# MULTIPLICATIVE GRÜSS' TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach $C^*$-algebra of bounded linear operators on Hilbert space $H$. For $A \in \mathcal{B}(H)$ we define the modulus of $A$ by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if $U$ is an unitary operator and $B : \Omega \to \mathcal{B}(H)$ is strongly $\mu$-measurable with $BU \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$ and such that

$$\left|B(s)U - \frac{\gamma + \Gamma}{2}U\right|^2 \le \frac{1}{4}|\Gamma - \gamma|^2 1_H \text{ for } \mu\text{-a.e. } s \in \Omega$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^* - B^*(s)\right)(B(s) - \gamma U)\right] \ge 0 \text{ for } \mu\text{-a.e. } s \in \Omega$$

for some complex constants $\gamma$, $\Gamma$ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then

$$\left(\int_\Omega w(s)|B(s)U|^2 d\mu(s)\right)^{1/2} \le \frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^*\left(\int_\Omega w(s)B(s)d\mu(s)\right)U\right|.$$

Applications for finite Fourier Transform are also given.

## 1. Introduction

For two Lebesgue integrable functions $f$, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt.$$

In 1934, G. Grüss [11] showed that

$$|D(f,g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided $m$, $M$, $n$, $N$ are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on } [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by the author in [2]:

**Theorem 1.** *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If $\varphi$, $\phi$, $\gamma$, $\Gamma \in \mathbb{K}$ and $x$, $y \in H$ are such that*

$$\operatorname{Re}\langle\phi e - x, x - \varphi e\rangle \ge 0 \quad and \quad \operatorname{Re}\langle\Gamma e - y, y - \gamma e\rangle \ge 0 \tag{1.3}$$

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*or, equivalently (see* [4]*)*

$$\left\|x-\frac{\varphi+\phi}{2}e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right| \quad and \quad \left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.4}$$

*then*

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leq\frac{1}{4}\left|\phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{1.5}$$

*The constant* $\frac{1}{4}$ *is best possible in (1.5).*

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by the author in 2001, [3].

**Theorem 2.** *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex Hilbert space,* $\Omega\subset\mathbb{R}^n$ *be a Lebesgue measurable set and* $\rho:\Omega\to[0,\infty)$ *a Lebesgue measurable function with* $\int_\Omega\rho(s)\,ds=1$. *We denote by* $L_{2,\rho}(\Omega,H)$ *the set of all Bochner measurable functions* $f$ *on* $\Omega$ *such that* $\|f\|_{2,\rho}^2:=\int_\Omega\rho(s)\|f(s)\|^2ds<\infty$. *If* $f$, $g$ *belong to* $L_{2,\rho}(\Omega,H)$ *and there exist the vectors* $x$, $X$, $y$, $Y\in H$ *such that*

$$\begin{aligned}&\int_\Omega\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt\geq0,\\&\int_\Omega\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt\geq0,\end{aligned} \tag{1.6}$$

*then we have the inequality*

$$\begin{aligned}&\left|\int_\Omega\rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega\rho(t)f(t)\,dt,\int_\Omega\rho(t)g(t)\,dt\right\rangle\right|\\&\leq\frac{1}{4}\|X-x\|\|Y-y\|.\end{aligned} \tag{1.7}$$

*The constant* $\frac{1}{4}$ *is sharp in the sense mentioned above.*

**Remark 1.** *A practical sufficient condition for (1.6) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle\geq0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle\geq0$$

*or, equivalently*

$$\left\|f(t)-\frac{X+x}{2}\right\|\leq\frac{1}{2}\|X-x\| \ \ and \ \left\|g(t)-\frac{Y+y}{2}\right\|\leq\frac{1}{2}\|Y-y\|,$$

*for a.e.* $t\in\Omega$.

For related results, see [1], [4]-[10] and [12]-[13].

