JENSEN TYPE INEQUALITIES FOR THE SQUARE MODULUS CONVEX FUNCTIONS OF OPERATORS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. We say that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$ if

$$|B((1-t)u+tv)|^2\leq(\geq)(1-t)|B(u)|^2+t|B(v)|^2$$

in the operator order of $\mathcal{B}(H)$, for all $u,v\in[a,b]$ and $t\in[0,1]$. In this paper, we show among others that, if $B:[m,M]\subset\mathbb{R}\to\mathcal{B}(H)$ is square modulus convex on $[m,M]$ and $f:\Omega\to[m,M]$ so that f, $|B\circ f|^2$, $\operatorname{Re}((B\circ f)^*(B'\circ f))$, $f\operatorname{Re}((B\circ f)^*(B'\circ f))\in L_w(\Omega,\mu,\mathcal{B}(H))$, where $w\geq 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$, then

$$\begin{aligned}0&\leq\int_\Omega w(s)|B\circ f|^2d\mu(s)-\left|B\left(\int_\Omega wfd\mu\right)\right|^2\\&\leq\frac{1}{2}(M-m)\left\|\left(\operatorname{Re}\left((B(M))^*B'_-(M)\right)-\operatorname{Re}\left((B(m))^*B'_+(m)\right)\right)\right\|.\end{aligned}$$

The discrete versions are also provided.

1. Introduction

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

The following Cauchy-Bunyakowsky-Schwarz inequality holds

$$\sum_{k=1}^n w_k|z_k|^2\sum_{k=1}^n w_k|A_k|^2\geq\left|\sum_{k=1}^n w_kz_kA_k\right|^2, \tag{1.1}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\geq 0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n w_k=1$.

Definition 1. *We say that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$ if*

$$|B((1-t)u+tv)|^2\leq(\geq)(1-t)|B(u)|^2+t|B(v)|^2 \tag{1.2}$$

in the operator order of $\mathcal{B}(H)$, for all $u,v\in[a,b]$ and $t\in[0,1]$.

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Let $A,\ B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (1.1), we get

$$\begin{aligned}\left|(1-\alpha)A+\alpha B\right|^2 &= \left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^2\\ &\le \left[\left((1-\alpha)^{1/2}\right)^2+\left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2+\left|\alpha^{1/2}B\right|^2\right]\\ &= (1-\alpha+\alpha)\left[(1-\alpha)|A|^2+\alpha|B|^2\right]\\ &= (1-\alpha)|A|^2+\alpha|B|^2.\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let $t_1,\ t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}\left|C\left((1-\alpha)t_1+\alpha t_2\right)\right|^2 &= \left|(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B\right|^2\\ &= \left|(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)\right|^2\\ &\le (1-\alpha)\left|(1-t_1)A+t_1B\right|^2+\alpha\left|(1-t_2)A+t_2B\right|^2\\ &= (1-\alpha)|C(t_1)|^2+\alpha|C(t_2)|^2,\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\le(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators $A,\ B$ with spectra in I.

For such function and $A,\ B$, we consider

$$D(t):=\left[f((1-t)A+tB)\right]^{1/2},\ t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$. Therefore for $A,\ B>0$, the function

$$B_r(t):=((1-t)A+tB)^{r/2},\ \ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\le r\le 2$ or $-1\le r\le 0$.

Let $B:[a,b]\to\mathbb{C}$ be defined by $B(t):=x(t)+y(t)i$, $t\in[a,b]$. Observe that $|B(t)|^2=x^2(t)+y^2(t)$, $t\in[a,b]$. Now, if $x^2,\ y^2$ are convex on $[a,b]$, then obviously that x^2+y^2 is convex on $[a,b]$. However, if we take $x(t)=t\sin t$, $y(t)=t\cos t$, $t\in[0,2\pi]$, then neither x^2 nor y^2 is convex on $[0,2\pi]$ but x^2+y^2 is convex on $[0,2\pi]$.

Proposition 1. *If the continuous function* $B:[a,b]\to\mathcal{B}(H)$ *is square modulus concave on* $[a,b]$ *then for* $p\in(0,1)$, $|B(\cdot)|^p$ *is also square modulus concave on* $[a,b]$.

Proof. By the operator monotonicity and operator concavity of the function $h(t)=t^p$ for $p\in(0,1)$, we have

$$\begin{aligned}\left|B((1-t)u+tv)\right|^{2p} &\ge \left((1-t)|B(u)|^2+t|B(v)|^2\right)^p\\ &\ge (1-t)|B(u)|^{2p}+t|B(v)|^{2p}\end{aligned}$$

for $t\in[0,1]$, which shows that $|B(\cdot)|^p$ is also square modulus concave on $[a,b]$. □

For A, $B>0$, the function

$$B_q(t):=((1-t)A+tB)^{q/2},\ t\in[0,1]$$

is *square modulus concave* on $[0,1]$ for $q\in(0,1)$.

Indeed, we have for t_1, $t_2\in[0,1]$ and $\alpha\in[0,1]$ that

$$\begin{aligned}|B_q((1-\alpha)t_1+\alpha t_2)|^2&=((1-(1-\alpha)t_1-+\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B)^q\\&=[(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)]^q\\&\geq(1-\alpha)((1-t_1)A+t_1B)^r+\alpha((1-t_2)A+t_2B)^q\\&=(1-\alpha)|B_q(t_1)|^2+\alpha|B_q(t_2)|^2,\end{aligned}$$

which shows that B_q is *square modulus concave* on $[0,1]$.

