GRÜSS' TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if $\alpha \in L_w^2(\Omega, \mu, \mathbb{C})$, $B \in L_2(\Omega, \mu, \mathcal{B}(H))$ with $w \geq 0$ and $\int_\Omega w(s) = 1$ and $X, Y \in H$ with $X \neq Y$ so that

$$\left| B(s) - \frac{X+Y}{2} \right|^2 \leq \frac{1}{4} |Y-X|^2 \text{ for } \mu\text{-a.e. } s \in \Omega,$$

or, equivalently,

$$\operatorname{Re}\left[\left(B^*(s) - Y^* \right) (X - B(s)) \right] \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega,$$

then

$$\left| \int_\Omega w(s) \alpha(s) B(s) d\mu(s) - \int_\Omega w(s) \alpha(s) d\mu(s) \int_\Omega w(s) B(s) d\mu(s) \right|^2$$
$$\leq \frac{1}{4} \left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) - \left| \int_\Omega w(s) \alpha(s) d\mu(s) \right|^2 \right) |Y-X|^2.$$

Applications for finite Fourier Transform are also given.

1. Introduction

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{(b-a)^2} \int_a^b f(t) dt \int_a^b g(t) dt.$$

In 1934, G. Grüss [11] showed that

$$|D(f,g)| \leq \frac{1}{4} (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by the author in [2]:

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Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e\in H$, $\|e\|=1$. If φ, ϕ, γ, $\Gamma\in\mathbb{K}$ and x, $y\in H$ are such that*

$$\operatorname{Re}\langle\phi e-x,x-\varphi e\rangle\geq 0 \quad \text{and} \quad \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.3}$$

or, equivalently (see [4])

$$\left\|x-\frac{\varphi+\phi}{2}e\right\|\leq\frac{1}{2}|\phi-\varphi| \quad \text{and} \quad \left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma|, \tag{1.4}$$

then

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\phi-\varphi||\Gamma-\gamma|. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in (1.5).

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by the author in 2001, [3].

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space, $\Omega\subset\mathbb{R}^n$ be a Lebesgue measurable set and $\rho:\Omega\to[0,\infty)$ a Lebesgue measurable function with $\int_\Omega\rho(s)\,ds=1$. We denote by $L_{2,\rho}(\Omega,H)$ the set of all Bochner measurable functions f on Ω such that $\|f\|_{2,\rho}^2:=\int_\Omega\rho(s)\|f(s)\|^2\,ds<\infty$. If f, g belong to $L_{2,\rho}(\Omega,H)$ and there exist the vectors x, X, y, $Y\in H$ such that*

$$\begin{aligned}\int_\Omega\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt&\geq 0,\\ \int_\Omega\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt&\geq 0,\end{aligned} \tag{1.6}$$

then we have the inequality

$$\begin{aligned}&\left|\int_\Omega\rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega\rho(t)f(t)\,dt,\int_\Omega\rho(t)g(t)\,dt\right\rangle\right|\\ &\leq\frac{1}{4}\|X-x\|\|Y-y\|.\end{aligned} \tag{1.7}$$

The constant $\frac{1}{4}$ is sharp in the sense mentioned above.

Remark 1. *A practical sufficient condition for (1.6) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle\geq 0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle\geq 0$$

or, equivalently

$$\left\|f(t)-\frac{X+x}{2}\right\|\leq\frac{1}{2}\|X-x\| \quad \text{and} \quad \left\|g(t)-\frac{Y+y}{2}\right\|\leq\frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$.

For related results, see [1], [4]-[10] and [12]-[13].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

In this paper we show among others that, if $\alpha \in L^2_w(\Omega,\mu,\mathbb{C})$, $B \in L_2(\Omega,\mu,\mathcal{B}(H))$ with $w \geq 0$ and $\int_\Omega w(s) = 1$ and $X, Y \in H$ with $X \neq Y$ so that

$$\left| B(s) - \frac{X+Y}{2} \right|^2 \leq \frac{1}{4} |Y-X|^2 \text{ for } \mu\text{-a.e. } s \in \Omega,$$

or, equivalently,

$$\operatorname{Re}\left[\left(B^*(s) - Y^*\right)(X - B(s)) \right] \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega,$$

then

$$\begin{aligned} &\left| \int_\Omega w(s)\alpha(s) B(s) d\mu(s) - \int_\Omega w(s)\alpha(s) d\mu(s) \int_\Omega w(s) B(s) d\mu(s) \right|^2 \\ &\leq \frac{1}{4} \left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) - \left| \int_\Omega w(s)\alpha(s) d\mu(s) \right|^2 \right) |Y-X|^2 . \end{aligned}$$

Applications for finite Fourier Transform are also given.