Denote by $\mathcal{B}(H)$ the Banach $C^*$-algebra of bounded linear operators on Hilbert space $H$. For $A\in\mathcal{B}(H)$ we define the modulus of $A$ by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

We have the following Cauchy-Bunyakowsky-Schwarz inequality:

If $\alpha\in L_w^2(\Omega)$ and

$$A\in L_{2,w}(\Omega,\mathcal{B}(H)):=\left\{A:\Omega\to B(H),\ \int_\Omega w(x)\|A(x)\|^2d\mu(x)<\infty\right\},$$

then

$$\int_{\Omega} w(x)|\alpha(x)|^2 d\mu(x) \int_{\Omega} w(x)|A(x)|^2 d\mu(x) \geq \left|\int_{\Omega} w(x)\alpha(x)A(x)d\mu(x)\right|^2, \tag{1.8}$$

in the operator order of $\mathcal{B}(H)$.

Indeed, we have for $\alpha \in L^2_w(\Omega)$ and $A \in L_{2,w}(\Omega, \mathcal{B}(H))$,

$$0 \leq \left|\overline{\alpha(x)}A(y) - \overline{\alpha(y)}A(x)\right|^2 = |\alpha(x)||A(y)|^2 - \alpha(y)\overline{\alpha(x)}A^*(x)A(y) - \alpha(x)\overline{\alpha(y)}A^*(y)A(x) + |\alpha(y)|^2|A(x)|^2,$$

which gives that

$$|\alpha(x)|^2|A(y)|^2 + |\alpha(y)|^2|A(x)|^2 \geq \alpha(y)\overline{\alpha(x)}A^*(x)A(y) + \alpha(x)\overline{\alpha(y)}A^*(y)A(x)$$

for all $y, x \in \Omega$.

Now, multiply this with $w(y)w(x) \geq 0$ to get

$$w(x)|\alpha(x)|^2 w(y)|A(y)|^2 + w(y)|\alpha(y)|^2 w(x)|A(x)|^2 \geq w(x)\overline{\alpha(x)}A^*(x)w(y)\alpha(y)A(y) + w(y)\overline{\alpha(y)}A^*(y)w(x)\alpha(x)A(x)$$

for all $y, x \in \Omega$.

Integrating over $x$ and $y$ on $\Omega$, then we get

$$\begin{aligned} &\int_a^b w(x)|\alpha(x)|^2 d\mu(x) \int_a^b |A(y)|^2 d\mu(y) \\ &+ \int_a^b |\alpha(y)|^2 d\mu(y) \int_a^b w(x)|A(x)|^2 d\mu(x) \\ &\geq \int_a^b w(x)\overline{\alpha(x)}A^*(x)d\mu(x) \int_a^b \alpha(y)A(y)d\mu(y) \\ &+ \int_a^b w(y)\overline{\alpha(y)}A^*(y)d\mu(y) \int_a^b \alpha(x)A(x)d\mu(x) \\ &= 2\left|\int_a^b w(y)\alpha(y)A(y)d\mu(y)\right|^2, \end{aligned}$$

and the inequality (1.8) is obtained.

In this paper we show among others that, if $U$ is an unitary operator, i.e., $U^*U = 1_H$ and $B : \Omega \to \mathcal{B}(H)$ is strongly $\mu$-measurable with $BU \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$ and such that

$$\left|B(s)U - \frac{\gamma+\Gamma}{2}U\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2 1_H \text{ for } \mu\text{-a.e. } s \in \Omega$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^* - B^*(s)\right)(B(s) - \gamma U)\right] \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega$$

for some complex constants $\gamma$, $\Gamma$ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then

$$\left(\int_{\Omega} w(s)|B(s)U|^2 d\mu(s)\right)^{1/2} \leq \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^*\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right|.$$

Applications for finite Fourier Transform are also given.

## 2. Main Results

We have the following identity of interest:

**Lemma 1.** *For any* $A$, $X$, $Y \in \mathcal{B}(H)$, *we have*

$$\left|A-\frac{X+Y}{2}\right|^{2}-\frac{1}{4}|X-Y|^{2}=\operatorname{Re}\left[\left(A^{*}-X^{*}\right)(A-Y)\right]. \tag{2.1}$$