We have the following Hermite-Hadamard type inequalities [7]:

Theorem 1. *Assume that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$. Then for all u, $v\in[a,b]$ we have*

$$\begin{aligned}\left|B\left(\frac{u+v}{2}\right)\right|^2&\leq(\geq)\int_0^1|B((1-t)u+tv)|^2dt\\&\leq(\geq)\frac{1}{2}\left[|B(u)|^2+|B(v)|^2\right].\end{aligned}\tag{1.3}$$

For more results of this type, see [7]. Recent inequalities for operator convex functions can be also found in [1]-[6] and [8]-[11].

In this paper, we show among others that, if $B:[m,M]\subset\mathbb{R}\to\mathcal{B}(H)$ is square modulus convex on $[m,M]$ and $f:\Omega\to[m,M]$ so that $f,|B\circ f|^2,\operatorname{Re}\left((B\circ f)^*(B'\circ f)\right)$, $f\operatorname{Re}\left((B\circ f)^*(B'\circ f)\right)\in L_w(\Omega,\mu,\mathcal{B}(H))$, where $w\geq0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$, then

$$\begin{aligned}0&\leq\int_\Omega w(s)|B\circ f|^2d\mu(s)-\left|B\left(\int_\Omega wfd\mu\right)\right|^2\\&\leq\frac{1}{2}(M-m)\left\|\left(\operatorname{Re}\left((B(M))^*B'_-(M)\right)-\operatorname{Re}\left((B(m))^*B'_+(m)\right)\right)\right\|.\end{aligned}$$

The discrete versions are also provided.

2. Some Preliminary Facts

Following Roberts and Varberg [12, p. 5], we recall that if $f:I\to\mathbb{R}$ is a convex function, then for any $s_0\in\mathring{I}$ (the interior of the interval I) the limits

$$f'_-(s_0):=\lim_{s\to s_0-}\frac{f(s)-f(s_0)}{s-s_0}\text{ and }f'_+(s_0):=\lim_{s\to s_0+}\frac{f(s)-f(s_0)}{s-s_0}$$

exists and $f'_-(s_0)\leq f'_+(s_0)$. The functions f'_- and f'_+ are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [12, p. 7]).

From the monotonicity of the lateral derivatives f'_- and f'_+ we also have *the gradient inequality*

$$f'_-(s)(s-\tau)\geq f(s)-f(\tau)\geq f'_+(\tau)(s-\tau)$$

for any s, $\tau\in\mathring{I}$.

If $I=[m,M]$, then at the end points we also have the inequalities

$$f(s)-f(m)\geq f'_+(m)(s-m)$$

for any $s\in(m,M]$ and

$$f(\tau)-f(M)\geq f'_{-}(M)(\tau-M)$$

for any $\tau\in[m,M)$.

For the operator $T\in\mathcal{B}(H)$ we define the selfadjoint operator

$$\operatorname{Re}(T)=\frac{1}{2}(T^{*}+T).$$

Assume that function $B:[m,M]\to\mathcal{B}(H)$ is continuous on $[m,M]$. The function $B:[m,M]\to\mathcal{B}(H)$ is square modulus convex on $[m,M]$ if and only if for all $x\in H\setminus\{0\}$ the auxiliary function $\varphi_{B,x}:[m,M]\to[0,\infty)$, $\varphi_{B,x}(u)=\|B(u)x\|^{2}$ is convex on $[m,M]$.

Indeed, condition (1.2) is equivalent to

$$\left\langle |B((1-t)u+tv)|^{2}x,x\right\rangle\leq(1-t)\left\langle |B(u)|^{2}x,x\right\rangle+t\left\langle |B(v)|^{2}x,x\right\rangle,$$

namely

$$\begin{aligned}&\left\langle [B((1-t)u+tv)]^{*}B((1-t)u+tv)x,x\right\rangle\\&\leq(1-t)\left\langle [B(u)]^{*}B(u)x,x\right\rangle+t\left\langle [B(v)]^{*}B(v)x,x\right\rangle,\end{aligned}$$

or

$$\|B((1-t)u+tv)x\|^{2}\leq(1-t)\|B(u)x\|^{2}+t\|B(v)x\|^{2}$$

for all $t\in[0,1]$ and $u,v\in[m,M]$.

We also have

$$\begin{aligned}\varphi'_{\pm B,x}(u)&=(\langle B(u)x,B(u)x\rangle)'_{\pm}=\left\langle B'_{\pm}(u)x,B(u)x\right\rangle+\left\langle B(u)x,B'_{\pm}(u)x\right\rangle\\&=\left\langle (B(u))^{*}B'_{\pm}(u)x,x\right\rangle+\left\langle \left(B'_{\pm}(u)\right)^{*}B(u)x,x\right\rangle\\&=\left\langle (B(u))^{*}B'_{\pm}(u)x,x\right\rangle+\left\langle \left((B(u))^{*}B'_{\pm}(u)\right)^{*}x,x\right\rangle\\&=\left\langle 2\operatorname{Re}\left((B(u))^{*}B'_{\pm}(u)\right)x,x\right\rangle\end{aligned}$$

and

$$\begin{aligned}\varphi'_{+B,x}(m)&=2\left\langle \operatorname{Re}\left((B(m))^{*}B'_{+}(m)\right)x,x\right\rangle,\\\varphi'_{-B,x}(M)&=2\left\langle \operatorname{Re}\left((B(M))^{*}B'_{-}(M)\right)x,x\right\rangle\end{aligned}$$

for all $x\in H\setminus\{0\}$.

