p-NORMS DISCRETE INEQUALITIES FOR TWO SEQUENCES RELATED TO LASOTA-OPIAL'S RESULT

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Abstract. For a sequence $\{x_i\}_{i=0}^{n}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. Assume that $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers. In this paper we show among others that, if $y_0 = 0$, then for $n \in \{2, ..., N\}$,

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| \leq \left(\sum_{i=0}^{n-1} i |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^q \right)^{1/q}$$

while, if $y_N = 0$, then for $n \in \{1, ..., N-1\}$,

$$\sum_{i=n}^{N-1} |\Delta x_i| |y_i| \leq \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^q \right)^{1/q}.$$

Some particular inequalities of interest are also provided.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a, b]$ and such that $u' \in L_2[a, b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \leq \frac{1}{4} (b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \leq t \leq \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \leq b \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \leq \frac{1}{2} (b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.2}$$

with equality if and only if $u(t) = c(t-a)$ for some constant c;

(iii) *If $\int_a^b u(t) \, dt = 0$, then the inequality (1.1) holds with equality if and only if*

$$u(t) = c\left(t - \frac{a+b}{2}\right)$$

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for any constant c.

The inequality (1.1) was obtained by Olech in [8] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [9].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

The inequality (1.1) in the case (iii), namely in the case that u satisfies the condition $\int_a^b u(t)\,dt = 0$ was obtained by Brown and Plum in [4].

As mentioned in [4] the inequality (1.1) also holds if $u(a) + u(b) = 0$.

For a sequence $\{x_i\}_{i=0}^n$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k. \tag{1.3}$$

In [7], Lasota provided discrete versions of Opial inequality (1.1) about the forward difference operator as follows:

Theorem 2. *Let $\{x_i\}_{i=0}^N$ be a sequence of numbers real numbers with $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.4}$$

where $\lfloor \cdot \rfloor$ is the integer part function. If N is even, then the inequality (1.4) is sharp.

Also, we have the following results, see [1]:

Theorem 3. *Let $\{x_i\}_{i=0}^N$ be a sequence of numbers real numbers. If $x_0 = 0$, then*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \le \frac{1}{2} (\tau - 1) \sum_{i=0}^{\tau-1} |\Delta x_i|^2, \ \tau \in \{2, ..., N\}. \tag{1.5}$$

If $x_N = 0$, then

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \le \frac{1}{2} (N - \tau + 1) \sum_{i=0}^{N-1} |\Delta x_i|^2, \ \tau \in \{1, ..., N-1\}. \tag{1.6}$$

For other discrete Opial type inequalities, see [5], [6] and [10]-[13].

Motivated by the above results, in this paper we obtain various p-norms upper bounds for the quantities

$$\sum_{i=1}^{\tau-1} |x_i \Delta y_i|, \ x_0 = 0, \ \tau \in \{2, ..., N\}$$

and

$$\sum_{i=\tau}^{N-1} |x_i \Delta y_i|, \ x_N = 0, \ \tau \in \{1, ..., N-1\},$$

where $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ are sequences of complex numbers.

2. Main Results

We have the following result for two sequences:

Theorem 4. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $y_0 = 0$, *then for* $n \in \{2, ..., N\}$,

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^p\right)^{1/p} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^q\right)^{1/q} \le \frac{1}{p} \sum_{i=0}^{n-1} i |\Delta x_i|^p + \frac{1}{q} \sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^q, \tag{2.1}$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

In particular,

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^2\right)^{1/2} \le \frac{1}{2} \sum_{i=0}^{n-1} \left(i |\Delta x_i|^2 + (n-i-1) |\Delta y_i|^2\right). \tag{2.2}$$

Proof. Let $n \in \{2, ..., N\}$. Since $y_0 = 0$, hence $y_i = \sum_{j=0}^{i-1} \Delta y_j$ for $i = 1, ..., n-1$. Then

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| = \sum_{i=1}^{n-1} |\Delta x_i| \left|\sum_{j=0}^{i-1} \Delta y_j\right| = \sum_{i=1}^{n-1} i^{1/p} |\Delta x_i| i^{-1/p} \left|\sum_{j=0}^{i-1} \Delta y_j\right| =: A.$$