2. Preliminary Facts

We have the following Cauchy-Bunyakowsky-Schwarz inequality for the operator modulus:

Lemma 1. *If* $\alpha \in L^2_w(\Omega,\mu,\mathbb{C})$ *and*

$$A \in L_{2,w}(\Omega,\mu,\mathcal{B}(H)) := \left\{ A : \Omega \to B(H), \ \int_\Omega w(s) \|A(s)\|^2 d\mu(s) < \infty \right\},$$

then

$$\begin{aligned} &\left| \int_\Omega w(s)\alpha(s) A(s) d\mu(s) \right|^2 \\ &\leq \int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \int_\Omega w(s) |A(s)|^2 d\mu(s) \end{aligned} \tag{2.1}$$

in the operator order of $\mathcal{B}(H)$.

Proof. We have for $\alpha \in L^2_w(\Omega,\mu,\mathbb{C})$ and $A \in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$,

$$\begin{aligned} 0 \leq \left| \overline{\alpha(s)} A(t) - \overline{\alpha(t)} A(s) \right|^2 &= |\alpha(s)| |A(t)|^2 - \alpha(t)\overline{\alpha(s)} A^*(s) A(t) \\ &- \alpha(s)\overline{\alpha(t)} A^*(t) A(s) + |\alpha(t)|^2 |A(s)|^2 , \end{aligned}$$

which gives that

$$\begin{aligned} &|\alpha(s)|^2 |A(t)|^2 + |\alpha(t)|^2 |A(s)|^2 \\ &\geq \alpha(t)\overline{\alpha(s)} A^*(s) A(t) + \alpha(s)\overline{\alpha(t)} A^*(t) A(s) \end{aligned}$$

for all $t, s \in \Omega$.

Now, multiply this with $w(t) w(s) \geq 0$ to get

$$\begin{aligned} &w(s) |\alpha(s)|^2 w(t) |A(t)|^2 + w(t) |\alpha(t)|^2 w(s) |A(s)|^2 \\ &\geq w(s)\overline{\alpha(s)} A^*(s) w(t)\alpha(t) A(t) + w(t)\overline{\alpha(t)} A^*(t) w(s)\alpha(s) A(s) \end{aligned}$$

for all $t, s \in \Omega$.

Integrating over s and t on Ω, then we get

$$\begin{aligned}
&\int_a^b w(s)|\alpha(s)|^2 d\mu(s) \int_a^b |A(t)|^2 d\mu(t) \\
&+ \int_a^b |\alpha(t)|^2 d\mu(t) \int_a^b w(s)|A(s)|^2 d\mu(s) \\
&\geq \int_a^b w(s)\overline{\alpha(s)}A^*(s) d\mu(s) \int_a^b \alpha(t) A(t) d\mu(t) \\
&+ \int_a^b w(t)\overline{\alpha(t)}A^*(t) d\mu(t) \int_a^b \alpha(s) A(s) d\mu(s) \\
&= 2\left|\int_a^b w(t)\alpha(t) A(t) d\mu(t)\right|^2,
\end{aligned}$$

and the inequality (2.1) is obtained. □

We recall Löwner-Heinz inequality which says that, if $0 \leq A \leq B$, then for all $p \in (0,1)$ we have $0 \leq A^p \leq B^p$. By using this property, we can state the following result as well:

Corollary 1. *With the assumptions of Lemma 2, we have the inequality*

$$\begin{aligned}
&\left|\int_\Omega w(s)\alpha(s) A(s) d\mu(s)\right| \\
&\leq \left(\int_\Omega w(s)|\alpha(s)|^2 d\mu(s)\right)^{1/2} \left(\int_\Omega w(s)|A(s)|^2 d\mu(s)\right)^{1/2},
\end{aligned} \tag{2.2}$$

in the operator order of $\mathcal{B}(H)$.

The proof follows by (2.1) by taking the operator square root.

Remark 2. *We remark that, if α is real valued and $A(s)$, $s \in \Omega$ are selfadjoint operators, then we have*

$$\begin{aligned}
&\left|\int_\Omega w(s)\alpha(s) A(s) d\mu(s)\right| \\
&\leq \left(\int_\Omega w(s)\alpha^2(s) d\mu(s)\right)^{1/2} \left(\int_\Omega w(s) A^2(s) d\mu(s)\right)^{1/2}.
\end{aligned} \tag{2.3}$$

We have the following lemma that is of interest in itself:

Lemma 2. *Assume that $f \in L_{2,w}(\Omega,\mu,H)$, then for all $v \in H$,*

$$\begin{aligned}
0 &\leq \int_\Omega w(s)\|f(s)\|^2 d\mu(s) - \left\|\int_\Omega w(s) f(s) d\mu(s)\right\|^2 \\
&\leq \int_\Omega w(s)\|f(s)-v\|^2 d\mu(s).
\end{aligned} \tag{2.4}$$