We have for $t\in(m,M)$ and small $h\neq0$ such that $t+h\in(m,M)$,

$$\begin{aligned}\left\langle \frac{|B(t+h)|^{2}-|B(t)|^{2}}{h}x,x\right\rangle&=\frac{1}{h}\left[\left\langle |B(t+h)|^{2}x,x\right\rangle-\left\langle |B(t)|^{2}x,x\right\rangle\right]\\&=\frac{1}{h}\left[\|B(t+h)x\|^{2}-\|B(t)x\|^{2}\right]\end{aligned}$$

for all $x\in H\setminus\{0\}$.

By taking the lateral limits, we get

$$\begin{aligned}\lim_{h\to\pm0}\left\langle \frac{|B(t+h)|^{2}-|B(t)|^{2}}{h}x,x\right\rangle&=\lim_{h\to\pm0}\frac{1}{h}\left[\|B(t+h)x\|^{2}-\|B(t)x\|^{2}\right]\\&=\left\langle 2\operatorname{Re}\left((B(t))^{*}B'_{\pm}(t)\right)x,x\right\rangle\end{aligned}$$

for all $x\in H\setminus\{0\}$.

Therefore for the function $\varphi(t) := |B(t)|^2$, $t \in (m, M)$ we have

$$\varphi'_{\pm}(t) = 2\operatorname{Re}\left((B(t))^* B'_{\pm}(t)\right)$$

and

$$\varphi'_{+}(m) = 2\operatorname{Re}\left((B(m))^* B_{+}(m)\right),\ \varphi'_{-}(M) = 2\operatorname{Re}\left((B(M))^* B'_{-}(M)\right).$$

We also have *the operator gradient inequality*

$$2(t-\tau)\operatorname{Re}\left((B(t))^* B'_{-}(t)\right) \geq |B(t)|^2 - |B(\tau)|^2 \geq 2(t-\tau)\operatorname{Re}\left((B(\tau))^* B'_{+}(\tau)\right) \tag{2.1}$$

for any t, $\tau \in (m, M)$.

Moreover, at the end points of the interval we have the operator inequalities

$$|B(t)|^2 - |B(m)|^2 \geq 2(t-m)\operatorname{Re}\left((B(m))^* B'_{+}(m)\right) \tag{2.2}$$

for any $t \in (m, M]$ and

$$2(M-\tau)\operatorname{Re}\left((B(M))^* B'_{-}(M)\right) \geq |B(M)|^2 - |B(\tau)|^2 \tag{2.3}$$

for any $\tau \in [m, M)$.

Finally, we notice that for $m \leq t_1 < t_2 \leq M$, we have

$$\begin{aligned}\operatorname{Re}\left((B(m))^* B'_{+}(m)\right) &\leq \operatorname{Re}\left((B(t_1))^* B'_{\pm}(t_1)\right) \\ &\leq \operatorname{Re}\left((B(t_2))^* B'_{\pm}(t_2)\right) \leq \operatorname{Re}\left((B(M))^* B'_{-}(M)\right)\end{aligned} \tag{2.4}$$

in the operator order of $\mathcal{B}(H)$.

3. Main Results

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(s) \geq 0$ for μ-a.e. (almost every) $s \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is μ-measurable and $\int_\Omega w(s)|f(s)|\, d\mu(s) < \infty\}$. For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(s)\, d\mu(s)$. We also assume that $\int_\Omega w d\mu = 1$.

We define $L_{w,p}(\Omega, \mu, \mathcal{B}(H))$, $p \geq 1$, to be the space of all strongly μ-measurable functions on $A : \Omega \to \mathcal{B}(H)$ such that the integral $\int_\Omega w(s)\|A(s)\|^p d\mu(s)$ is finite. For $p = 1$ we denote $L_w(\Omega, \mu, \mathcal{B}(H))$.

Theorem 2. *Let $B : [m, M] \subset \mathbb{R} \to \mathcal{B}(H)$ be square modulus convex on $[m, M]$ and $f : \Omega \to [m, M]$ so that f, $|B \circ f|^2$, $\operatorname{Re}\left((B \circ f)^* (B' \circ f)\right)$, $f \operatorname{Re}\left((B \circ f)^* (B' \circ f)\right) \in L_w(\Omega, \mu, \mathcal{B}(H))$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality*

$$\begin{aligned}0 &\leq \int_\Omega w(s)|B \circ f|^2 d\mu(s) - \left|B\left(\int_\Omega wf d\mu\right)\right|^2 \\ &\leq 2\left[\int_\Omega wf \operatorname{Re}\left((B \circ f)^* (B' \circ f)\right) d\mu \right. \\ &\quad \left. - \int_\Omega wf d\mu \int_\Omega w \operatorname{Re}\left((B \circ f)^* (B'_{-} \circ f)\right) d\mu(s)\right]\end{aligned} \tag{3.1}$$

in the operator order of $\mathcal{B}(H)$.

Proof. We have by the operator gradient inequality (2.1) that

$$2\left(t-\int_{\Omega} wfd\mu\right)\operatorname{Re}\left((B(t))^{*}B'(t)\right) \tag{3.2}$$
$$\geq |B(t)|^{2}-\left|B\left(\int_{\Omega} wfd\mu\right)\right|^{2}$$
$$\geq 2\left(t-\int_{\Omega} wfd\mu\right)\operatorname{Re}\left(\left(B\left(\int_{\Omega} wfd\mu\right)\right)^{*}B'\left(\int_{\Omega} wfd\mu\right)\right)$$

for almost every $t\in(m,M)$.