Using the discrete Hölder inequality for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$A \le \left(\sum_{i=1}^{n-1} i |\Delta x_i|^p\right)^{1/p} \left[\sum_{i=1}^{n-1} i^{-q/p} \left|\sum_{j=0}^{i-1} \Delta y_j\right|^q\right]^{1/q} =: B.$$

By Hölder's inequality we have

$$i^{-/p} \left|\sum_{j=0}^{i-1} \Delta y_j\right| \le \left(\sum_{j=0}^{i-1} |\Delta y_j|^q\right)^{1/q},$$

which implies that

$$i^{-q/p} \left|\sum_{j=0}^{i-1} \Delta y_j\right|^q \le \sum_{j=0}^{i-1} |\Delta y_j|^q,$$

which implies that

$$B \le \left(\sum_{i=1}^{n-1} i |\Delta x_i|^p\right)^{1/p} \left[\sum_{i=1}^{n-1} \left(\sum_{j=0}^{i-1} |\Delta y_j|^q\right)\right]^{1/q}. \tag{2.3}$$

From (1.3), we have for $m = 1$ and n is replaced by $n-1$ that

$$\sum_{i=1}^{n-1} a_i \Delta b_i = a_{n-1} b_n - a_1 b_1 - \sum_{i=1}^{n-2} b_{i+1} \Delta a_i,$$

which by taking $a_i = \sum_{j=0}^{i-1} |\Delta y_j|^q$, $b_i = i$, produces that

$$\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)$$
$$= n\sum_{j=0}^{n-2}|\Delta y_j|^q - |\Delta y_0|^q - \sum_{i=1}^{n-2}(i+1)|\Delta y_i|^q$$
$$= n|\Delta y_0|^q - |\Delta y_0|^q + n\sum_{j=1}^{n-2}|\Delta y_j|^q - \sum_{i=1}^{n-2}(i+1)|\Delta y_i|^q$$
$$= (n-1)|\Delta y_0|^q + \sum_{i=1}^{n-2}(n-i-1)|\Delta y_i|^q = \sum_{i=0}^{n-2}(n-i-1)|\Delta y_i|^q .$$

Now, it is obvious that

$$\sum_{i=1}^{n-1} i|\Delta x_i|^q = \sum_{i=0}^{n-1} i|\Delta x_i|^q$$

and

$$\sum_{i=0}^{n-2}(n-i-1)|\Delta y_i|^q = \sum_{i=0}^{n-1}(n-i-1)|\Delta y_i|^q .$$

By making use of (2.3), we derive the first inequality in (2.1).

The second inequality follows by the elementary inequality

$$\alpha^{1/p}\beta^{1/q} \le \frac{1}{p}\alpha + \frac{1}{q}\beta,\ \ \alpha,\ \beta \ge 0. \tag{2.4}$$

□

Corollary 1. *With the assumptions of Theorem 4 we have*

$$\sum_{i=1}^{n-1}|\Delta x_i||y_i| \le \left(\sum_{i=0}^{n-1} i|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{n-1}(n-i-1)|\Delta y_i|^q\right)^{1/q} \tag{2.5}$$
$$\le \begin{cases} \frac{(n-1)n}{2}\max_{i\in\{0,\dots,n-1\}}|\Delta x_i|\max_{i\in\{0,\dots,n-1\}}|\Delta y_i|, \\ \\ (n-1)\left(\sum_{i=1}^{n-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=1}^{n-1}|\Delta y_i|^p\right)^{1/q}, \end{cases}$$

where $p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1.$

In particular,

$$\sum_{i=1}^{n-1}|\Delta x_i||y_i| \le \left(\sum_{i=0}^{n-1} i|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)|\Delta y_i|^2\right)^{1/2} \tag{2.6}$$
$$\le (n-1)\left(\sum_{i=0}^{n-1}|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}|\Delta y_i|^2\right)^{1/2}.$$

Proof. We have

$$\sum_{i=0}^{n-1} i|\Delta x_i|^p \le \max_{i\in\{1,\dots,n-1\}}|\Delta x_i|^p\sum_{i=0}^{n-1} i = \frac{(n-1)n}{2}\max_{i\in\{0,\dots,n-1\}}|\Delta x_i|^p$$

and

$$\sum_{i=0}^{n-1}(n-i-1)\left|\Delta y_i\right|^q \leq \max_{i\in\{0,\dots,n-1\}}\left|\Delta y_i\right|^p \sum_{i=0}^{n-1}(n-i-1) = \frac{(n-1)\,n}{2}\max_{i\in\{0,\dots,n-1\}}\left|\Delta y_i\right|^p .$$

