DISCRETE INEQUALITIES FOR TWO SEQUENCES RELATED TO LASOTA-OPIAL'S RESULT

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Abstract. For a sequence $\{x_i\}_{i=0}^{n}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. Assume that $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers. In this paper we show among others that, if $y_0 = 0$, then for $n \in \{2, ..., N\}$,

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^2 \right)^{1/2} \le \frac{1}{2} \sum_{i=0}^{n-1} \left[i |\Delta x_i|^2 + (n-i-1) |\Delta y_i|^2 \right],$$

while, if $y_N = 0$, then for $n \in \{1, ..., N-1\}$,

$$\sum_{i=n}^{N-1} |\Delta x_i| |y_i| \le \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^2 \right)^{1/2} \le \frac{1}{2} \sum_{i=n}^{N-1} \left[(N-i) |\Delta x_i|^2 + (i+1-n) |\Delta y_i|^2 \right].$$

Some particular inequalities of interest are also provided.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u' \in L_2[a,b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \le \frac{1}{4} (b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \le t \le \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \le b \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \le \frac{1}{2} (b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.2}$$

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with equality if and only if $u(t) = c(t-a)$ *for some constant* c;

(iii) *If* $\int_a^b u(t)\,dt = 0$, *then the inequality (1.1) holds with equality if and only if*

$$u(t) = c\left(t - \frac{a+b}{2}\right)$$

for any constant c.

The inequality (1.1) was obtained by Olech in [8] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [9].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

The inequality (1.1) in the case (iii), namely in the case that u satisfies the condition $\int_a^b u(t)\,dt = 0$ was obtained by Brown and Plum in [4].

As mentioned in [4] the inequality (1.1) also holds if $u(a) + u(b) = 0$.

For a sequence $\{x_i\}_{i=0}^{n}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k. \tag{1.3}$$

In [7], Lasota provided discrete versions of Opial inequality (1.1) about the forward difference operator as follows:

Theorem 2. *Let* $\{x_i\}_{i=0}^{N}$ *be a sequence of real numbers with* $x_0 = x_N = 0$. *Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.4}$$

where $\lfloor \cdot \rfloor$ *is the integer part function. If* N *is even, then the inequality (1.4) is sharp.*

Also, we have the following results, see [1]:

Theorem 3. *Let* $\{x_i\}_{i=0}^{N}$ *be a sequence of real numbers. If* $x_0 = 0$, *then*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \le \frac{1}{2} (\tau - 1) \sum_{i=0}^{\tau-1} |\Delta x_i|^2, \ \tau \in \{2, ..., N\}. \tag{1.5}$$

If $x_N = 0$, *then*

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \le \frac{1}{2} (N - \tau + 1) \sum_{i=0}^{N-1} |\Delta x_i|^2, \ \tau \in \{1, ..., N-1\}. \tag{1.6}$$

For other discrete Opial type inequalities, see [5], [6] and [10]-[13].

Motivated by the above results, in this paper we obtain various upper bounds for the quantities

$$\sum_{i=1}^{\tau-1} |x_i \Delta y_i|, \ x_0 = 0, \ \tau \in \{2, ..., N\}$$

and

$$\sum_{i=\tau}^{N-1} |x_i \Delta y_i|, \ x_N = 0, \ \tau \in \{1, ..., N-1\},$$

where $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ are sequences of complex numbers. In particular, we obtain refinements of (1.4)-(1.6) above.

2. Main Results

We have the following result for two sequences:

Theorem 4. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $y_0 = 0$, *then for* $n \in \{2, ..., N\}$,

$$\begin{aligned}
\sum_{i=1}^{n-1} |\Delta x_i| |y_i| &\le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^2\right)^{1/2} \\
&\le \frac{1}{2} \sum_{i=0}^{n-1} \left[i |\Delta x_i|^2 + (n-i-1) |\Delta y_i|^2\right] \\
&\le \frac{1}{2} \times \begin{cases} n(n-1) \max_{i\in\{0,\dots,n-1\}} \left\{|\Delta x_i|^2, |\Delta y_i|^2\right\}, \\ (n-1) \sum_{i=0}^{n-1} \left(|\Delta x_i|^2 + |\Delta y_i|^2\right). \end{cases}
\end{aligned} \tag{2.1}$$

Proof. Let $n \in \{2, ..., N\}$. Since $y_0 = 0$, hence $y_i = \sum_{j=0}^{i-1} \Delta y_j$ for $i = 1, ..., n-1$. Then

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| = \sum_{i=1}^{n-1} |\Delta x_i| \left|\sum_{j=0}^{i-1} \Delta y_j\right| = \sum_{i=1}^{n-1} \sqrt{i} |\Delta x_i| \frac{1}{\sqrt{i}} \left|\sum_{j=0}^{i-1} \Delta y_j\right| =: A.$$

By the discrete Cauchy-Bunyakowsky-Schwarz (CBS) inequality, we have

$$A \le \left(\sum_{i=1}^{n-1} i |\Delta x_i|^2\right)^{1/2} \left[\sum_{i=1}^{n-1} \frac{1}{i} \left|\sum_{j=0}^{i-1} \Delta y_j\right|^2\right]^2 =: B.$$

By (CBS) inequality we also have

$$\frac{1}{i} \left|\sum_{j=0}^{i-1} \Delta y_j\right|^2 \le \sum_{j=0}^{i-1} |\Delta y_j|^2,$$

which gives

$$B \le \left(\sum_{i=1}^{n-1} i |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} |\Delta y_j|^2\right)\right)^{1/2}. \tag{2.2}$$

From (1.3), we have for $m = 1$ and n is replaced by $n-1$ that

$$\sum_{i=1}^{n-1} a_i \Delta b_i = a_{n-1} b_n - a_1 b_1 - \sum_{i=1}^{n-2} b_{i+1} \Delta a_i,$$

which by taking $a_i = \sum_{j=0}^{i-1} |\Delta y_j|^2$, $b_i = i$, produces that

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}|\Delta y_j|^2\right) \\
&= n\sum_{j=0}^{n-2}|\Delta y_j|^2 - |\Delta y_0|^2 - \sum_{i=1}^{n-2}(i+1)|\Delta y_i|^2 \\
&= n|\Delta y_0|^2 - |\Delta y_0|^2 + n\sum_{j=1}^{n-2}|\Delta y_j|^2 - \sum_{i=1}^{n-2}(i+1)|\Delta y_i|^2 \\
&= (n-1)|\Delta y_0|^2 + \sum_{i=1}^{n-2}(n-i-1)|\Delta y_i|^2 = \sum_{i=0}^{n-2}(n-i-1)|\Delta y_i|^2 .
\end{aligned}$$

Now, it is obvious that

$$\sum_{i=1}^{n-1} i|\Delta x_i|^2 = \sum_{i=0}^{n-1} i|\Delta x_i|^2$$

and

$$\sum_{i=0}^{n-2}(n-i-1)|\Delta y_i|^2 = \sum_{i=0}^{n-1}(n-i-1)|\Delta y_i|^2 .$$

By utilising (2.2) we derive the first part of (2.1). The second part follows by the A-G-means inequality,

$$\sqrt{ab} \le \frac{a+b}{2},\ a,\ b \ge 0.$$

Now, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left[i|\Delta x_i|^2 + (n-i-1)|\Delta y_i|^2\right] \\
&\le \max_{i\in\{0,\dots,n-1\}}\left\{|\Delta x_i|^2, |\Delta y_i|^2\right\}\sum_{i=0}^{n-1}[i+(n-i-1)] \\
&= \max_{i\in\{0,\dots,n-1\}}\left\{|\Delta x_i|^2, |\Delta y_i|^2\right\} n(n-1),
\end{aligned}$$

which proves the first branch.

For the second branch, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left[i|\Delta x_i|^2 + (n-i-1)|\Delta y_i|^2\right] \\
&\le \max_{i\in\{0,\dots,n\}}\{i, n-i-1\}\sum_{i=0}^{n-1}\left[|\Delta x_i|^2 + |\Delta y_i|^2\right] \\
&= (n-1)\sum_{i=0}^{n-1}\left[|\Delta x_i|^2 + |\Delta y_i|^2\right].
\end{aligned}$$

□

The case of one sequence, is as follows:

Corollary 1. *Assume that* $\{x_i\}_{i=0}^N$ *is a sequence of complex numbers. If* $x_0 = 0$, *then for* $n \in \{2, ..., N\}$, *we have the refinement of (1.5)*

$$\sum_{i=1}^{n-1} |\Delta x_i| |x_i| \le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta x_i|^2\right)^{1/2} \le \frac{1}{2}(n-1)\sum_{i=0}^{n-1} |\Delta x_i|^2 . \tag{2.3}$$

We also have:

Theorem 5. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $y_N = 0$, *then for* $n \in \{1, ..., N-1\}$, *then*

$$\begin{aligned}
&\sum_{i=n}^{N-1} |\Delta x_i| |y_i| \\
&\le \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^2\right)^{1/2} \\
&\le \frac{1}{2}\sum_{i=n}^{N-1} \left[(N-i) |\Delta x_i|^2 + (i+1-n) |\Delta y_i|^2\right] \\
&\le \frac{1}{2} \times \begin{cases} (N-n+1)(N-n) \max_{i\in\{n,...,N-1\}} \left\{|\Delta x_i|^2, |\Delta y_i|^2\right\}, \\ (N-n)\left[\sum_{i=n}^{N-1}\left(|\Delta x_i|^2 + |\Delta y_i|^2\right)\right]. \end{cases}
\end{aligned} \tag{2.4}$$

Proof. If $y_N = 0$, then $y_i = -\sum_{j=i}^{N-1} \Delta y_j$ for $i = n+1, ..., N-1$. Then

$$\begin{aligned}
\sum_{i=n}^{N-1} |\Delta x_i| |y_i| &= \sum_{i=n}^{N-1} |\Delta x_i| \left|\sum_{j=i}^{N-1} \Delta y_j\right| \\
&= \sum_{i=n}^{N-1} \sqrt{N-i} |\Delta x_i| \frac{1}{\sqrt{N-i}} \left|\sum_{j=i}^{N-1} \Delta y_j\right| =: C.
\end{aligned}$$

By the discrete Cauchy-Bunyakowsky-Schwarz (CBS) inequality, we have

$$\begin{aligned}
C &\le \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=n}^{N-1} \frac{1}{N-i} \left|\sum_{j=i}^{N-1} \Delta y_j\right|^2\right)^{1/2} \\
&\le \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=n}^{N-1} \left(\sum_{j=i}^{N-1} |\Delta y_j|^2\right)\right)^{1/2} =: D
\end{aligned} \tag{2.5}$$

From (1.3) we have

$$\sum_{i=n}^{N-1} a_i \Delta b_i = a_{N-1} b_N - a_n b_n - \sum_{i=n}^{N-2} b_{i+1} \Delta a_i,$$

and by $a_i = \sum_{j=i}^{N-1} |\Delta y_i|^2$ and $b_i = i$, we have

$$\begin{aligned}
&\sum_{i=n}^{N-1}\left(\sum_{j=i}^{N-1}|\Delta y_i|^2\right) \\
&= N|\Delta y_{N-1}|^2 - n\sum_{j=n}^{N-1}|\Delta y_i|^2 - \sum_{i=n}^{N-2}(i+1)\left(\sum_{j=i+1}^{N-1}|\Delta y_i|^2 - \sum_{j=i}^{N-1}|\Delta y_j|^2\right) \\
&= N|\Delta y_{N-1}|^2 + \sum_{i=n}^{N-2}(i+1)|\Delta y_i|^2 - n\sum_{j=n}^{N-1}|\Delta y_i|^2 . \\
&= N|\Delta y_{N-1}|^2 + \sum_{i=n}^{N-2}(i+1)|\Delta y_i|^2 - n\sum_{j=n}^{N-2}|\Delta y_i|^2 - n|\Delta y_{N-1}|^2 \\
&= \sum_{i=n}^{N-2}(i+1)|\Delta y_i|^2 - n\sum_{j=n}^{N-2}|\Delta y_i|^2 \\
&= \sum_{i=n}^{N-2}(i+1-n)|\Delta y_i|^2 + (N-n)|\Delta y_{N-1}|^2 = \sum_{i=n}^{N-1}(i+1-n)|\Delta y_i|^2 .
\end{aligned}$$

Then

$$D = \left(\sum_{i=n}^{N-1}(N-i)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)|\Delta y_i|^2\right)^{1/2},$$

which, by (2.5), proves the first inequality in (2.4).

The second part follows by A-G-means inequality. The last part is obvious. □

Corollary 2. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $x_N = 0$, *then for* $n \in \{1, ..., N-1\}$, *we have the refinement of (1.6)*

$$\begin{aligned}
\text{(2.6)} \qquad \sum_{i=n}^{N-1}|\Delta x_i||x_i| &\le \left(\sum_{i=n}^{N-1}(N-i)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)|\Delta x_i|^2\right)^{1/2} \\
&\le \frac{1}{2}(N-n+1)\sum_{i=n}^{N-1}|\Delta x_i|^2 .
\end{aligned}$$

We also have the following result that incorporates both cases:

Theorem 6. *Assume that* $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $y_0 = y_N = 0$, *then for* $n \in \{2, ..., N-1\}$,

$$\begin{aligned}
\text{(2.7)} \qquad \sum_{i=1}^{N-1}|\Delta x_i||y_i| &\le \left(\sum_{i=0}^{N-1}p_i(n)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i(n)|\Delta y_i|^2\right)^{1/2} \\
&\le \frac{1}{2}\sum_{i=0}^{N-1}\left(p_i(n)|\Delta x_i|^2 + q_i(n)|\Delta y_i|^2\right),
\end{aligned}$$

where

$$p_i(n) := \begin{cases} i, & \text{if } 0 \le i \le n-1, \\ N-i, & \text{if } n \le i \le N-1 \end{cases}$$

and

$$q_i(n) := \begin{cases} n-i-1, & \text{if } 0 \le i \le n-1, \\ i+1-n, & \text{if } n \le i \le N-1. \end{cases}$$

Proof. We have for $n \in \{2, ..., N-1\}$ that

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^2 \right)^{1/2}$$

and

$$\sum_{i=n}^{N-1} |\Delta x_i| |y_i| \le \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^2 \right)^{1/2}.$$

If we add these inequalities, then we get, by the elementary inequality

$$ab + cd \le \left(a^2 + c^2\right)^{1/2} \left(b^2 + d^2\right)^{1/2}, \; a, \; b, \; c, \; d \ge 0$$

that

$$\begin{aligned} \sum_{i=1}^{N-1} |\Delta x_i| |y_i| &\le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^2 \right)^{1/2} \\ &\quad + \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^2 \right)^{1/2} \\ &\le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2 + \sum_{i=n}^{N-1} (N-i) |\Delta x_i|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^2 + \sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^2 \right)^{1/2} \\ &= \left(\sum_{i=0}^{N-1} p_i |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} q_i |\Delta y_i|^2 \right)^{1/2}, \end{aligned}$$

which proves the first inequality in (2.7).

The second inequality follows by A-G-means inequality. □

Corollary 3. *Assume that* $\{x_i\}_{i=0}^N$ *is a sequence of complex numbers with* $x_0 = x_N = 0$*, then*

$$\begin{aligned} \sum_{i=1}^{N-1} |\Delta x_i| |x_i| &\le \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} q(n)_i |\Delta x_i|^2 \right)^{1/2} \\ &\le \frac{1}{2} \sum_{i=0}^{N-1} s_i(n) |\Delta x_i|^2 \end{aligned} \tag{2.8}$$

where

$$s_i(n) := \begin{cases} n-1, & \text{if } 0 \le i \le n-1, \\ N-n+1, & \text{if } n \le i \le N-1. \end{cases}$$

Remark 1. *If we take in (2.8)* $n = \left\lfloor \frac{N+1}{2} \right\rfloor + 1$, *then by (2.8) we get*

$$\begin{aligned} (2.9) \quad & \sum_{i=1}^{N-1} |\Delta x_i| \, |x_i| \\ & \le \left(\sum_{i=0}^{N-1} p_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} q \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \right)^{1/2} \\ & \le \frac{1}{2} \sum_{i=0}^{N-1} s_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta x_i|^2 \le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2 , \end{aligned}$$

where

$$s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \le i \le \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \le i \le N-1 \end{cases} \le \left\lfloor \frac{N+1}{2} \right\rfloor .$$

The inequality (2.9) is a refinement of Lasota's result (1.4).

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