If we take in (3.2) $t=f(s)$, multiply with $w(s)\geq 0$ and integrate, then we get

$$2\int_{\Omega} w(s)\left(f(s)-\int_{\Omega} wfd\mu\right)\operatorname{Re}\left((B(f(s)))^{*}B'(f(s))\right)d\mu(s) \tag{3.3}$$
$$\geq \int_{\Omega} w(s)\left[|B(f(s))|^{2}-\left|B\left(\int_{\Omega} wfd\mu\right)\right|^{2}\right]d\mu(s)$$
$$\geq 2\int_{\Omega} w(s)\left(f(s)-\int_{\Omega} wfd\mu\right)$$
$$\times\operatorname{Re}\left(\left(B\left(\int_{\Omega} wfd\mu\right)\right)^{*}B'\left(\int_{\Omega} wfd\mu\right)\right)d\mu(s).$$

Since

$$\int_{\Omega} w(s)\left(f(s)-\int_{\Omega} wfd\mu\right)\operatorname{Re}\left((B(f(s)))^{*}B'(f(s))\right)d\mu(s)$$
$$=\int_{\Omega} w(s)f(s)\operatorname{Re}\left((B(f(s)))^{*}B'(f(s))\right)d\mu(s)$$
$$-\int_{\Omega} wfd\mu\int_{\Omega} w(s)\operatorname{Re}\left((B(f(s)))^{*}B'(f(s))\right)d\mu(s),$$

$$\int_{\Omega} w(s)\left[|B(f(s))|^{2}-\left|B\left(\int_{\Omega} wfd\mu\right)\right|^{2}\right]d\mu(s)$$
$$=\int_{\Omega} w(s)|B(f(s))|^{2}d\mu(s)-\int_{\Omega} w(s)d\mu(s)\left|B\left(\int_{\Omega} wfd\mu\right)\right|^{2}$$

and

$$\int_{\Omega} w(s)\left(f(s)-\int_{\Omega} wfd\mu\right)\operatorname{Re}\left(\left(B\left(\int_{\Omega} wfd\mu\right)\right)^{*}B'\left(\int_{\Omega} wfd\mu\right)\right)d\mu(s)$$
$$=\int_{\Omega} w(s)\left(f(s)-\int_{\Omega} wfd\mu\right)d\mu(s)\operatorname{Re}\left(\left(B\left(\int_{\Omega} wfd\mu\right)\right)^{*}B'\left(\int_{\Omega} wfd\mu\right)\right)$$
$$=0,$$

hence by (3.3) we get (3.1). □

We have the following Cauchy-Bunyakowsky-Schwarz inequality:

Lemma 1. *If* $\alpha\in L_{w}^{2}(\Omega,\mu,\mathbb{C})$ *and*

$$A\in L_{2,w}(\Omega,\mu,\mathcal{B}(H)):=\left\{A:\Omega\to B(H),\ \int_{\Omega} w(s)\|A(s)\|^{2}d\mu(s)<\infty\right\},$$

then

$$\left|\int_{\Omega} w\alpha A d\mu\right|^{2} \leq \int_{\Omega} w\left|\alpha\right|^{2} d\mu \int_{\Omega} w\left|A\right|^{2} d\mu \tag{3.4}$$

in the operator order of $\mathcal{B}(H)$.

Proof. We have for $\alpha \in L_{w}^{2}(\Omega, \mu, \mathbb{C})$ and $A \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$,

$$\begin{aligned} 0 \leq \left|\overline{\alpha(x)} A(y) - \overline{\alpha(y)} A(x)\right|^{2} &= \left|\alpha(x)\right| \left|A(y)\right|^{2} - \alpha(y) \overline{\alpha(x)} A^{*}(x) A(y) \\ &- \alpha(x) \overline{\alpha(y)} A^{*}(y) A(x) + \left|\alpha(y)\right|^{2} \left|A(x)\right|^{2}, \end{aligned}$$

which gives that

$$\begin{aligned} &\left|\alpha(x)\right|^{2} \left|A(y)\right|^{2} + \left|\alpha(y)\right|^{2} \left|A(x)\right|^{2} \\ &\geq \alpha(y) \overline{\alpha(x)} A^{*}(x) A(y) + \alpha(x) \overline{\alpha(y)} A^{*}(y) A(x) \end{aligned}$$

for all $y, x \in \Omega$.

Now, multiply this with $w(y) w(x) \geq 0$ to get

$$\begin{aligned} &w(x) \left|\alpha(x)\right|^{2} w(y) \left|A(y)\right|^{2} + w(y) \left|\alpha(y)\right|^{2} w(x) \left|A(x)\right|^{2} \\ &\geq w(x) \overline{\alpha(x)} A^{*}(x) w(y) \alpha(y) A(y) + w(y) \overline{\alpha(y)} A^{*}(y) w(x) \alpha(x) A(x) \end{aligned}$$

for all $y, x \in \Omega$.