Proof. Observe that, for any $v \in H$

$$
\begin{aligned}
(2.5)\qquad 0 &\leq \int_{\Omega} w(s)\|f(s)\|^2 d\mu(s) - \left\|\int_{\Omega} w(s) f(s)\, d\mu(s)\right\|^2 \\
&= \int_{\Omega} w(s)\langle f(s), f(s)\rangle\, d\mu(s) \\
&\quad - \left\langle \int_{\Omega} w(s) f(s)\, d\mu(s), \int_{\Omega} w(s) f(s)\, d\mu(s)\right\rangle \\
&= \int_{\Omega} w(s)\left\langle f(s) - \int_{\Omega} w(u) f(u)\, d\mu(u), f(s) - v\right\rangle d\mu(s) =: K.
\end{aligned}
$$

Therefore, by Schwarz inequality in Hilbert spaces and the CBS integral inequality, we have

$$
\begin{aligned}
(2.6)\qquad K &\leq \int_{\Omega} w(s)\left|\left\langle f(s) - \int_{\Omega} w(u) f(u)\, d\mu(u), f(s) - v\right\rangle\right| d\mu(s) \\
&\leq \int_{\Omega} w(s)\left\|f(s) - \int_{\Omega} w(u) f(u)\, d\mu(u)\right\| \|f(s) - v\|\, d\mu(s) \\
&\leq \left(\int_{\Omega} w(s)\left\|f(s) - \int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2 d\mu(s)\right)^{1/2} \\
&\quad \times \left(\int_{\Omega} w(s)\|f(s) - v\|^2 d\mu(s)\right)^{1/2}.
\end{aligned}
$$

Since, by the properties of inner product and integral,

$$
\begin{aligned}
&\int_{\Omega} w(s)\left\|f(s) - \int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2 d\mu(s) \\
&= \int_{\Omega} w(s)\left[\|f(s)\|^2 - 2\operatorname{Re}\left\langle f(s), \int_{\Omega} w(u) f(u)\, d\mu(u)\right\rangle \right.\\
&\quad \left. + \left\|\int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2\right] d\mu(s) \\
&\quad = \int_{\Omega} w(s)\|f(s)\|^2 d\mu(s) \\
&\quad - 2\operatorname{Re}\left\langle \int_{\Omega} w(s) f(s)\, d\mu(u), \int_{\Omega} w(u) f(u)\, d\mu(u)\right\rangle \\
&\quad + \left\|\int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2 \\
&\quad = \int_{\Omega} w(s)\|f(s)\|^2 d\mu(s) - 2\left\|\int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2 \\
&\quad + \left\|\int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2 \\
&\quad = \int_{\Omega} w(s)\|f(s)\|^2 d\mu(s) - \left\|\int_{\Omega} w(u) f(u)\, d\mu(u)\right\|^2,
\end{aligned}
$$

hence by (2.5) and (2.6) we get

$$\begin{aligned}
0 &\leq \int_{\Omega} w(s)\|f(s)\|^{2} d\mu(s)-\left\|\int_{\Omega} w(s) f(s) d\mu(s)\right\|^{2} \\
&\leq\left(\int_{\Omega} w(s)\|f(s)\|^{2} d\mu(s)-\left\|\int_{\Omega} w(u) f(u) d\mu(u)\right\|^{2}\right)^{1/2} \\
&\times\left(\int_{\Omega} w(s)\|f(s)-v\|^{2} d\mu(s)\right)^{1/2},
\end{aligned}$$

which is equivalent to (2.4). □

Lemma 3. *For $x,\ y,\ z \in H$ we have the equality*

$$\left\|z-\frac{x+y}{2}\right\|^{2}-\frac{1}{4}\|y-x\|^{2}=\operatorname{Re}\langle z-x, z-y\rangle . \tag{2.7}$$

Proof. We have

$$\begin{aligned}
&\left\|z-\frac{x+y}{2}\right\|^{2}-\frac{1}{4}\|y-x\|^{2} \\
&=\|z\|^{2}-2 \operatorname{Re}\left\langle z, \frac{x+y}{2}\right\rangle+\frac{1}{4}\|x+y\|^{2}-\frac{1}{4}\|y-x\|^{2} \\
&=\|z\|^{2}-\operatorname{Re}\langle z, x\rangle-\operatorname{Re}\langle z, y\rangle+\frac{1}{4}\left(\|x\|^{2}+2 \operatorname{Re}\langle x, y\rangle+\|y\|^{2}\right) \\
&-\frac{1}{4}\left(\|x\|^{2}-2 \operatorname{Re}\langle x, y\rangle+\|y\|^{2}\right) \\
&=\|z\|^{2}-\operatorname{Re}\langle z, x\rangle-\operatorname{Re}\langle z, y\rangle+\operatorname{Re}\langle x, y\rangle .
\end{aligned}$$

Also

$$\begin{aligned}
\operatorname{Re}\langle z-x, z-y\rangle &=\operatorname{Re}\left[\|z\|^{2}-\langle x, z\rangle-\langle z, y\rangle+\langle x, y\rangle\right] \\
&=\|z\|^{2}-\operatorname{Re}\langle x, z\rangle-\operatorname{Re}\langle z, y\rangle+\operatorname{Re}\langle x, y\rangle \\
&=\|z\|^{2}-\operatorname{Re}\overline{\langle z, x\rangle}-\operatorname{Re}\langle z, y\rangle+\operatorname{Re}\langle x, y\rangle \\
&=\|z\|^{2}-\operatorname{Re}\langle z, x\rangle-\operatorname{Re}\langle z, y\rangle+\operatorname{Re}\langle x, y\rangle,
\end{aligned}$$

which proves the equality (2.7). □

Lemma 4. *Let $f \in L_2(\Omega, \mu, H)$ and $x,\ y \in H$ with $x \neq y$. If*

$$\left\|f(s)-\frac{x+y}{2}\right\|^{2} \leq \frac{1}{4}\|y-x\|^{2} \text{ for } \mu\text{-a.e. } s \in \Omega, \tag{2.8}$$

or, equivalently,

$$\operatorname{Re}\langle x-f(s), f(s)-y\rangle \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega, \tag{2.9}$$

then

$$0 \leq \int_{\Omega} w(s)\|f(s)\|^{2} d\mu(s)-\left\|\int_{\Omega} w(s) f(s) d\mu(s)\right\|^{2} \leq \frac{1}{4}\|y-x\|^{2} . \tag{2.10}$$

The constant $\frac{1}{4}$ is best possible.

Proof. The equivalence of the statements (2.8) and (2.9) follows by Lemma 3.

Now, if we use the inequality (2.4) for $v = \frac{x+y}{2}$, then we get

$$\begin{aligned} 0 &\leq \int_\Omega w(s)\|f(s)\|^2 d\mu(s) - \left\|\int_\Omega w(s) f(s) d\mu(s)\right\|^2 \\ &\leq \int_\Omega w(s)\left\|f(s) - \frac{x+y}{2}\right\|^2 d\mu(s) \leq \frac{1}{4}\|y-x\|^2 \int_\Omega w(s) d\mu(s) \\ &= \frac{1}{4}\|y-x\|^2, \end{aligned}$$

which proves (2.10). □

Remark 3. *The inequality (2.10) was proved firstly in* [3] *in a different way and it was also shown that the constant* $\frac{1}{4}$ *is best possible.*

3. Main Results

For $T \in \mathcal{B}(H)$ we define the selfadjoint operator

$$\operatorname{Re}(T) = \frac{T^* + T}{2}.$$

Observe that $\operatorname{Re}(T^*) = \operatorname{Re}(T)$ and for all $x \in H$,

$$\langle \operatorname{Re}(T)x, x\rangle = \operatorname{Re}\langle Tx, x\rangle.$$

We have the following reverse of the Cauchy-Bunyakowsky-Schwarz integral inequality for the operator modulus:

Theorem 3. *Let* $B \in L_2(\Omega, \mu, \mathcal{B}(H))$ *and* $X, Y \in H$ *with* $X \neq Y$. *If*

$$\left|B(s) - \frac{X+Y}{2}\right|^2 \leq \frac{1}{4}|Y-X|^2 \text{ for } \mu\text{-a.e. } s \in \Omega, \tag{3.1}$$

or, equivalently,

$$\operatorname{Re}\left[(B^*(s) - Y^*)(X - B(s))\right] \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega, \tag{3.2}$$

then

$$0 \leq \int_\Omega w(s)|B(s)|^2 d\mu(s) - \left|\int_\Omega w(s) B(s) d\mu(s)\right|^2 \leq \frac{1}{4}|Y-X|^2. \tag{3.3}$$

The constant $\frac{1}{4}$ *is best possible.*

Proof. Let $s \in \Omega$. Observe that

$$\left|B(s) - \frac{X+Y}{2}\right|^2 \leq \frac{1}{4}|Y-X|^2 \tag{3.4}$$

is equivalent to

$$\left\langle \left|B(s) - \frac{X+Y}{2}\right|^2 x, x\right\rangle \leq \frac{1}{4}\left\langle |Y-X|^2 x, x\right\rangle$$

for all $x \in H$, or to

$$\left\langle \left(B(s) - \frac{X+Y}{2}\right)^* \left(B(s) - \frac{X+Y}{2}\right) x, x\right\rangle \leq \frac{1}{4}\left\langle (Y-X)^*(Y-X)x, x\right\rangle,$$

namely

$$\left\| B(s)x - \frac{Xx + Yx}{2} \right\|^2 \leq \frac{1}{4} \|Yx - Xx\| \tag{3.5}$$

for all $x \in H$.

By making use of the first part of Lemma 4, we obtain that (3.5) is equivalent to

$$\operatorname{Re} \langle Xx - B(s)x, B(s)x - Yx \rangle \geq 0 \tag{3.6}$$

for all $x \in H$.

Observe that

$$\begin{aligned} \operatorname{Re} \langle Xx - B(s)x, B(s)x - Yx \rangle &= \operatorname{Re} \left\langle (B(s) - Y)^* (X - B(s)) x, x \right\rangle \\ &= \operatorname{Re} \left\langle (B(s) - Y)^* (X - B(s)) x, x \right\rangle \\ &= \left\langle \operatorname{Re} \left[(B(s) - Y)^* (X - B(s)) \right] x, x \right\rangle \end{aligned}$$

for all $x \in H$.

Therefore the condition (3.6) is equivalent to

$$\operatorname{Re} \left[(B(s) - Y)^* (X - B(s)) \right] \geq 0,$$

which proves the equivalence of the statements (3.1) and (3.2).

Now by utilising (3.3) for $f(s) = B(s)x$, $x \in H$, then we get

$$0 \leq \int_\Omega w(s) \|B(s)x\|^2 d\mu(s) - \left\| \int_\Omega w(s) B(s) d\mu(s) x \right\|^2 \leq \frac{1}{4} \|Yx - Xx\|^2,$$

or

$$\begin{aligned} 0 &\leq \int_\Omega w(s) \langle B(s)x, B(s)x \rangle d\mu(s) \\ &\quad - \left\langle \int_\Omega w(s) B(s) d\mu(s) x, \int_\Omega w(s) B(s) d\mu(s) x \right\rangle \\ &\leq \frac{1}{4} \langle (Y - X)x, (Y - X)x \rangle, \end{aligned}$$

namely

$$\begin{aligned} 0 &\leq \int_\Omega w(s) \left\langle (B(s))^* B(s)x, x \right\rangle d\mu(s) \\ &\quad - \left\langle \left(\int_\Omega w(s) B(s) d\mu(s) \right)^* \int_\Omega w(s) B(s) d\mu(s) x, x \right\rangle \\ &\leq \frac{1}{4} \left\langle (Y - X)^* (Y - X) x, x \right\rangle, \end{aligned}$$

which is equivalent to (3.3). □

We can improve (3.3) as follows:

Theorem 4. *Let $B \in L_2(\Omega, \mu, \mathcal{B}(H))$ and $X, Y \in H$ with $X \neq Y$. If either (3.1) or (3.2) holds, then*

$$0 \leq \int_\Omega w(s) |B(s)|^2 d\mu(s) - \left| \int_\Omega w(s) f(s) d\mu(s) \right|^2 \tag{3.7}$$
$$\leq \operatorname{Re}\left[\left(\int_\Omega w(s) B^*(s) d\mu(s) - Y^* \right) \left(X - \int_\Omega w(s) B(s) d\mu(s) \right) \right]$$
$$\leq \frac{1}{4} |Y - X|^2 .$$

The constant $\frac{1}{4}$ is best possible.

Proof. We have

$$K_1 := \operatorname{Re}\left[\left(\int_\Omega w(s) B(s) d\mu(s) - Y \right)^* \left(X - \int_\Omega w(s) B(s) d\mu(s) \right) \right]$$
$$= \operatorname{Re}\left[\left(\int_\Omega w(s) B^*(s) d\mu(s) - Y^* \right) \left(X - \int_\Omega w(s) B(s) d\mu(s) \right) \right]$$
$$= \operatorname{Re}\left[\left(\int_\Omega w(s) B^*(s) d\mu(s) \right) X \right] - \left| \int_\Omega w(s) B(s) d\mu(s) \right|^2$$
$$- \operatorname{Re}(Y^* X) - \operatorname{Re}\left[Y^* \int_\Omega w(s) B(s) d\mu(s) \right]$$
$$= \left(\int_\Omega w(s) \operatorname{Re}(B^*(s) X) d\mu(s) \right) - \left| \int_\Omega w(s) B(s) d\mu(s) \right|^2$$
$$- \operatorname{Re}(Y^* X) - \int_\Omega w(s) \operatorname{Re}(Y^* B(s)) d\mu(s)$$

and

$$K_2 := \int_\Omega w(s) \operatorname{Re}\left[(B(s) - Y)^* (X - B(s)) \right] d\mu(s)$$
$$= \int_\Omega w(s) \left[\operatorname{Re}(B^*(s) X) - \operatorname{Re}(Y^* X) - |B(s)|^2 + \operatorname{Re}(Y^* B(s)) \right] d\mu(s)$$
$$= \int_\Omega w(s) \operatorname{Re}(B^*(s) X) d\mu(s) - \operatorname{Re}(Y^* X)$$
$$- \int_\Omega w(s) |B(s)|^2 d\mu(s) + \int_\Omega w(s) \operatorname{Re}(Y^* B(s)) d\mu(s) .$$