*Proof.* We have

$$\begin{aligned}
&\left|A-\frac{X+Y}{2}\right|^{2}-\frac{1}{4}|X-Y|^{2}\\
&=|A|^{2}-\frac{X^{*}+Y^{*}}{2}A-A^{*}\frac{X+Y}{2}+\frac{1}{4}\left(|X|^{2}+X^{*}Y+Y^{*}X+|Y|^{2}\right)\\
&-\frac{1}{4}\left(|X|^{2}-X^{*}Y-Y^{*}X+|Y|^{2}\right)\\
&=|A|^{2}-\frac{X^{*}+Y^{*}}{2}A-A^{*}\frac{X+Y}{2}+\frac{1}{2}\left(X^{*}Y+Y^{*}X\right)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\left[\left(A^{*}-X^{*}\right)(A-Y)\right]\\
&=\operatorname{Re}\left[|A|^{2}-X^{*}A-A^{*}Y+X^{*}Y\right]\\
&=|A|^{2}-\operatorname{Re}\left(X^{*}A\right)-\operatorname{Re}\left(A^{*}Y\right)+\operatorname{Re}\left(X^{*}Y\right)\\
&=|A|^{2}-\frac{1}{2}\left(X^{*}A+A^{*}X\right)-\frac{1}{2}\left(A^{*}Y+Y^{*}A\right)+\frac{1}{2}\left(X^{*}Y+Y^{*}X\right)\\
&=|A|^{2}-\frac{1}{2}\left(X^{*}+Y^{*}\right)A-\frac{1}{2}A^{*}(X+Y)+\frac{1}{2}\left(X^{*}Y+Y^{*}X\right),
\end{aligned}$$

which proves the desired identity (2.1). □

**Corollary 1.** *Let* $A$, $X$, $Y \in \mathcal{B}(H)$. *The following statements are equivalent*

$$\left|A-\frac{X+Y}{2}\right|^{2}\leq\frac{1}{4}|X-Y|^{2}$$

*and*

$$\operatorname{Re}\left[\left(X^{*}-A^{*}\right)(A-Y)\right]\geq 0.$$

We have the following reverse inequality of Cauchy-Bunyakowsky-Schwarz integral inequality for operator modulus:

**Theorem 3.** *Let* $U$ *be an unitary operator and* $B:\Omega\to\mathcal{B}(H)$ *strongly* $\mu$-*measurable with* $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ *and such that*

$$\left|B(s)U-\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}|\Gamma-\gamma|^{2}1_{H}\text{ for }\mu\text{-a.e. }s\in\Omega \tag{2.2}$$

*or, equivalently*

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^{*}-B^{*}(s)\right)(B(s)-\gamma U)\right]\geq 0\text{ for }\mu\text{-a.e. }s\in\Omega \tag{2.3}$$

*for some complex constants* $\gamma$, $\Gamma$ *with* $\operatorname{Re}(\Gamma\overline{\gamma})>0$. *Then*

$$\left(\int_{\Omega}w(s)\left|B(s)U\right|^{2}d\mu(s)\right)^{1/2} \leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^{*}\left(\int_{\Omega}w(s)B(s)d\mu(s)\right)U\right] \leq \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^{*}\left(\int_{\Omega}w(s)B(s)d\mu(s)\right)U\right|. \tag{2.4}$$

*Proof.* The equivalence of the statements (2.2) and (2.3) follows by Corollary 1 for $A=B(s)$, $X=\Gamma U$ and $Y=\gamma U$ and taking into account that $|U|^{2}=1_{H}$.

By the properties of operator modulus, we have

$$\left|B(s)U\right|^{2}-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}U\right)^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H},$$

namely

$$\left|B(s)U\right|^{2}-2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H},$$

or

$$\left|B(s)U\right|^{2}+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}-\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}\leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right], \tag{2.5}$$

for $\mu$-a.e. $s\in\Omega$.

Observe that

$$\frac{1}{4}\left|\Gamma+\gamma\right|^{2}-\frac{1}{4}\left|\Gamma-\gamma\right|^{2}=\frac{1}{4}\left(\left|\Gamma\right|^{2}+2\operatorname{Re}(\Gamma\overline{\gamma})+\left|\gamma\right|^{2}\right)-\frac{1}{4}\left(\left|\Gamma\right|^{2}-2\operatorname{Re}(\Gamma\overline{\gamma})+\left|\gamma\right|^{2}\right)=\operatorname{Re}(\Gamma\overline{\gamma})>0,$$

then by (2.5) we get

$$\left|B(s)U\right|^{2}+\operatorname{Re}(\Gamma\overline{\gamma})1_{H}\leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right], \tag{2.6}$$

for $\mu$-a.e. $s\in\Omega$.

If we multiply (2.6) by $w(s)\geq 0$ and integrate, then we get

$$\int_{\Omega}w(s)\left|B(s)U\right|^{2}d\mu(s)+\operatorname{Re}(\Gamma\overline{\gamma})1_{H} \leq 2\int_{\Omega}w(s)\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]d\mu(s) = 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}\left(\int_{\Omega}w(s)B(s)d\mu(s)\right)U\right]. \tag{2.7}$$

Using the elementary operator inequality

$$2aA\leq A^{2}+a^{2}1_{H},$$

where $A \geq 0$ in the operator order and $a \geq 0$, then we also have

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\left(\int_{\Omega} w(s)\left|B(s)U\right|^{2} d\mu(s)\right)^{1/2} \leq \int_{\Omega} w(s)\left|B(s)U\right|^{2} d\mu(s) + \operatorname{Re}(\Gamma\overline{\gamma})1_{H}, \tag{2.8}$$

and by (2.7) and (2.8) we derive the first part of (2.4).

For an operator $T$ we consider the selfadjoint operators

$$\operatorname{Re}(T) := \frac{T^{*}+T}{2},\ \ \operatorname{Im}(T) := \frac{T-T^{*}}{2i}.$$

Then $T = \operatorname{Re}(T) + i\operatorname{Im}(T)$, $|T|^{2} = (\operatorname{Re}(T))^{2} + (\operatorname{Im}(T))^{2}$. We have $|T|^{2} \geq (\operatorname{Re}(T))^{2}$ which implies, by taking the square root, that $|T| \geq |\operatorname{Re}(T)|$.

Therefore

$$\begin{aligned}
0 &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right] \\
&\leq \left|\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right| \\
&= \left|\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\right|\left|U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right| \\
&= \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right|,
\end{aligned}$$

and the last part of (2.4) is thus proved. □

**Remark 2.** *Observe that for $z = \alpha + i\beta$ and $A \in \mathcal{B}(H)$, we have*

$$\begin{aligned}
\operatorname{Re}(\overline{z}A) &= \operatorname{Re}\left[(\alpha - i\beta)(\operatorname{Re}A + i\operatorname{Im}A)\right] \\
&= \operatorname{Re}\left[\alpha\operatorname{Re}A + \beta\operatorname{Im}A - i(\beta\operatorname{Re}A - \alpha\operatorname{Im}A)\right] \\
&= \alpha\operatorname{Re}A + \beta\operatorname{Im}A = \operatorname{Re}z\operatorname{Re}A + \operatorname{Im}z\operatorname{Im}A
\end{aligned}$$

*and then*

$$\begin{aligned}
&\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right] \\
&= \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\left[U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right] \\
&+ \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\left[U^{*}\left(\int_{\Omega} w(s)B(s)d\mu(s)\right)U\right] \\
&= \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\operatorname{Re}(\gamma+\Gamma)U^{*}\left(\int_{\Omega} w(s)\operatorname{Re}(B(s))d\mu(s)\right)U \\
&+ \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\operatorname{Im}(\gamma+\Gamma)U^{*}\left(\int_{\Omega} w(s)\operatorname{Im}(B(s))d\mu(s)\right)U.
\end{aligned}$$

*Therefore by (2.4) we have the unpacked inequality*

$$\begin{aligned}(2.9)\qquad &\left(\int_\Omega w(s)\left|B(s)U\right|^2 d\mu(s)\right)^{1/2}\\ &\le \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left[\operatorname{Re}(\gamma+\Gamma)U^*\left(\int_\Omega w(s)\operatorname{Re}(B(s))\,d\mu(s)\right)U\right.\\ &\left.+\operatorname{Im}(\gamma+\Gamma)U^*\left(\int_\Omega w(s)\operatorname{Im}(B(s))\,d\mu(s)\right)U\right]\\ &\le \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|.\end{aligned}$$

*From (2.9) we also derive the additive reverse of Cauchy-Bunyakowsky-Schwarz inequality*

$$\begin{aligned}(2.10)\qquad 0&\le \left(\int_\Omega w(s)\left|B(s)U\right|^2 d\mu(s)\right)^{1/2}-\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|\\ &\le \frac{|\gamma+\Gamma|-2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|,\end{aligned}$$

*provided that $B$ satisfies the conditions from Theorem 3.*

**Corollary 2.** *Let $U$ be an unitary operator and $B:\Omega\to\mathcal{B}(H)$ strongly $\mu$-measurable with $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that*

$$(2.11)\qquad \left|B(s)U-\frac{m+M}{2}U\right|^2\le\frac{1}{4}(M-m)^2 1_H \text{ for } \mu\text{-a.e. } s\in\Omega$$

*or, equivalently*

$$(2.12)\qquad \operatorname{Re}\left[(MU^*-B^*(s))(B(s)-mU)\right]\ge 0 \text{ for } \mu\text{-a.e. } s\in\Omega$$

*for some real numbers $M>m>0$. Then*

$$\begin{aligned}(2.13)\qquad &\left(\int_\Omega w(s)\left|B(s)U\right|^2 d\mu(s)\right)^{1/2}\\ &\le \frac{m+M}{2\sqrt{mM}}\operatorname{Re}\left[U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right]\\ &\le \frac{m+M}{2\sqrt{Mm}}\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|.\end{aligned}$$

*Also, we have the additive inequality*

$$\begin{aligned}(2.14)\qquad 0&\le \left(\int_\Omega w(s)\left|B(s)U\right|^2 d\mu(s)\right)^{1/2}-\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|\\ &\le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}}\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|.\end{aligned}$$

## 3. Grüss Type Inequalities

We have the following result that is of interest in itself:

**Lemma 2.** *Assume that* $\alpha \in L_w^2(\Omega, \mu, \mathbb{C})$ *and* $A \in L_2(\Omega, \mu, \mathcal{B}(H))$, *then*

$$\left| \int_\Omega w(s)\alpha(s) A(s) d\mu(s) - \int_\Omega w(s)\alpha(s) d\mu(s) \int_\Omega w(s) A(s) d\mu(s) \right|^2 \tag{3.1}$$
$$\leq \left( \int_\Omega w(s) |\alpha(s)|^2 d\mu(s) - \left| \int_\Omega w(s)\alpha(s) d\mu(s) \right|^2 \right)$$
$$\times \left( \int_\Omega w(s) |A(s)|^2 d\mu(s) - \left| \int_\Omega w(s) A(s) d\mu(s) \right|^2 \right)$$
$$\leq \left[ \left( \int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \right)^{1/2} \left( \int_\Omega w(s) |A(s)|^2 d\mu(s) \right)^{1/2} \right.$$
$$\left. - \left| \int_\Omega w(s)\alpha(s) d\mu(s) \right| \left| \int_\Omega w(s) A(s) d\mu(s) \right| \right]^2.$$

*Proof.* We use the following Sonin type identity that can be proved by performing the calculations in the right side

$$\int_\Omega w(s)\alpha(s) A(s) d\mu(s) - \int_\Omega w(s)\alpha(s) d\mu(s) \int_\Omega w(s) A(s) d\mu(s)$$
$$= \int_\Omega w(s) \left( \alpha(s) - \int_\Omega w(t)\alpha(t) d\mu(t) \right)$$
$$\times \left( A(s) - \int_\Omega w(t) A(t) d\mu(t) \right) d\mu(s).$$

By using (1.8) we have

$$\left| \int_\Omega w(s) \left( \alpha(s) - \int_\Omega w(t)\alpha(t) d\mu(t) \right) \right. \tag{3.2}$$
$$\left. \times \left( A(s) - \int_\Omega w(t) A(t) d\mu(t) \right) d\mu(s) \right|^2$$
$$\leq \left[ \int_\Omega w(s) \left| \alpha(s) - \int_\Omega w(t)\alpha(t) d\mu(t) \right|^2 d\mu(s) \right]$$
$$\times \left[ \int_\Omega w(s) \left| A(s) - \int_\Omega w(t) A(t) d\mu(t) \right|^2 d\mu(t) \right].$$

Since

$$\int_\Omega w(s) \left| \alpha(s) - \int_\Omega w(t)\alpha(t) d\mu(t) \right|^2 d\mu(s) \tag{3.3}$$
$$= \int_\Omega w(s) |\alpha(s)|^2 d\mu(s) - \left| \int_\Omega w(s)\alpha(s) d\mu(s) \right|^2$$

and

$$\int_\Omega w(s) \left| A(s) - \int_\Omega w(t) A(t) d\mu(t) \right|^2 d\mu(t) \tag{3.4}$$
$$= \int_\Omega w(s) |A(s)|^2 d\mu(s) - \left| \int_\Omega w(s) A(s) d\mu(s) \right|^2,$$

hence by (3.2), (3.3) and (3.4) we derive the first part of (3.1).

Now, observe that for real numbers $a$, $b$ and selfadjoint operators $A$, $B$ we have the operator inequality

$$\left(b^2-a^2\right)\left(B^2-A^2\right)\le (bB-aA)^2 .$$

Indeed, we have

$$\begin{aligned}&(bB-aA)^2-\left(b^2-a^2\right)\left(B^2-A^2\right)\\&=b^2B^2-ba\left(AB+BA\right)+a^2A^2-b^2B^2+a^2B^2+b^2A^2-a^2A^2\\&=a^2B^2+b^2A^2-ba\left(AB+BA\right)=(aB-bA)^2\ge 0.\end{aligned}$$

Therefore by

$$b=\left(\int_\Omega w(s)\left|\alpha(s)\right|^2 d\mu(s)\right)^{1/2},\ a=\left|\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\right|$$

and

$$B=\left(\int_\Omega w(s)\left|A(s)\right|^2 d\mu(s)\right)^{1/2},\ A=\left|\int_\Omega w(s)\,A(s)\,d\mu(s)\right|$$

we deduce the second part of (3.1). □

By taking the square root in (3.1) we also have:

**Corollary 3.** *With the assumptions of Lemma 2 we have*

$$\begin{aligned}&\left|\int_\Omega w(s)\,\alpha(s)\,A(s)\,d\mu(s)-\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\int_\Omega w(s)\,A(s)\,d\mu(s)\right|\\&\le\left(\int_\Omega w(s)\left|\alpha(s)\right|^2 d\mu(s)\right)^{1/2}\left(\int_\Omega w(s)\left|A(s)\right|^2 d\mu(s)\right)^{1/2}\\&-\left|\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\right|\left|\int_\Omega w(s)\,A(s)\,d\mu(s)\right|.\end{aligned}\tag{3.5}$$

We have the following Gruss' type inequality

**Theorem 4.** *Assume that $B:\Omega\to\mathcal{B}(H)$ is strongly $\mu$-measurable with $B\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that either*

$$\left|B(s)-\frac{\gamma+\Gamma}{2}1_H\right|^2\le\frac{1}{4}\left|\Gamma-\gamma\right|^2 1_H \text{ for } \mu\text{-a.e. } s\in\Omega\tag{3.6}$$

*or, equivalently*

$$\operatorname{Re}\left[\left(\overline{\Gamma}1_H-B^*(s)\right)\left(B(s)-\gamma 1_H\right)\right]\ge 0 \text{ for } \mu\text{-a.e. } s\in\Omega\tag{3.7}$$

*for some complex constants $\gamma,\Gamma$ with $\operatorname{Re}(\Gamma\overline{\gamma})>0$. Also assume that $\alpha\in L_w^2(\Omega,\mu,\mathbb{C})$ satisfies either the condition*

$$\left|\alpha(s)-\frac{\delta+\Delta}{2}\right|^2\le\frac{1}{4}\left|\Delta-\delta\right|^2 \text{ for } \mu\text{-a.e. } s\in\Omega\tag{3.8}$$

*or, equivalently*

$$\operatorname{Re}\left[\left(\overline{\Delta}-\overline{\alpha(s)}\right)\left(\alpha(s)-\delta\right)\right]\ge 0 \text{ for } \mu\text{-a.e. } s\in\Omega\tag{3.9}$$

*for some complex constants* $\delta$, $\Delta$ *with* $\operatorname{Re}\left(\Delta\overline{\delta}\right)>0$. *Then*

$$\left|\int_{\Omega}w(s)\alpha(s)B(s)d\mu(s)-\int_{\Omega}w(s)\alpha(s)d\mu(s)\int_{\Omega}w(s)B(s)d\mu(s)\right|^{2}\tag{3.10}$$
$$\leq\left(\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}}-1\right)^{2}$$
$$\times\left|\int_{\Omega}w(s)\alpha(s)d\mu(s)\right|^{2}\left|\int_{\Omega}w(s)B(s)d\mu(s)\right|^{2}.$$

*Proof.* From (3.5) we get

$$\left|\int_{\Omega}w(s)\alpha(s)B(s)d\mu(s)-\int_{\Omega}w(s)\alpha(s)d\mu(s)\int_{\Omega}w(s)B(s)d\mu(s)\right|^{2}\tag{3.11}$$
$$\leq\left[\left(\int_{\Omega}w(s)|\alpha(s)|^{2}d\mu(s)\right)^{1/2}\left(\int_{\Omega}w(s)|B(s)|^{2}d\mu(s)\right)^{1/2}\right.$$
$$\left.-\left|\int_{\Omega}w(s)\alpha(s)d\mu(s)\right|\left|\int_{\Omega}w(s)B(s)d\mu(s)\right|\right]^{2}=:K.$$

From (2.4) we have for $U=1_{H}$ that

$$\left(\int_{\Omega}w(s)|B(s)|^{2}d\mu(s)\right)^{1/2}\leq\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|\int_{\Omega}w(s)B(s)d\mu(s)\right|.\tag{3.12}$$

Also, we have the scalar inequality

$$\left(\int_{\Omega}w(s)|\alpha(s)|^{2}d\mu(s)\right)^{1/2}\leq\frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}}\left|\int_{\Omega}w(s)\alpha(s)d\mu(s)\right|.\tag{3.13}$$

Therefore

$$K\leq\left[\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}}\left|\int_{\Omega}w(s)\alpha(s)d\mu(s)\right|\left|\int_{\Omega}w(s)B(s)d\mu(s)\right|\right.$$
$$\left.-\left|\int_{\Omega}w(s)\alpha(s)d\mu(s)\right|\left|\int_{\Omega}w(s)B(s)d\mu(s)\right|\right]^{2}$$
$$=\left(\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}}-1\right)^{2}$$
$$\times\left|\int_{\Omega}w(s)\alpha(s)d\mu(s)\right|^{2}\left|\int_{\Omega}w(s)B(s)d\mu(s)\right|^{2},$$

which proves the desired result (3.10). □

**Remark 3.** *If the conditions (3.6)-(3.9) hold for $\gamma = \delta = \phi$ and $\Gamma = \Delta = \Phi$, then we have the simpler bounds*

$$\left| \int_\Omega w(s)\,\alpha(s)\,B(s)\,d\mu(s) - \int_\Omega w(s)\,\alpha(s)\,d\mu(s) \int_\Omega w(s)\,B(s)\,d\mu(s) \right|^2 \leq \left( \frac{|\Phi - \phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} \right)^2 \left| \int_\Omega w(s)\,\alpha(s)\,d\mu(s) \right|^2 \left| \int_\Omega w(s)\,B(s)\,d\mu(s) \right|^2. \tag{3.14}$$

*Observe that for $\gamma = \delta = \phi$ and $\Gamma = \Delta = \Phi$,*

$$\frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \frac{|\delta + \Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} - 1 = \frac{|\phi + \Phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} - 1 = \frac{|\phi + \Phi|^2 - 4\operatorname{Re}\left(\Phi\overline{\phi}\right)}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} = \frac{|\Phi - \phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)},$$

*where $\operatorname{Re}\left(\Phi\overline{\phi}\right) > 0$ and by (3.10) we derive (3.14).*

*By taking the square root in (3.14), we get*

$$\left| \int_\Omega w(s)\,\alpha(s)\,B(s)\,d\mu(s) - \int_\Omega w(s)\,\alpha(s)\,d\mu(s) \int_\Omega w(s)\,B(s)\,d\mu(s) \right| \leq \frac{|\Phi - \phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} \left| \int_\Omega w(s)\,\alpha(s)\,d\mu(s) \right| \left| \int_\Omega w(s)\,B(s)\,d\mu(s) \right|. \tag{3.15}$$

## 4. Applications for Finite Fourier Transform

Let $B : [a, b] \to \mathcal{B}(H)$ be a Bochner integrable mapping defined on the finite interval $[a, b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(B)(t) := \int_a^b e^{-2\pi its} B(s)\,ds.$$

Let $E$ be the *exponential mean* of two complex numbers defined by

$$E(z, w) := \begin{cases} \dfrac{e^z - e^w}{z - w}, & \text{if } z \neq w \\ \exp(w) & \text{if } z = w \end{cases}, \quad z, w \in \mathbb{C}. \tag{4.1}$$

Observe that

$$\int_a^b e^{-2\pi its} ds = (b - a)\,E(-2\pi ita, -2\pi itb),$$

$$\left| e^{2\pi its} \right|^2 = 1,$$

$$\int_a^b e^{2\pi its} ds = \frac{1}{2\pi it}\left[ e^{2\pi itb} - e^{2\pi ita} \right],$$

and

$$\left|\int_a^b e^{2\pi its}ds\right|^2 = \left(\frac{1}{2\pi|t|}\right)^2\left[\left|e^{2\pi itb}\right|^2 - 2\operatorname{Re}\left[e^{2\pi itb}e^{-2\pi ita}\right] + \left|e^{2\pi ita}\right|^2\right]$$
$$= \frac{1}{4\pi^2t^2}\left[1 - 2\operatorname{Re}\left[e^{2\pi it(b-a)}\right] + 1\right]$$
$$= \frac{1}{2\pi^2t^2}\left[1 - \operatorname{Re}\left[\cos(2\pi t(b-a)) + i\sin(2\pi t(b-a))\right]\right]$$
$$= \frac{1}{2\pi^2t^2}\left[1 - \cos(2\pi t(b-a))\right]$$
$$= \frac{1}{2\pi^2|t|^2}\left[1 - \left(1 - 2\sin^2(\pi t(b-a))\right)\right] = \frac{\sin^2[\pi t(b-a)]}{\pi^2t^2}.$$

From the inequality (3.5) for $\Omega = [a,b]$, $w(s) = \frac{1}{b-a}$ and $\alpha(s) = e^{-2\pi its}$, we have for $t \in \mathbb{R}$, $t \neq 0$, that

$$\left|\frac{1}{b-a}\int_a^b e^{-2\pi its}B(s)\,ds - \frac{1}{b-a}\int_a^b e^{-2\pi its}ds\frac{1}{b-a}\int_a^b B(s)\,ds\right| \tag{4.2}$$
$$\leq \left(\frac{1}{b-a}\int_a^b |A(s)|^2\,d\mu(s)\right)^{1/2} - \frac{|\sin[\pi t(b-a)]|}{\pi|t|(b-a)}\left|\frac{1}{b-a}\int_a^b A(s)\,d\mu(s)\right|,$$

namely

$$\left|\mathcal{F}(B)(t) - E(-2\pi ita, -2\pi itb)\int_a^b B(s)\,ds\right| \tag{4.3}$$
$$\leq \left((b-a)\int_a^b |B(s)|^2\,d\mu(s)\right)^{1/2} - \frac{|\sin[\pi t(b-a)]|}{\pi|t|(b-a)}\left|\int_a^b B(s)\,d\mu(s)\right|,$$

provided that $B \in L_2([a,b], \mathcal{B}(H))$.

If

$$\left|B(s) - \frac{\gamma+\Gamma}{2}1_H\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2\,1_H \text{ for } \mu\text{-a.e. } s \in \Omega, \tag{4.4}$$

then by (2.4) we get

$$\left(\frac{1}{b-a}\int_a^b |B(s)|^2\,ds\right)^{1/2} \leq \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|\frac{1}{b-a}\int_a^b B(s)\,ds\right|.$$

Then by (4.3) we derive

$$\left|\mathcal{F}(B)(t) - E(-2\pi ita, -2\pi itb)\int_a^b B(s)\,ds\right| \tag{4.5}$$
$$\leq \left(\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} - \frac{|\sin[\pi t(b-a)]|}{\pi|t|(b-a)}\right)\left|\int_a^b B(s)\,ds\right|,$$

for $t \in \mathbb{R}$, $t \neq 0$, provided that $B$ satisfies the condition (4.4).

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