Integrating over x and y on Ω, then we get

$$\begin{aligned} &\int_{a}^{b} w(x) \left|\alpha(x)\right|^{2} d\mu(x) \int_{a}^{b} \left|A(y)\right|^{2} d\mu(y) \\ &+ \int_{a}^{b} \left|\alpha(y)\right|^{2} d\mu(y) \int_{a}^{b} w(x) \left|A(x)\right|^{2} d\mu(x) \\ &\geq \int_{a}^{b} w(x) \overline{\alpha(x)} A^{*}(x) d\mu(x) \int_{a}^{b} \alpha(y) A(y) d\mu(y) \\ &+ \int_{a}^{b} w(y) \overline{\alpha(y)} A^{*}(y) d\mu(y) \int_{a}^{b} \alpha(x) A(x) d\mu(x) \\ &= 2\left|\int_{a}^{b} w(y) \alpha(y) A(y) d\mu(y)\right|^{2}, \end{aligned}$$

and the inequality (3.4) is obtained. □

We recall Löwner-Heinz inequality which says that, if $0 \leq A \leq B$, then for all $p \in (0, 1)$ we have $0 \leq A^{p} \leq B^{p}$. By using this property, we can state the following result as well:

Corollary 1. *With the assumptions of Lemma 1, we have the inequality*

$$\left|\int_{\Omega} w\alpha A d\mu\right| \leq \left(\int_{\Omega} w\left|\alpha\right|^{2} d\mu\right)^{1/2} \left(\int_{\Omega} w\left|A\right|^{2} d\mu\right)^{1/2}, \tag{3.5}$$

in the operator order of $\mathcal{B}(H)$.

The proof follows by (3.4) by taking the operator square root.

Remark 1. *We remark that, if α is real valued and $A(s)$, $s\in\Omega$ are selfadjoint operators, then we have*

$$\left|\int_\Omega w\alpha A d\mu\right| \le \left(\int_\Omega w\alpha^2 d\mu\right)^{1/2}\left(\int_\Omega wA^2 d\mu\right)^{1/2}. \tag{3.6}$$

Lemma 2. *If $\alpha:\Omega\to[m,M]$ and $A(s)$, $s\in\Omega$ are selfadjoint operators such that $\alpha\in L_w^2(\Omega,\mu,\mathbb{C})$, $A\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$, then we have*

$$\begin{aligned}\left|\int_\Omega w\alpha A d\mu - \int_\Omega w\alpha d\mu\int_\Omega wAd\mu\right| &\le \left[\int_\Omega w\alpha^2 d\mu - \left(\int_\Omega w\alpha d\mu\right)^2\right]^{1/2}\left(\int_\Omega wA^2 d\mu - \left(\int_\Omega wAd\mu\right)^2\right)^{1/2} \\ &\le \frac{1}{2}(M-m)\left[\int_\Omega wA^2 d\mu - \left(\int_\Omega wAd\mu\right)^2\right]^{1/2}.\end{aligned} \tag{3.7}$$

Proof. We use the following Sonin type identity that can be proved by performing the calculations in the right side

$$\int_\Omega w\alpha A d\mu - \int_\Omega w\alpha d\mu\int_\Omega wAd\mu = \int_\Omega w\left(\alpha-\int_\Omega w\alpha d\mu\right)\left(A-\int_\Omega wAd\mu\right)d\mu.$$

By using (3.6) we have

$$\begin{aligned}&\left|\int_\Omega w\left(\alpha-\int_\Omega w\alpha d\mu\right)\left(A-\int_\Omega wAd\mu\right)d\mu\right| \\ &\le \left[\int_\Omega w\left(\alpha-\int_\Omega w\alpha d\mu\right)^2 d\mu\right]^{1/2}\left[\int_\Omega w\left(A-\int_\Omega wAd\mu\right)^2 d\mu\right]^{1/2}.\end{aligned} \tag{3.8}$$

Since

$$\begin{aligned}\int_\Omega w\left(\alpha-\int_\Omega w\alpha d\mu\right)^2 d\mu &= \int_\Omega w\alpha^2 d\mu - \left(\int_\Omega w\alpha d\mu\right)^2 \\ &\le \frac{1}{4}(M-m)^2\end{aligned} \tag{3.9}$$

and

$$\int_\Omega w\left(A-\int_\Omega wAd\mu\right)^2 d\mu = \int_\Omega wA^2 d\mu - \left(\int_\Omega wAd\mu\right)^2,$$

hence by (3.8) and (3.9) we derive the desired result (3.7). □

Corollary 2. *With the assumption of Theorem 2, we have*

$$\begin{aligned}0 &\le \int_\Omega w(s)|B\circ f|^2 d\mu(s) - \left|B\left(\int_\Omega wfd\mu\right)\right|^2 \\ &\le (M-m)\left[\int_\Omega w\left(\operatorname{Re}\left((B\circ f)^*(B'\circ f)\right)\right)^2 d\mu \right.\\ &\quad \left. -\left(\int_\Omega w\operatorname{Re}\left((B\circ f)^*(B'\circ f)\right)d\mu\right)^2\right]^{1/2}.\end{aligned} \tag{3.10}$$

Proof. Using (3.7) for $\alpha = f$, we get

$$\begin{aligned}
0 &\le \int_\Omega wf \operatorname{Re}\left((B\circ f)^* (B'\circ f)\right) d\mu \\
&\quad - \int_\Omega wf d\mu \int_\Omega w \operatorname{Re}\left((B\circ f)^* (B'\circ f)\right) d\mu (s) \Big] \\
&\le \frac{1}{2}(M-m) \left[\int_\Omega w \left(\operatorname{Re}\left((B\circ f)^* (B'\circ f)\right)\right)^2 d\mu \right. \\
&\quad \left. - \left(\int_\Omega w \operatorname{Re}\left((B\circ f)^* (B'\circ f)\right) d\mu\right)^2 \right]^{1/2}.
\end{aligned}$$

