dt - \frac{(b - E(w,[a,b]))B(a) + (E(w,[a,b]) - a)B(b)}{b-a} \right|^2 \\
&\le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2} w(t)\, dt \int_a^b \left| B'(t) - \frac{B(b) - B(a)}{b-a} \right|^2 dt \\
&\le \frac{1}{4}(b-a) \int_a^b \left| B'(t) - \frac{B(b) - B(a)}{b-a} \right|^2 ds.
\end{aligned} \tag{3.12}$$

We have:

Proposition 4. *Assume that* $B : [a,b] \to H$ *is strongly differentiable and* $B' \in L_2([a,b], H)$. *Then*

$$\left| B\left(\frac{a+b}{2}\right) - \frac{B(a)+B(b)}{2}\right|^2 \le \frac{1}{8}(b-a)\int_a^b |B'(t) - B'(a+b-t)|^2 ds \tag{3.13}$$

and

$$\left| B\left(\frac{a+b}{2}\right) - \frac{B(a)+B(b)}{2}\right|^2 \le \frac{1}{4}(b-a)\int_a^b \left|B'(t) - \frac{B(b)-B(a)}{b-a}\right|^2 ds. \tag{3.14}$$

The proof is similar to the one from Proposition 2.

For two *Lebesgue integrable* functions f, $g : [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt. \tag{3.15}$$

In 1935, Grüss [14] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{3.16}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]. \tag{3.17}$$

The constant $\frac{1}{4}$ is best possible in (3.15) in the sense that it cannot be replaced by a smaller quantity.

Theorem 6. *Assume that* $h : [a,b] \to H$ *is absolutely continuous and* $h' \in L_2([a,b], H)$. *If* $u : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned} \|C(h,u)\|^2 &\le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2} du(t) \\ &\quad \times \left(\frac{1}{b-a}\int_a^b \|h(t)\|^2 dt - \left\|\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2\right) \\ &\le \frac{1}{4}[u(b) - u(a)] \\ &\quad \times \left(\frac{1}{b-a}\int_a^b \|h(t)\|^2 dt - \left\|\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2\right). \end{aligned} \tag{3.18}$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned} &\int_a^b \left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds\right) du(t) \\ &= \left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds\right) u(t)\Big|_a^b \\ &- \int_a^b u(t)\,d\left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds\right) \\ &= -\int_a^b u(t)\,h(t)\,dt + \frac{1}{b-a}\int_a^b h(s)\,ds\int_a^b u(t)\,dt, \end{aligned}$$

which gives that

$$C(h,u)=\frac{1}{b-a}\int_a^b\left(\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right)du(t)\,. \tag{3.19}$$

Consider

$$g(t):=\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds,\ \ t\in[a,b]\,,$$

then g is absolutely continuous, $g(a)=g(b)=0$,

$$g'(t):=\frac{1}{b-a}\int_a^b h(s)\,ds-h(t)\,,\ \ t\in[a,b]$$

and by (2.1) we get

$$\begin{aligned}&\int_a^b\left\|\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right\|^2 du(t) \\ &\le\frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t)\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt \\ &\le\frac{1}{4}(b-a)\,[u(b)-u(a)]\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt.\end{aligned} \tag{3.20}$$

Using (CBS) inequality, we have

$$\begin{aligned}&[u(b)-u(a)]\int_a^b\left\|\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right\|^2 du(t) \\ &\ge\left\|\int_a^b\left(\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right)du(t)\right\|^2 \\ &=(b-a)^2\,\|C(h,u)\|^2\,.\end{aligned} \tag{3.21}$$

By (3.20) and (3.21) we get

$$\begin{aligned}&(b-a)^2\,\|C(h,u)\|^2 \\ &\le\frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t)\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt \\ &\le\frac{1}{4}(b-a)\,[u(b)-u(a)]\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt,\end{aligned}$$

which gives

$$\begin{aligned}&\|C(h,u)\|^2\le\frac{1}{2}\frac{1}{b-a}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t) \\ &\times\frac{1}{b-a}\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt \\ &\le\frac{1}{4}\,[u(b)-u(a)]\,\frac{1}{b-a}\int_a^b\left\|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right\|^2 dt.\end{aligned}$$