Since

$$K_1 - K_2 = \int_\Omega w(s) |B(s)|^2 d\mu(s) - \left| \int_\Omega w(s) B(s) d\mu(s) \right|^2 ,$$

hence we derive the following identity of interest

$$\int_\Omega w(s) |B(s)|^2 d\mu(s) - \left| \int_\Omega w(s) B(s) d\mu(s) \right|^2 \tag{3.8}$$
$$= \operatorname{Re}\left[\left(\int_\Omega w(s) B^*(s) d\mu(s) - Y^* \right) \left(X - \int_\Omega w(s) B(s) d\mu(s) \right) \right]$$
$$- \int_\Omega w(s) \operatorname{Re}\left[(B^*(s) - Y^*) (X - B(s)) \right] d\mu(s) .$$

Now, if condition (3.2) holds, then

$$\int_{\Omega} w(s) \operatorname{Re}\left[\left(B^{*}(s)-Y^{*}\right)(X-B(s))\right] d\mu(s) \geq 0,$$

which proves the first inequality in (3.7).

Observe that we have the following operator inequality

$$4 \operatorname{Re}\left(C^{*} D\right) \leq |C+D|^{2} \tag{3.9}$$

for all C, $D \in \mathcal{B}(H)$.

Indeed, we have

$$\begin{aligned}
& |C+D|^{2}-4 \operatorname{Re}\left(C^{*} D\right) \\
& =(C+D)^{*}(C+D)-4 \frac{C^{*} D+D^{*} C}{2} \\
& =\left(C^{*}+D^{*}\right)(C+D)-2\left(C^{*} D+D^{*} C\right) \\
& =|C|^{2}+D^{*} C+C^{*} D+|D|^{2}-2\left(C^{*} D+D^{*} C\right) \\
& =|C|^{2}+|D|^{2}-C^{*} D-D^{*} C=|C-D|^{2} \geq 0 .
\end{aligned}$$

By utilising (3.9) we get

$$\begin{aligned}
& \operatorname{Re}\left[\left(\int_{\Omega} w(s) B(s) d\mu(s)-Y\right)^{*}\left(X-\int_{\Omega} w(s) B(s) d\mu(s)\right)\right] \\
& \leq \frac{1}{4}\left|\int_{\Omega} w(s) B(s) d\mu(s)-Y+X-\int_{\Omega} w(s) B(s) d\mu(s)\right|^{2} \\
& =\frac{1}{4}|Y-X|^{2},
\end{aligned}$$

which proves the last part of (3.7). □

We have the following Grüss type operator inequality:

Theorem 5. *Let* $\alpha \in L_{w}^{2}(\Omega, \mu, \mathbb{C})$, $B \in L_{2}(\Omega, \mu, \mathcal{B}(H))$ *and* X, $Y \in H$ *with* $X \neq Y$*. If either (3.1) or (3.2) holds, then*

$$\begin{aligned}
& \left|\int_{\Omega} w(s) \alpha(s) B(s) d\mu(s)-\int_{\Omega} w(s) \alpha(s) d\mu(s) \int_{\Omega} w(s) B(s) d\mu(s)\right|^{2} \\
& \leq\left(\int_{\Omega} w(s)|\alpha(s)|^{2} d\mu(s)-\left|\int_{\Omega} w(s) \alpha(s) d\mu(s)\right|^{2}\right) \\
& \times \operatorname{Re}\left[\left(\int_{\Omega} w(s) B^{*}(s) d\mu(s)-Y^{*}\right)\left(X-\int_{\Omega} w(s) B(s) d\mu(s)\right)\right] \\
& \leq \frac{1}{4}\left(\int_{\Omega} w(s)|\alpha(s)|^{2} d\mu(s)-\left|\int_{\Omega} w(s) \alpha(s) d\mu(s)\right|^{2}\right)|Y-X|^{2} .
\end{aligned} \tag{3.10}$$

Proof. We use the following Sonin type identity that can be proved by performing the calculations in the right side

$$\begin{aligned}
&\int_{\Omega} w(s)\,\alpha(s)\,B(s)\,d\mu(s) - \int_{\Omega} w(s)\,\alpha(s)\,d\mu(s)\int_{\Omega} w(s)\,B(s)\,d\mu(s) \\
&= \int_{\Omega} w(s)\left(\alpha(s) - \int_{\Omega} w(t)\,\alpha(t)\,d\mu(t)\right) \\
&\times\left(B(s) - \int_{\Omega} w(t)\,B(t)\,d\mu(t)\right) d\mu(s).
\end{aligned}$$

By using (2.1) we have

$$\begin{aligned}
&\left|\int_{\Omega} w(s)\left(\alpha(s) - \int_{\Omega} w(t)\,\alpha(t)\,d\mu(t)\right)\right. \\
&\left.\times\left(B(s) - \int_{\Omega} w(t)\,B(t)\,d\mu(t)\right) d\mu(s)\right|^2 \\
&\le \left[\int_{\Omega} w(s)\left|\alpha(s) - \int_{\Omega} w(t)\,\alpha(t)\,d\mu(t)\right|^2 d\mu(s)\right] \\
&\times\left[\int_{\Omega} w(s)\left|B(s) - \int_{\Omega} w(t)\,B(t)\,d\mu(t)\right|^2 d\mu(t)\right].
\end{aligned} \tag{3.11}$$

Since

$$\begin{aligned}
&\int_{\Omega} w(s)\left|\alpha(s) - \int_{\Omega} w(t)\,\alpha(t)\,d\mu(t)\right|^2 d\mu(s) \\
&= \int_{\Omega} w(s)\,|\alpha(s)|^2\,d\mu(s) - \left|\int_{\Omega} w(s)\,\alpha(s)\,d\mu(s)\right|^2
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
&\int_{\Omega} w(s)\left|B(s) - \int_{\Omega} w(t)\,B(t)\,d\mu(t)\right|^2 d\mu(t) \\
&= \int_{\Omega} w(s)\,|B(s)|^2\,d\mu(s) - \left|\int_{\Omega} w(s)\,B(s)\,d\mu(s)\right|^2,
\end{aligned} \tag{3.13}$$

hence by (3.11), (3.12) and (3.13) we derive

$$\begin{aligned}
&\left|\int_{\Omega} w(s)\,\alpha(s)\,B(s)\,d\mu(s) - \int_{\Omega} w(s)\,\alpha(s)\,d\mu(s)\int_{\Omega} w(s)\,B(s)\,d\mu(s)\right|^2 \\
&\le \left(\int_{\Omega} w(s)\,|\alpha(s)|^2\,d\mu(s) - \left|\int_{\Omega} w(s)\,\alpha(s)\,d\mu(s)\right|^2\right) \\
&\times\left(\int_{\Omega} w(s)\,|B(s)|^2\,d\mu(s) - \left|\int_{\Omega} w(s)\,B(s)\,d\mu(s)\right|^2\right).
\end{aligned}$$

By making use of (3.7) we derive the desired result (3.10). □

Corollary 2. *With the assumptions of Theorem 5 and if there exist the constant* β, $\gamma \in \mathbb{C}$ *such that*

$$\left|\alpha(s) - \frac{\beta+\gamma}{2}\right|^2 \le \frac{1}{4}\,|\gamma-\beta|^2 \text{ for } \mu\text{-a.e. } s \in \Omega, \tag{3.14}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\overline{\alpha(s)}-\bar{\gamma}\right)(\beta-\alpha(s))\right]\geq 0 \text{ for } \mu\text{-a.e. } s\in\Omega, \tag{3.15}$$

then we have

$$\begin{aligned}&\left|\int_{\Omega}w(s)\alpha(s)B(s)d\mu(s)-\int_{\Omega}w(s)\alpha(s)d\mu(s)\int_{\Omega}w(s)B(s)d\mu(s)\right|^{2}\\&\leq\frac{1}{4}|\gamma-\beta|^{2}\\&\times\operatorname{Re}\left[\left(\int_{\Omega}w(s)B^{*}(s)d\mu(s)-Y^{*}\right)\left(X-\int_{\Omega}w(s)B(s)d\mu(s)\right)\right]\\&\leq\frac{1}{16}|\gamma-\beta|^{2}|Y-X|^{2}.\end{aligned}\tag{3.16}$$

Remark 4. *By taking the square root in (3.16), we get the Grüss type inequality*

$$\begin{aligned}&\left|\int_{\Omega}w(s)\alpha(s)B(s)d\mu(s)-\int_{\Omega}w(s)\alpha(s)d\mu(s)\int_{\Omega}w(s)B(s)d\mu(s)\right|\\&\leq\frac{1}{4}|\gamma-\beta|\\&\times\left(\operatorname{Re}\left[\left(\int_{\Omega}w(s)B^{*}(s)d\mu(s)-Y^{*}\right)\left(X-\int_{\Omega}w(s)B(s)d\mu(s)\right)\right]\right)^{1/2}\\&\leq\frac{1}{4}|\gamma-\beta||Y-X|.\end{aligned}\tag{3.17}$$

4. Applications for Finite Fourier Transform

Let $B:[a,b]\to\mathcal{B}(H)$ be a Bochner integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(B)(t):=\int_{a}^{b}e^{-2\pi its}B(s)ds.$$

Let E be the *exponential mean* of two complex numbers defined by

$$E(z,w):=\begin{cases}\dfrac{e^{z}-e^{w}}{z-w}, & \text{if } z\neq w\\ \exp(w) & \text{if } z=w\end{cases},\quad z,w\in\mathbb{C}. \tag{4.1}$$

Observe that

$$\int_{a}^{b}e^{-2\pi its}ds=(b-a)E(-2\pi ita,-2\pi itb),$$

$$\left|e^{2\pi its}\right|^{2}=1,$$

$$\int_{a}^{b}e^{2\pi its}ds=\frac{1}{2\pi it}\left[e^{2\pi itb}-e^{2\pi ita}\right],$$

and

$$\begin{aligned}\left|\int_a^b e^{2\pi its}ds\right|^2 &= \left(\frac{1}{2\pi|t|}\right)^2\left[\left|e^{2\pi itb}\right|^2 - 2\operatorname{Re}\left[e^{2\pi itb}e^{-2\pi ita}\right] + \left|e^{2\pi ita}\right|^2\right] \\ &= \frac{1}{4\pi^2t^2}\left[1 - 2\operatorname{Re}\left[e^{2\pi it(b-a)}\right] + 1\right] \\ &= \frac{1}{2\pi^2t^2}\left[1 - \operatorname{Re}\left[\cos(2\pi t(b-a)) + i\sin(2\pi t(b-a))\right]\right] \\ &= \frac{1}{2\pi^2t^2}\left[1 - \cos(2\pi t(b-a))\right] \\ &= \frac{1}{2\pi^2|t|^2}\left[1 - \left(1 - 2\sin^2(\pi t(b-a))\right)\right] = \frac{\sin^2[\pi t(b-a)]}{\pi^2t^2}.\end{aligned}$$

Let $B \in L_2([a,b],\mathcal{B}(H))$ and $X, Y \in H$ with $X \neq Y$. If either

$$\left|B(s) - \frac{X+Y}{2}\right|^2 \leq \frac{1}{4}|Y-X|^2 \text{ for a.e. } s \in [a,b],$$

or, equivalently,

$$\operatorname{Re}\left[(B^*(s) - Y^*)(X - B(s))\right] \geq 0 \text{ for a.e. } s \in [a,b],$$

then by (3.10) for $w(s) = \frac{1}{b-a}$ and $\alpha(s) = e^{-2\pi its}$, $s \in [a,b]$, $t \in \mathbb{R}$ we get

$$\begin{aligned}&\left|\int_a^b e^{2\pi its}B(s)\,ds - \frac{1}{b-a}\int_a^b e^{2\pi its}ds\int_a^b B(s)\,ds\right|^2 \\ &\leq (b-a)\left(\frac{1}{b-a}\int_a^b\left|e^{2\pi its}\right|^2ds - \left|\frac{1}{b-a}\int_a^b e^{2\pi its}ds\right|^2\right) \\ &\times \operatorname{Re}\left[\left(\frac{1}{b-a}\int_a^b B^*(s)\,ds - Y^*\right)\left(X - \frac{1}{b-a}\int_a^b B(s)\,ds\right)\right] \\ &\leq \frac{1}{4}(b-a)\left(\frac{1}{b-a}\int_a^b\left|e^{2\pi its}\right|^2ds - \left|\frac{1}{b-a}\int_a^b e^{2\pi its}ds\right|^2\right)|Y-X|^2.\end{aligned}$$

Therefore we have the inequalities

$$\begin{aligned}&\left|\mathcal{F}(B)(t) - E(-2\pi ita, -2\pi itb)\int_a^b B(s)\,ds\right|^2 \\ &\leq (b-a)\left(1 - \frac{\sin^2[\pi t(b-a)]}{\pi^2t^2(b-a)^2}\right) \\ &\times \operatorname{Re}\left[\left(\frac{1}{b-a}\int_a^b B^*(s)\,ds - Y^*\right)\left(X - \frac{1}{b-a}\int_a^b B(s)\,ds\right)\right] \\ &\leq \frac{1}{4}(b-a)\left(1 - \frac{\sin^2[\pi t(b-a)]}{\pi^2t^2(b-a)^2}\right)|Y-X|^2,\end{aligned} \tag{4.2}$$

for $t \in \mathbb{R}$.

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