By utilising (3.1) we then obtain the desired result (3.10). □

We have the following reverse of Cauchy-Bunyakowsky-Schwarz inequality that is of interest in itself:

Lemma 3. *Assume that $f \in L_{2,w}(\Omega, \mu, H)$, then for all $v \in H$,*

$$\begin{aligned}
0 &\le \int_\Omega w(s) \|f(s)\|^2 d\mu(s) - \left\| \int_\Omega w(s) f(s) d\mu(s) \right\|^2 \\
&\le \int_\Omega w(s) \|f(s) - v\|^2 d\mu(s).
\end{aligned} \tag{3.11}$$

Proof. Observe that, for any $v \in H$

$$\begin{aligned}
0 &\le \int_\Omega w(s) \|f(s)\|^2 d\mu(s) - \left\| \int_\Omega w(s) f(s) d\mu(s) \right\|^2 \\
&= \int_\Omega w(s) \langle f(s), f(s)\rangle d\mu(s) \\
&\quad - \left\langle \int_\Omega w(s) f(s) d\mu(s), \int_\Omega w(s) f(s) d\mu(s) \right\rangle \\
&= \int_\Omega w(s) \left\langle f(s) - \int_\Omega w(u) f(u) d\mu(u), f(s) - v \right\rangle d\mu(s) =: K.
\end{aligned} \tag{3.12}$$

Therefore, by Schwarz inequality in Hilbert spaces and the CBS integral inequality, we have

$$\begin{aligned}
K &\le \int_\Omega w(s) \left| \left\langle f(s) - \int_\Omega w(u) f(u) d\mu(u), f(s) - v \right\rangle \right| d\mu(s) \\
&\le \int_\Omega w(s) \left\| f(s) - \int_\Omega w(u) f(u) d\mu(u) \right\| \|f(s) - v\| d\mu(s) \\
&\le \left(\int_\Omega w(s) \left\| f(s) - \int_\Omega w(u) f(u) d\mu(u) \right\|^2 d\mu(s) \right)^{1/2} \\
&\quad \times \left(\int_\Omega w(s) \|f(s) - v\|^2 d\mu(s) \right)^{1/2}.
\end{aligned} \tag{3.13}$$

Since, by the properties of inner product and integral,

$$\begin{aligned}
&\int_{\Omega} w(s) \left\| f(s) - \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2 d\mu(s) \\
&= \int_{\Omega} w(s) \left[\| f(s) \|^2 - 2 \operatorname{Re} \left\langle f(s), \int_{\Omega} w(u) f(u) \, d\mu(u) \right\rangle \right. \\
&+ \left. \left\| \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2 \right] d\mu(s) \\
&= \int_{\Omega} w(s) \| f(s) \|^2 d\mu(s) \\
&- 2 \operatorname{Re} \left\langle \int_{\Omega} w(s) f(s) \, d\mu(u), \int_{\Omega} w(u) f(u) \, d\mu(u) \right\rangle \\
&+ \left\| \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2 \\
&= \int_{\Omega} w(s) \| f(s) \|^2 d\mu(s) - 2 \left\| \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2 \\
&+ \left\| \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2 \\
&= \int_{\Omega} w(s) \| f(s) \|^2 d\mu(s) - \left\| \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2,
\end{aligned}$$

hence by (3.12) and (3.13) we get

$$\begin{aligned}
0 &\le \int_{\Omega} w(s) \| f(s) \|^2 d\mu(s) - \left\| \int_{\Omega} w(s) f(s) \, d\mu(s) \right\|^2 \\
&\le \left(\int_{\Omega} w(s) \| f(s) \|^2 d\mu(s) - \left\| \int_{\Omega} w(u) f(u) \, d\mu(u) \right\|^2 \right)^{1/2} \\
&\times \left(\int_{\Omega} w(s) \| f(s) - v \|^2 d\mu(s) \right)^{1/2},
\end{aligned}$$

which is equivalent to (3.11). □

We can also state the following simpler upper bound for the Jensen's gap:

Corollary 3. *With the assumption of Theorem 2, we have*

$$\begin{aligned}
(3.14) \qquad 0 &\le \int_{\Omega} w(s) |B \circ f|^2 d\mu(s) - \left| B \left(\int_{\Omega} w f d\mu \right) \right|^2 \\
&\le \frac{1}{2} (M - m) \left\| \left(\operatorname{Re} \left((B(M))^* B'_-(M) \right) - \operatorname{Re} \left((B(m))^* B'_+(m) \right) \right) \right\|
\end{aligned}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Observe that, by (2.4)

$$(3.15) \qquad \operatorname{Re} \left((B(m))^* B'_+(m) \right) \le \operatorname{Re} \left((B(t))^* B'(t) \right) \le \operatorname{Re} \left((B(M))^* B'_-(M) \right)$$

for almost every $t \in [m, M]$.

This implies that

$$\begin{aligned}\left\langle \operatorname{Re}\left((B(m))^* B'_+(m)\right)x,x\right\rangle &\le \left\langle \operatorname{Re}\left((B(t))^* B'(t)\right)x,x\right\rangle\\ &\le \left\langle \operatorname{Re}\left((B(M))^* B'_-(M)\right)x,x\right\rangle\end{aligned}$$

for all $x \in H$ and for almost every $t \in [m, M]$.

This inequality is equivalent to

$$\begin{aligned}&\Big|\Big\langle\Big(\operatorname{Re}\left((B(t))^* B'(t)\right)\\ &-\frac{1}{2}\operatorname{Re}\left((B(m))^* B'_+(m)\right)+\operatorname{Re}\left((B(M))^* B'_-(M)\right)\Big)x,x\Big\rangle\Big|\\ &\le \frac{1}{2}\left\langle\left[\left(\operatorname{Re}\left((B(M))^* B'_-(M)\right)-\operatorname{Re}\left((B(m))^* B'_+(m)\right)\right)\right]x,x\right\rangle\end{aligned}$$

for all $x \in H$ and for almost every $t \in [m, M]$.

By taking the supremum over $x \in H$, $\|x\| = 1$, we derive the norm inequality

$$\begin{aligned}&\left\|\operatorname{Re}\left((B(t))^* B'(t)\right)-\frac{\operatorname{Re}\left((B(m))^* B'_+(m)\right)+\operatorname{Re}\left((B(M))^* B'_-(M)\right)}{2}\right\|\\ &\le \frac{1}{2}\left\|\left(\operatorname{Re}\left((B(M))^* B'_-(M)\right)-\operatorname{Re}\left((B(m))^* B'_+(m)\right)\right)\right\|.\end{aligned}$$

Observe that for $x \in H$, we have

$$\begin{aligned}&\left\|\operatorname{Re}\left((B(t))^* B'(t)\right)x-\frac{\operatorname{Re}\left((B(m))^* B'_+(m)\right)x+\operatorname{Re}\left((B(M))^* B'_-(M)\right)x}{2}\right\|^2\\ &=\Big\|\Big(\operatorname{Re}\left((B(t))^* B'(t)\right)\\ &-\frac{1}{2}\operatorname{Re}\left((B(m))^* B'_+(m)\right)+\operatorname{Re}\left((B(M))^* B'_-(M)\right)\Big)x\Big\|^2\\ &\le\Big\|\Big(\operatorname{Re}\left((B(t))^* B'(t)\right)\\ &-\frac{1}{2}\operatorname{Re}\left((B(m))^* B'_+(m)\right)+\operatorname{Re}\left((B(M))^* B'_-(M)\right)\Big)\Big\|^2\|x\|^2\\ &\le\frac{1}{4}\left\|\left(\operatorname{Re}\left((B(M))^* B'_-(M)\right)-\operatorname{Re}\left((B(m))^* B'_+(m)\right)\right)\right\|^2\|x\|^2.\end{aligned}$$

From (3.11) we get for

$$v=\frac{\operatorname{Re}\left((B(m))^* B'_+(m)\right)x+\operatorname{Re}\left((B(M))^* B'_-(M)\right)x}{2}$$

that

$$
\begin{aligned}
(3.16)\qquad 0 &\leq \int_{\Omega} w(s)\left\|\operatorname{Re}\left((B(t))^{*} B'(t)\right) x\right\|^{2} d\mu(s) \\
&-\left\|\int_{\Omega} w(s) \operatorname{Re}\left((B(t))^{*} B'(t)\right) x d\mu(s)\right\|^{2} \\
&\leq \int_{\Omega} w(s) \left\| \operatorname{Re}\left((B(t))^{*} B'(t)\right) x \right.\\
&\left. -\frac{1}{2}\left[\operatorname{Re}\left((B(m))^{*} B'_{+}(m)\right) x+\operatorname{Re}\left((B(M))^{*} B'_{-}(M)\right)\right] x\right\|^{2} d\mu(s) \\
&\leq \frac{1}{4}\left\|\left(\operatorname{Re}\left((B(M))^{*} B'_{-}(M)\right)-\operatorname{Re}\left((B(m))^{*} B'_{+}(m)\right)\right)\right\|^{2}\|x\|^{2}
\end{aligned}
$$

for $x \in H$.

Since

$$
\begin{aligned}
&\int_{\Omega} w(s)\left\|\operatorname{Re}\left((B(t))^{*} B'(t)\right) x\right\|^{2} d\mu(s) \\
&-\left\|\int_{\Omega} w(s) \operatorname{Re}\left((B(t))^{*} B'(t)\right) x d\mu(s)\right\|^{2} \\
&=\left\langle\int_{\Omega} w(s)\left|\operatorname{Re}\left((B(t))^{*} B'(t)\right)\right|^{2} d\mu(s) x, x\right\rangle \\
&-\left\langle\left|\int_{\Omega} w(s) \operatorname{Re}\left((B(t))^{*} B'(t)\right) d\mu(s)\right|^{2} x, x\right\rangle,
\end{aligned}
$$

hence

$$
\begin{aligned}
0 &\leq \int_{\Omega} w(s)\left(\operatorname{Re}\left((B(t))^{*} B'(t)\right)\right)^{2} d\mu(s) \\
&-\left(\int_{\Omega} w(s) \operatorname{Re}\left((B(t))^{*} B'(t)\right) d\mu(s)\right)^{2} \\
&\leq \frac{1}{4}\left\|\left(\operatorname{Re}\left((B(M))^{*} B'_{-}(M)\right)-\operatorname{Re}\left((B(m))^{*} B'_{+}(m)\right)\right)\right\|^{2},
\end{aligned}
$$

which, by taking the square root in the operator inequality, gives that

$$
\begin{aligned}
0 &\leq \left[\int_{\Omega} w(s)\left(\operatorname{Re}\left((B(t))^{*} B'(t)\right)\right)^{2} d\mu(s)\right. \\
&\left.-\left(\int_{\Omega} w(s) \operatorname{Re}\left((B(t))^{*} B'(t)\right) d\mu(s)\right)^{2}\right]^{1/2} \\
&\leq \frac{1}{2}\left\|\left(\operatorname{Re}\left((B(M))^{*} B'_{-}(M)\right)-\operatorname{Re}\left((B(m))^{*} B'_{+}(m)\right)\right)\right\| .
\end{aligned}
$$