Since, by the properties of inner product

$$\frac{1}{b-a}\int_a^b \left\| h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds \right\|^2 dt$$
$$= \frac{1}{b-a}\int_a^b \|h(t)\|^2 dt - \left\| \frac{1}{b-a}\int_a^b h(s)\,ds \right\|^2,$$

the theorem is thus proved. □

Finally, we can prove in a similar way the operator inequality:

Theorem 7. *Assume that* $A : [a,b] \to H$ *belongs to* $L_2([a,b], H)$. *If* $u : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$|C(A,u)|^2 \le \frac{1}{2}\int_a^b (t-a)^{1/2}(b-t)^{1/2}\,du(t) \times \left(\frac{1}{b-a}\int_a^b |A(t)|^2 dt - \left| \frac{1}{b-a}\int_a^b A(s)\,ds \right|^2 \right)$$
$$\le \frac{1}{4}[u(b)-u(a)] \times \left(\frac{1}{b-a}\int_a^b |A(t)|^2 dt - \left| \frac{1}{b-a}\int_a^b A(s)\,ds \right|^2 \right). \tag{3.22}$$

References

[1] M. W. Alomari, On Beesack-Wirtinger Inequality, *Results Math.*, Online First 2017 Springer International Publishing DOI 10.1007/s00025-016-0644-6.

[2] T. M. Apostol, *Mathematical Analysis,* Addison-Wesley Publishing Company, Second Edition, 1981.

[3] P. R. Beesack, Extensions of Wirtinger's inequality. *Trans. R. Soc. Can.* **53**, 21–30 (1959)

[4] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.*, **38**(1) (2007), 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Article 14. [Online: `http://rgmia.vu.edu.au/v5n2.html`].

[5] J. B. Diaz and F. T. Metcalf, Variations on Wirtinger's inequality, in: *Inequalities* Academic Press, New York, 1967, pp. 79–103.

[6] S. S. Dragomir, Accurate approximations for the Riemann-Stieltjes integral via theory of inequalities. *J. Math. Inequal.* **3** (2009), no. 4, 663–681

[7] S. S. Dragomir, Approximating the Riemann-Stieltjes integral via a Chebyshev type functional. *Acta Comment. Univ. Tartu. Math.* **18** (2014), No. 2, 239–259. Preprint *RGMIA Res. Rep. Coll.* **10** (2007), Suplement Art. 18 [Online `http://rgmia.org/papers/v10e/ARSICTF.pdf`].

[8] S. S. Dragomir, Integral inequalities related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 59, 16 pp. [Online `https://rgmia.org/papers/v21/v21a59.pdf`].

[9] S. S. Dragomir, Integral inequalities for functions with values in Hilbert spaces related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art. 45, 19 pp. [Online `https://rgmia.org/papers/v24/v24a45.pdf`].

[10] S. S. Dragomir, Absolute value integral inequalities for functions with operator values in Hilbert spaces related to Wirtinger's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art. 47, 15 pp. [Online `https://rgmia.org/papers/v24/v24a47.pdf`].

[11] S. S. Dragomir, Riemann-Stieltjes integral inequalities of Wirtinger type for vector and operator valued functions in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art. ,

[12] L. Fejér, Über die Fourierreihen, II, (In Hungarian) *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[13] R. Giova, An estimate for the best constant in the L_p-Wirtinger inequality with weights, *J. Func. Spaces Appl.*,Volume **6**, Number 1 (2008), 1-16.

[14] G. Grüss, Über das Maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b g(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b g(x)dx\int_a^b g(x)dx$, *Math. Z.* , **39**(1935), 215-226.

[15] J. Jaroš, On an integral inequality of the Wirtinger type, *Appl. Math. Letters,* **24** (2011) 1389–1392.

[16] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[17] C. F. Lee, C. C. Yeh, C. H. Hong and R. P. Agarwal, Lyapunov and Wirtinger inequalities, *Appl. Math. Lett.* **17** (2004) 847–853.

[18] C. A. Swanson, Wirtinger's inequality, *SIAM J. Math. Anal.* **9** (1978) 484–491.

[19] T. Ricciardi, A sharp weighted Wirtinger inequality, *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat.* (8), **8** (1) (2005), 259–267.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa