ADDITIVE GRÜSS' TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if U is an unitary operator and $B : \Omega \to \mathcal{B}(H)$ is strongly μ-measurable with $BU \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$ and such that

$$\left|B(s)U - \frac{\gamma+\Gamma}{2}U\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2 1_H \text{ for } \mu\text{-a.e. } s \in \Omega$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^* - B^*(s)\right)(B(s) - \gamma U)\right] \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega$$

for some complex constants γ, Γ with$\gamma + \Gamma \neq 0$, then

$$\left(\int_\Omega w(s)|B(s)U|^2 d\mu(s)\right)^{1/2} \leq \left|U^*\left(\int_\Omega w(s)B(s)d\mu(s)\right)U\right| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}1_H.$$

Applications for finite Fourier Transform are also given.

1. Introduction

For two Lebesgue integrable functions f, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt.$$

In 1934, G. Grüss [11] showed that

$$|D(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by the author in [2]:

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Theorem 1. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e\in H$, $\|e\|=1$. If φ, ϕ, γ, $\Gamma\in\mathbb{K}$ and x, $y\in H$ are such that*

$$\operatorname{Re}\langle\phi e-x,x-\varphi e\rangle\geq 0 \quad \textit{and} \quad \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.3}$$

or, equivalently (see [4])

$$\left\|x-\frac{\varphi+\phi}{2}e\right\|\leq\frac{1}{2}\left|\phi-\varphi\right| \quad \textit{and} \quad \left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.4}$$

then

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right|\leq\frac{1}{4}\left|\phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in (1.5).

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by the author in 2001, [3].

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space, $\Omega\subset\mathbb{R}^n$ be a Lebesgue measurable set and $\rho:\Omega\to[0,\infty)$ a Lebesgue measurable function with $\int_\Omega\rho(s)\,ds=1$. We denote by $L_{2,\rho}(\Omega,H)$ the set of all Bochner measurable functions f on Ω such that $\|f\|_{2,\rho}^2:=\int_\Omega\rho(s)\|f(s)\|^2ds<\infty$. If f, g belong to $L_{2,\rho}(\Omega,H)$ and there exist the vectors x, X, y, $Y\in H$ such that*

$$\begin{aligned}&\int_\Omega\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt\geq 0,\\&\int_\Omega\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt\geq 0,\end{aligned} \tag{1.6}$$

then we have the inequality

$$\begin{aligned}&\left|\int_\Omega\rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega\rho(t)f(t)\,dt,\int_\Omega\rho(t)g(t)\,dt\right\rangle\right|\\&\leq\frac{1}{4}\|X-x\|\|Y-y\|.\end{aligned} \tag{1.7}$$

The constant $\frac{1}{4}$ is sharp in the sense mentioned above.

Remark 1. *A practical sufficient condition for (1.6) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle\geq 0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle\geq 0$$

or, equivalently

$$\left\|f(t)-\frac{X+x}{2}\right\|\leq\frac{1}{2}\|X-x\| \ \textit{and}\ \left\|g(t)-\frac{Y+y}{2}\right\|\leq\frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$.

For related results, see [1], [4]-[10] and [12]-[13].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

In this paper we show among others that, if U is an unitary operator and $B:\Omega\to\mathcal{B}(H)$ is strongly μ-measurable with $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that

$$\left|B(s)U-\frac{\gamma+\Gamma}{2}U\right|^2\le\frac{1}{4}|\Gamma-\gamma|^2 1_H \text{ for } \mu\text{-a.e. } s\in\Omega$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^*-B^*(s)\right)(B(s)-\gamma U)\right]\ge 0 \text{ for } \mu\text{-a.e. } s\in\Omega$$

for some complex constants γ, Γ with$\gamma+\Gamma\neq 0$, then

$$\left(\int_\Omega w(s)|B(s)U|^2 d\mu(s)\right)^{1/2}\le\left|U^*\left(\int_\Omega w(s)B(s)d\mu(s)\right)U\right| +\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}1_H.$$

Applications for finite Fourier Transform are also given.

2. Main Results

We have the following identity of interest:

Lemma 1. *For any A, X, $Y\in\mathcal{B}(H)$, we have*

$$\left|A-\frac{X+Y}{2}\right|^2-\frac{1}{4}|X-Y|^2=\operatorname{Re}\left[(A^*-X^*)(A-Y)\right]. \tag{2.1}$$

Proof. We have

$$\begin{aligned}&\left|A-\frac{X+Y}{2}\right|^2-\frac{1}{4}|X-Y|^2\\&=|A|^2-\frac{X^*+Y^*}{2}A-A^*\frac{X+Y}{2}+\frac{1}{4}\left(|X|^2+X^*Y+Y^*X+|Y|^2\right)\\&-\frac{1}{4}\left(|X|^2-X^*Y-Y^*X+|Y|^2\right)\\&=|A|^2-\frac{X^*+Y^*}{2}A-A^*\frac{X+Y}{2}+\frac{1}{2}(X^*Y+Y^*X)\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Re}\left[(A^*-X^*)(A-Y)\right]\\&=\operatorname{Re}\left[|A|^2-X^*A-A^*Y+X^*Y\right]\\&=|A|^2-\operatorname{Re}(X^*A)-\operatorname{Re}(A^*Y)+\operatorname{Re}(X^*Y)\\&=|A|^2-\frac{1}{2}(X^*A+A^*X)-\frac{1}{2}(A^*Y+Y^*A)+\frac{1}{2}(X^*Y+Y^*X)\\&=|A|^2-\frac{1}{2}(X^*+Y^*)A-\frac{1}{2}A^*(X+Y)+\frac{1}{2}(X^*Y+Y^*X),\end{aligned}$$

which proves the desired identity (2.1). □

Corollary 1. *Let A, X, $Y\in\mathcal{B}(H)$. The following statements are equivalent*

$$\left|A-\frac{X+Y}{2}\right|^2\le\frac{1}{4}|X-Y|^2$$

and

$$\operatorname{Re}\left[\left(X^{*}-A^{*}\right)(A-Y)\right]\geq 0.$$

We have the following reverse inequality of Cauchy-Bunyakowsky-Schwarz integral inequality for operator modulus:

Theorem 3. *Let U be an unitary operator and $B:\Omega\rightarrow\mathcal{B}(H)$ strongly μ-measurable with $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that*

$$\left|B(s)U-\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}\text{ for }\mu\text{-a.e. }s\in\Omega \tag{2.2}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^{*}-B^{*}(s)\right)\left(B(s)-\gamma U\right)\right]\geq 0\text{ for }\mu\text{-a.e. }s\in\Omega \tag{2.3}$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma})>0$. Then

$$\begin{aligned}&\left(\int_{\Omega}w(s)\left|B(s)U\right|^{2}d\mu(s)\right)^{1/2}\\&\leq\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right]+\frac{1}{4}\frac{|\Gamma-\gamma|^{2}}{|\gamma+\Gamma|}1_{H}\\&\leq\left|U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^{2}}{|\gamma+\Gamma|}1_{H}.\end{aligned}\tag{2.4}$$

Proof. The equivalence of the statements (2.2) and (2.3) follows by Corollary 1 for $A=B(s)$, $X=\Gamma U$ and $Y=\gamma U$ and taking into account that $|U|^{2}=1_{H}$.

By the properties of operator modulus, we have

$$\left|B(s)U\right|^{2}-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}U\right)^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H},$$

namely

$$\left|B(s)U\right|^{2}-2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H},$$

or

$$\left|B(s)U\right|^{2}+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}, \tag{2.5}$$

for μ-a.e. $s\in\Omega$.

f we multiply (2.5) by $w(s)\geq 0$ and integrate, then we get

$$\begin{aligned}&\int_{\Omega}w(s)\left|B(s)U\right|^{2}d\mu(s)+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\\&\leq 2\int_{\Omega}w(s)\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]d\mu(s)+\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}\\&=2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}.\end{aligned}\tag{2.6}$$

Using the elementary operator inequality

$$2aA\leq A^{2}+a^{2}1_{H},$$

where $A \geq 0$ in the operator order and $a \geq 0$, then we also have

$$2\left|\frac{\gamma+\Gamma}{2}\right|\left(\int_{\Omega} w(s)|B(s)U|^{2} d\mu(s)\right)^{1/2} \leq \int_{\Omega} w(s)|B(s)U|^{2} d\mu(s)+\left|\frac{\gamma+\Gamma}{2}\right|^{2} 1_{H}. \tag{2.7}$$

By utilising (2.6) and (2.7) we deduce

$$\begin{aligned} & 2\left|\frac{\gamma+\Gamma}{2}\right|\left(\int_{\Omega} w(s)|B(s)U|^{2} d\mu(s)\right)^{1/2} \\ & \leq 2 \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2} U^{*}\left(\int_{\Omega} w(s) B(s) d\mu(s)\right) U\right]+\frac{1}{4}|\Gamma-\gamma|^{2} 1_{H}. \end{aligned}$$

Since $\gamma+\Gamma \neq 0$, hence by dividing with $|\gamma+\Gamma| \neq 0$, we get

$$\begin{aligned} & \left(\int_{\Omega} w(s)|B(s)U|^{2} d\mu(s)\right)^{1/2} \\ & \leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|} U^{*}\left(\int_{\Omega} w(s) B(s) d\mu(s)\right) U\right]+\frac{1}{4} \frac{|\Gamma-\gamma|^{2}}{|\gamma+\Gamma|} 1_{H}, \end{aligned}$$

which proves the first inequality in (2.4).

For an operator T we consider the selfadjoint operators

$$\operatorname{Re}(T):=\frac{T^{*}+T}{2}, \ \operatorname{Im}(T):=\frac{T-T^{*}}{2i}.$$

Then $T=\operatorname{Re}(T)+i\operatorname{Im}(T)$, $|T|^{2}=(\operatorname{Re}(T))^{2}+(\operatorname{Im}(T))^{2}$. We have $|T|^{2} \geq (\operatorname{Re}(T))^{2}$ which implies, by taking the square root, that $|T| \geq |\operatorname{Re}(T)| \geq \operatorname{Re}(T)$.

Therefore

$$\begin{aligned} & \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|} U^{*}\left(\int_{\Omega} w(s) B(s) d\mu(s)\right) U\right] \\ & \leq\left|\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|} U^{*}\left(\int_{\Omega} w(s) B(s) d\mu(s)\right) U\right| \\ & =\left|\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}\right|\left|U^{*}\left(\int_{\Omega} w(s) B(s) d\mu(s)\right) U\right| \\ & =\left|U^{*}\left(\int_{\Omega} w(s) B(s) d\mu(s)\right) U\right|, \end{aligned}$$

which proves the last part of (2.4). □

Corollary 2. *Let U be an unitary operator and $B: \Omega \rightarrow \mathcal{B}(H)$ strongly μ-measurable with $BU \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$ and such that*

$$\left|B(s)U-\frac{m+M}{2} U\right|^{2} \leq \frac{1}{4}(M-m)^{2} 1_{H} \text{ for } \mu\text{-a.e. } s \in \Omega \tag{2.8}$$

or, equivalently

$$\operatorname{Re}\left[\left(MU^{*}-B^{*}(s)\right)(B(s)-mU)\right] \geq 0 \text{ for } \mu\text{-a.e. } s \in \Omega \tag{2.9}$$

for some real numbers $M > m > 0$*. Then*

$$\begin{aligned}(2.10)\quad & \left(\int_{\Omega} w(s)\left|B(s)U\right|^{2}d\mu(s)\right)^{1/2} \\ & \leq \operatorname{Re}\left[U^{*}\left(\int_{\Omega} w(s)\,B(s)\,d\mu(s)\right)U\right]+\frac{1}{4}\frac{(M-m)^{2}}{m+M}1_{H} \\ & \leq \left|U^{*}\left(\int_{\Omega} w(s)\,B(s)\,d\mu(s)\right)U\right|+\frac{1}{4}\frac{(M-m)^{2}}{m+M}1_{H}.\end{aligned}$$

3. Grüss Type Inequalities

We have the following Cauchy-Bunyakowsky-Schwarz inequality:

If $\alpha \in L_{w}^{2}(\Omega)$ and

$$A \in L_{2,w}(\Omega,\mathcal{B}(H)) := \left\{A:\Omega\rightarrow B(H)\,,\ \int_{\Omega} w(x)\left\|A(x)\right\|^{2}d\mu(x)<\infty\right\},$$

then

$$\begin{aligned}(3.1)\quad & \int_{\Omega} w(x)\left|\alpha(x)\right|^{2}d\mu(x)\int_{\Omega} w(x)\left|A(x)\right|^{2}d\mu(x) \\ & \geq \left|\int_{\Omega} w(x)\,\alpha(x)\,A(x)\,d\mu(x)\right|^{2},\end{aligned}$$

in the operator order of $\mathcal{B}(H)$.

Indeed, we have for $\alpha \in L_{w}^{2}(\Omega)$ and $A \in L_{2,w}(\Omega,\mathcal{B}(H))$,

$$\begin{aligned}0 & \leq \left|\overline{\alpha(x)}A(y)-\overline{\alpha(y)}A(x)\right|^{2} = \left|\alpha(x)\right|\left|A(y)\right|^{2}-\alpha(y)\,\overline{\alpha(x)}A^{*}(x)\,A(y) \\ & -\alpha(x)\,\overline{\alpha(y)}A^{*}(y)\,A(x)+\left|\alpha(y)\right|^{2}\left|A(x)\right|^{2},\end{aligned}$$

which gives that

$$\begin{aligned}& \left|\alpha(x)\right|^{2}\left|A(y)\right|^{2}+\left|\alpha(y)\right|^{2}\left|A(x)\right|^{2} \\ & \geq \alpha(y)\,\overline{\alpha(x)}A^{*}(x)\,A(y)+\alpha(x)\,\overline{\alpha(y)}A^{*}(y)\,A(x)\end{aligned}$$

for all y, $x \in \Omega$.

Now, multiply this with $w(y)\,w(x) \geq 0$ to get

$$\begin{aligned}& w(x)\left|\alpha(x)\right|^{2}w(y)\left|A(y)\right|^{2}+w(y)\left|\alpha(y)\right|^{2}w(x)\left|A(x)\right|^{2} \\ & \geq w(x)\,\overline{\alpha(x)}A^{*}(x)\,w(y)\,\alpha(y)\,A(y)+w(y)\,\overline{\alpha(y)}A^{*}(y)\,w(x)\,\alpha(x)\,A(x)\end{aligned}$$

for all y, $x \in \Omega$.

Integrating over x and y on Ω, then we get

$$\begin{aligned}
&\int_a^b w(x)|\alpha(x)|^2 d\mu(x) \int_a^b |A(y)|^2 d\mu(y) \\
&+ \int_a^b |\alpha(y)|^2 d\mu(y) \int_a^b w(x)|A(x)|^2 d\mu(x) \\
&\geq \int_a^b w(x)\overline{\alpha(x)}A^*(x) d\mu(x) \int_a^b \alpha(y) A(y) d\mu(y) \\
&+ \int_a^b w(y)\overline{\alpha(y)}A^*(y) d\mu(y) \int_a^b \alpha(x) A(x) d\mu(x) \\
&= 2\left|\int_a^b w(y)\alpha(y) A(y) d\mu(y)\right|^2,
\end{aligned}$$

and the inequality (3.1) is obtained.

We have the following result that is of interest in itself:

Lemma 2. *Assume that* $\alpha \in L_w^2(\Omega, \mu, \mathbb{C})$ *and* $A \in L_2(\Omega, \mu, \mathcal{B}(H))$, *then*

$$\begin{aligned}
&\left|\int_\Omega w(s)\alpha(s) A(s) d\mu(s) - \int_\Omega w(s)\alpha(s) d\mu(s) \int_\Omega w(s) A(s) d\mu(s)\right|^2 \\
&\leq \left(\int_\Omega w(s)|\alpha(s)|^2 d\mu(s) - \left|\int_\Omega w(s)\alpha(s) d\mu(s)\right|^2\right) \\
&\times \left(\int_\Omega w(s)|A(s)|^2 d\mu(s) - \left|\int_\Omega w(s) A(s) d\mu(s)\right|^2\right) \\
&\leq \left[\left(\int_\Omega w(s)|\alpha(s)|^2 d\mu(s)\right)^{1/2} \left(\int_\Omega w(s)|A(s)|^2 d\mu(s)\right)^{1/2}\right. \\
&\left.- \left|\int_\Omega w(s)\alpha(s) d\mu(s)\right| \left|\int_\Omega w(s) A(s) d\mu(s)\right|\right]^2.
\end{aligned} \tag{3.2}$$

Proof. We use the following Sonin type identity that can be proved by performing the calculations in the right side

$$\begin{aligned}
&\int_\Omega w(s)\alpha(s) A(s) d\mu(s) - \int_\Omega w(s)\alpha(s) d\mu(s) \int_\Omega w(s) A(s) d\mu(s) \\
&= \int_\Omega w(s)\left(\alpha(s) - \int_\Omega w(t)\alpha(t) d\mu(t)\right) \\
&\times \left(A(s) - \int_\Omega w(t) A(t) d\mu(t)\right) d\mu(s).
\end{aligned}$$

By using (3.1) we have

$$\begin{aligned}(3.3)\qquad & \left|\int_{\Omega} w(s)\left(\alpha(s)-\int_{\Omega} w(t)\alpha(t)\,d\mu(t)\right)\right. \\ & \left.\times\left(A(s)-\int_{\Omega} w(t)A(t)\,d\mu(t)\right)d\mu(s)\right|^{2} \\ & \leq\left[\int_{\Omega} w(s)\left|\alpha(s)-\int_{\Omega} w(t)\alpha(t)\,d\mu(t)\right|^{2}d\mu(s)\right] \\ & \times\left[\int_{\Omega} w(s)\left|A(s)-\int_{\Omega} w(t)A(t)\,d\mu(t)\right|^{2}d\mu(t)\right].\end{aligned}$$

Since

$$\begin{aligned}(3.4)\qquad & \int_{\Omega} w(s)\left|\alpha(s)-\int_{\Omega} w(t)\alpha(t)\,d\mu(t)\right|^{2}d\mu(s) \\ & =\int_{\Omega} w(s)|\alpha(s)|^{2}d\mu(s)-\left|\int_{\Omega} w(s)\alpha(s)\,d\mu(s)\right|^{2}\end{aligned}$$

and

$$\begin{aligned}(3.5)\qquad & \int_{\Omega} w(s)\left|A(s)-\int_{\Omega} w(t)A(t)\,d\mu(t)\right|^{2}d\mu(t) \\ & =\int_{\Omega} w(s)|A(s)|^{2}d\mu(s)-\left|\int_{\Omega} w(s)A(s)\,d\mu(s)\right|^{2},\end{aligned}$$

hence by (3.3), (3.4) and (3.5) we derive the first part of (3.2).

Now, observe that for real numbers a, b and selfadjoint operators A, B we have the operator inequality

$$\left(b^{2}-a^{2}\right)\left(B^{2}-A^{2}\right)\leq(bB-aA)^{2}.$$

Indeed, we have

$$\begin{aligned}& (bB-aA)^{2}-\left(b^{2}-a^{2}\right)\left(B^{2}-A^{2}\right) \\ & =b^{2}B^{2}-ba(AB+BA)+a^{2}A^{2}-b^{2}B^{2}+a^{2}B^{2}+b^{2}A^{2}-a^{2}A^{2} \\ & =a^{2}B^{2}+b^{2}A^{2}-ba(AB+BA)=(aB-bA)^{2}\geq 0.\end{aligned}$$

Therefore by

$$b=\left(\int_{\Omega} w(s)|\alpha(s)|^{2}d\mu(s)\right)^{1/2},\ a=\left|\int_{\Omega} w(s)\alpha(s)\,d\mu(s)\right|$$

and

$$B=\left(\int_{\Omega} w(s)|A(s)|^{2}d\mu(s)\right)^{1/2},\ A=\left|\int_{\Omega} w(s)A(s)\,d\mu(s)\right|$$

we deduce the second part of (3.2). □

By taking the square root in (3.2) we also have:

Corollary 3. *With the assumptions of Lemma 2 we have*

$$\begin{aligned}(3.6)\qquad &\left|\int_\Omega w(s)\,\alpha(s)\,A(s)\,d\mu(s)-\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\int_\Omega w(s)\,A(s)\,d\mu(s)\right|\\ &\le\left(\int_\Omega w(s)\,|\alpha(s)|^2\,d\mu(s)\right)^{1/2}\left(\int_\Omega w(s)\,|A(s)|^2\,d\mu(s)\right)^{1/2}\\ &-\left|\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\right|\left|\int_\Omega w(s)\,A(s)\,d\mu(s)\right|.\end{aligned}$$

We have the following Gruss' type inequality

Theorem 4. *Assume that* $B:\Omega\to\mathcal{B}(H)$ *is strongly* μ*-measurable with* $B\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ *and such that either*

$$(3.7)\qquad \left|B(s)-\frac{\gamma+\Gamma}{2}1_H\right|^2\le\frac{1}{4}|\Gamma-\gamma|^2\,1_H \text{ for } \mu\text{-a.e. } s\in\Omega$$

or, equivalently

$$(3.8)\qquad \operatorname{Re}\left[\left(\overline{\Gamma}1_H-B^*(s)\right)\left(B(s)-\gamma 1_H\right)\right]\ge 0 \text{ for } \mu\text{-a.e. } s\in\Omega$$

for some complex constants γ, Γ *with* $\operatorname{Re}(\Gamma\overline{\gamma})>0$.

If $\alpha\in L_w^2(\Omega,\mu,\mathbb{C})$, *then*

$$\begin{aligned}(3.9)\qquad &\left|\int_\Omega w(s)\,\alpha(s)\,B(s)\,d\mu(s)-\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\int_\Omega w(s)\,B(s)\,d\mu(s)\right|^2\\ &\le\Bigg[\left(\left(\int_\Omega w(s)\,|\alpha(s)|^2\,d\mu(s)\right)^{1/2}-\left|\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\right|\right)\\ &\times\left|\int_\Omega w(s)\,B(s)\,d\mu(s)\right|\\ &+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}\left(\int_\Omega w(s)\,|\alpha(s)|^2\,d\mu(s)\right)^{1/2}1_H\Bigg]^2.\end{aligned}$$

Proof. From (3.6) we get

$$\begin{aligned}(3.10)\qquad &\left|\int_\Omega w(s)\,\alpha(s)\,B(s)\,d\mu(s)-\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\int_\Omega w(s)\,B(s)\,d\mu(s)\right|^2\\ &\le\Bigg[\left(\int_\Omega w(s)\,|\alpha(s)|^2\,d\mu(s)\right)^{1/2}\left(\int_\Omega w(s)\,|B(s)|^2\,d\mu(s)\right)^{1/2}\\ &-\left|\int_\Omega w(s)\,\alpha(s)\,d\mu(s)\right|\left|\int_\Omega w(s)\,B(s)\,d\mu(s)\right|\Bigg]^2=:K.\end{aligned}$$

From (2.4) we get

$$\left(\int_\Omega w(s)\,|B(s)|^2\,d\mu(s)\right)^{1/2}\le\left|\int_\Omega w(s)\,B(s)\,d\mu(s)\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}1_H.$$

Therefore, by

$$K \leq \left[\left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \right)^{1/2} \left(\left| \int_\Omega w(s) B(s) d\mu(s) \right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\gamma + \Gamma|} 1_H \right) \right.$$
$$\left. - \left| \int_\Omega w(s) \alpha(s) d\mu(s) \right| \left| \int_\Omega w(s) B(s) d\mu(s) \right| \right]^2$$
$$\leq \left[\left(\left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \right)^{1/2} - \left| \int_\Omega w(s) \alpha(s) d\mu(s) \right| \right) \right.$$
$$\times \left| \int_\Omega w(s) B(s) d\mu(s) \right|$$
$$\left. + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\gamma + \Gamma|} \left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \right)^{1/2} 1_H \right]^2,$$

which proves (3.9). □

Remark 2. *By taking the square root in (3.9), we get*

$$\left| \int_\Omega w(s) \alpha(s) B(s) d\mu(s) - \int_\Omega w(s) \alpha(s) d\mu(s) \int_\Omega w(s) B(s) d\mu(s) \right| \tag{3.11}$$
$$\leq \left[\left(\left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \right)^{1/2} - \left| \int_\Omega w(s) \alpha(s) d\mu(s) \right| \right) \right.$$
$$\times \left| \int_\Omega w(s) B(s) d\mu(s) \right|$$
$$+ \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\gamma + \Gamma|} \left(\int_\Omega w(s) |\alpha(s)|^2 d\mu(s) \right)^{1/2} 1_H.$$