By (3.10) we derive (3.14). □

Now, if we consider the discrete measure, then for $B : [m, M] \subset \mathbb{R} \to \mathcal{B}(H)$, a square modulus convex function on $[m, M]$ that is also strongly differentiable on (m, M), $t_i \in [m, M]$, $w_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$, then by (3.1) we get

$$0 \leq \sum_{i=1}^{n} w_i |B(t_i)|^2 - \left| B\left(\sum_{i=1}^{n} w_i t_i\right)\right|^2 \tag{3.17}$$
$$\leq 2\left[\sum_{i=1}^{n} w_i t_i \operatorname{Re}\left[(B(t_i))^* (B'(t_i))\right] - \sum_{i=1}^{n} w_i t_i \sum_{i=1}^{n} w_i \operatorname{Re}\left[(B(t_i))^* (B'(t_i))\right]\right].$$

From (3.10) we get

$$0 \leq \sum_{i=1}^{n} w_i |B(t_i)|^2 - \left| B\left(\sum_{i=1}^{n} w_i t_i\right)\right|^2 \tag{3.18}$$
$$\leq (M-m)\left[\sum_{i=1}^{n} w_i \left(\operatorname{Re}\left[(B(t_i))^* (B'(t_i))\right]\right)^2 d\mu - \left(\sum_{i=1}^{n} w_i \operatorname{Re}\left[(B(t_i))^* (B'(t_i))\right] d\mu\right)^2\right]^{1/2},$$

while from (3.14) we derive

$$0 \leq \sum_{i=1}^{n} w_i |B(t_i)|^2 - \left| B\left(\sum_{i=1}^{n} w_i t_i\right)\right|^2 \tag{3.19}$$
$$\leq \frac{1}{2}(M-m)\left\|\left(\operatorname{Re}\left((B(M))^* B'_-(M)\right) - \operatorname{Re}\left((B(m))^* B'_+(m)\right)\right)\right\|.$$

4. An Examples

For distinct operators $A, B \in \mathcal{B}(H)$ we consider the function $\varphi_{A,B} : [0,1] \to \mathcal{B}(H)$ defined by $\varphi_{A,B}(t) = |(1-t)A + tB|^2$. Let $t \in (0,1)$ and small $h \neq 0$ such that $t + h \in (0,1)$, then we have

$$\begin{aligned}
&\varphi_{A,B}(t+h) \\
&= |(1-t-h)A + (t+h)B|^2 \\
&= (1-t-h)^2 |A|^2 + (1-t-h)(t+h)(A^*B + B^*A) + (t+h)^2 |B|^2
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{A,B}(t) &= |(1-t)A + tB|^2 \\
&= (1-t)^2 |A|^2 + (1-t)t(A^*B + B^*A) + t^2 |B|^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\varphi_{A,B}(t+h)-\varphi_{A,B}(t) \\
&=\left[(1-t-h)^2-(1-t)^2\right]|A|^2 \\
&+[(1-t-h)(t+h)-(1-t)t](A^*B+B^*A)+\left[(t+h)^2-t^2\right]|B|^2 \\
&=h[-2(1-t)+h]|A|^2+h(1-2t-h)(A^*B+B^*A)+h(h+2t)|B|^2,
\end{aligned}$$

which gives that

$$\begin{aligned}
&\frac{1}{h}\left[\varphi_{A,B}(t+h)-\varphi_{A,B}(t)\right] \\
&=[-2(1-t)+h]|A|^2+(1-2t-h)(A^*B+B^*A)+(h+2t)|B|^2
\end{aligned}$$

for small $h\neq 0$.

Taking the strong limit over $h\to 0$, we get

$$\begin{aligned}
\varphi'_{A,B}(t)&=-2(1-t)|A|^2+(1-2t)(A^*B+B^*A)+2t|B|^2 \\
&=2t\left[|A|^2-(A^*B+B^*A)+|B|^2\right]+A^*B+B^*A-2|A|^2 \\
&=2t|A-B|^2+A^*B+B^*A-|A|^2-|B|^2+|B|^2-|A|^2 \\
&=2t|A-B|^2-|A-B|^2+|B|^2-|A|^2 \\
&=(2t-1)|A-B|^2+|B|^2-|A|^2
\end{aligned}$$

for $t\in(0,1)$.

We also have the lateral derivatives

$$\varphi'_{+A,B}(0)=|B|^2-|A|^2-|A-B|^2$$

and

$$\varphi'_{-A,B}(1)=|A-B|^2+|B|^2-|A|^2.$$

Consider the μ-measurable function $f:\Omega\to[0,1]$ such that $f\in L_{2,w}(\Omega,\mu)$. Then by using the inequality (3.14) we derive

$$\begin{aligned}
(4.1)\qquad &0\leq\int_\Omega w(s)|(1-f(s))A+f(s)B|^2\,d\mu(s) \\
&-\left|\left(1-\int_\Omega w(s)f(s)\,d\mu(s)\right)A+\left(\int_\Omega w(s)f(s)\,d\mu(s)\right)B\right|^2 \\
&\leq\|A-B\|^2
\end{aligned}$$

for all distinct A, $B\in\mathcal{B}(H)$.

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