4. Applications for Finite Fourier Transform

Let $B : [a,b] \to \mathcal{B}(H)$ be a Bochner integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(B)(t) := \int_a^b e^{-2\pi i t s} B(s) ds.$$

Let E be the *exponential mean* of two complex numbers defined by

$$E(z,w) := \begin{cases} \dfrac{e^z - e^w}{z - w}, & \text{if } z \neq w \\ \exp(w) & \text{if } z = w \end{cases}, \quad z, w \in \mathbb{C}. \tag{4.1}$$

Observe that

$$\int_a^b e^{-2\pi i t s} ds = (b-a) E(-2\pi i t a, -2\pi i t b),$$

$$\left| e^{2\pi i t s} \right|^2 = 1,$$

$$\int_a^b e^{2\pi i t s} ds = \frac{1}{2\pi i t} \left[e^{2\pi i t b} - e^{2\pi i t a} \right],$$

and

$$\begin{aligned}\left|\int_a^b e^{2\pi its}ds\right|^2 &= \left(\frac{1}{2\pi|t|}\right)^2\left[\left|e^{2\pi itb}\right|^2 - 2\operatorname{Re}\left[e^{2\pi itb}e^{-2\pi ita}\right] + \left|e^{2\pi ita}\right|^2\right] \\ &= \frac{1}{4\pi^2t^2}\left[1 - 2\operatorname{Re}\left[e^{2\pi it(b-a)}\right] + 1\right] \\ &= \frac{1}{2\pi^2t^2}\left[1 - \operatorname{Re}\left[\cos\left(2\pi t\left(b-a\right)\right) + i\sin\left(2\pi t\left(b-a\right)\right)\right]\right] \\ &= \frac{1}{2\pi^2t^2}\left[1 - \cos\left(2\pi t\left(b-a\right)\right)\right] \\ &= \frac{1}{2\pi^2|t|^2}\left[1 - \left(1 - 2\sin^2\left(\pi t\left(b-a\right)\right)\right)\right] = \frac{\sin^2\left[\pi t\left(b-a\right)\right]}{\pi^2t^2}.\end{aligned}$$

From the inequality (3.6) for $\Omega = [a,b]$, $w(s) = \frac{1}{b-a}$ and $\alpha(s) = e^{-2\pi its}$, we have for $t \in \mathbb{R}$, $t \neq 0$, that

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b e^{-2\pi its}B(s)\,ds - \frac{1}{b-a}\int_a^b e^{-2\pi its}ds\frac{1}{b-a}\int_a^b B(s)\,ds\right| \\ &\leq \left(1 - \frac{\left|\sin\left[\pi t\left(b-a\right)\right]\right|}{\pi|t|\left(b-a\right)}\right)\left|\frac{1}{b-a}\int_a^b B(s)\,d\mu(s)\right| + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}1_H,\end{aligned}$$

namely

$$\begin{aligned}&\left|\mathcal{F}(B)(t) - E\left(-2\pi ita, -2\pi itb\right)\int_a^b B(s)\,ds\right| \\ &\leq \left(1 - \frac{\left|\sin\left[\pi t\left(b-a\right)\right]\right|}{\pi|t|\left(b-a\right)}\right)\left|\int_a^b B(s)\,d\mu(s)\right| + \frac{1}{4}\left(b-a\right)\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}1_H,\end{aligned} \tag{4.2}$$

provided that $B \in L_2([a,b], \mathcal{B}(H))$ and

$$\left|B(s) - \frac{\gamma+\Gamma}{2}1_H\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2 1_H \text{ for } \mu\text{-a.e. } s \in \Omega \tag{4.3}$$

with $\gamma + \Gamma \neq 0$.

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