Also,

$$\sum_{i=0}^{n-1} i\left|\Delta x_i\right|^p \leq (n-1)\sum_{i=0}^{n-1}\left|\Delta x_i\right|^p, \quad \sum_{i=0}^{n-1}(n-i-1)\left|\Delta y_i\right|^q \leq (n-1)\sum_{i=0}^{n-1}\left|\Delta y_i\right|^p .$$

These complete the proof. □

Remark 1. *Assume that $\{x_i\}_{i=0}^N$ is a sequence of complex numbers with $x_0=0$, then for $n\in\{2,\dots,N\}$,*

$$\sum_{i=1}^{n-1}\left|\Delta x_i\right|\left|x_i\right| \leq \left(\sum_{i=0}^{n-1} i\left|\Delta x_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left|\Delta x_i\right|^2\right)^{1/2} \leq \frac{1}{2}(n-1)\sum_{i=0}^{n-1}\left|\Delta x_i\right|^2 . \tag{2.7}$$

Also, we have the complementary result:

Theorem 5. *Assume that $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ are sequences of complex numbers. If $y_N=0$, then for $n\in\{1,\dots,N-1\}$, then*

$$\sum_{i=n}^{N-1}\left|\Delta x_i\right|\left|y_i\right| \leq \left(\sum_{i=n}^{N-1}(N-i)\left|\Delta x_i\right|^p\right)^{1/p}\left(\sum_{i=n}^{N-1}(i+1-n)\left|\Delta y_i\right|^q\right)^{1/q} \leq \frac{1}{p}\sum_{i=n}^{N-1}(N-i)\left|\Delta x_i\right|^p + \frac{1}{q}\sum_{i=n}^{N-1}(i+1-n)\left|\Delta y_i\right|^q , \tag{2.8}$$

where $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

In particular,

$$\sum_{i=n}^{N-1}\left|\Delta x_i\right|\left|y_i\right| \leq \left(\sum_{i=n}^{N-1}(N-i)\left|\Delta x_i\right|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\left|\Delta y_i\right|^2\right)^{1/2} \leq \frac{1}{2}\sum_{i=n}^{N-1}\left[(N-i)\left|\Delta x_i\right|^2 + (i+1-n)\left|\Delta y_i\right|^2\right]. \tag{2.9}$$

Proof. If $y_N=0$, then $y_i=-\sum_{j=i}^{N-1}\Delta y_j$ for $i=n+1,\dots,N-1$. Then

$$\sum_{i=n}^{N-1}\left|\Delta x_i\right|\left|y_i\right| = \sum_{i=n}^{N-1}\left|\Delta x_i\right|\left|\sum_{j=i}^{N-1}\Delta y_j\right| = \sum_{i=n}^{N-1}(N-i)^{1/p}\left|\Delta x_i\right|\frac{1}{(N-i)^{1/p}}\left|\sum_{j=i}^{N-1}\Delta y_j\right| =: C.$$

By the discrete Hölder's inequality, we have

$$C \leq \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=n}^{N-1} \frac{1}{(N-i)^{q/p}} \left| \sum_{j=i}^{N-1} \Delta y_j \right|^q \right)^{1/q} \tag{2.10}$$

$$= \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left[\sum_{i=n}^{N-1} \left(\frac{1}{(N-i)^{1/p}} \left| \sum_{j=i}^{N-1} \Delta y_j \right| \right)^q \right]^{1/q}$$

$$\leq \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left[\sum_{i=n}^{N-1} \left(\left(\sum_{j=i}^{N-1} \|\Delta y_j\|^q \right)^{1/q} \right)^q \right]^{1/q}$$

$$= \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left[\sum_{i=n}^{N-1} \left(\sum_{j=i}^{N-1} |\Delta y_j|^q \right) \right]^{1/q} =: D.$$

From (1.3) we have

$$\sum_{i=n}^{N-1} a_i \Delta b_i = a_{N-1} b_N - a_n b_n - \sum_{i=n}^{N-2} b_{i+1} \Delta a_i,$$

and by $a_i = \sum_{j=i}^{N-1} |\Delta y_i|^q$ and $b_i = i$, we have

$$\sum_{i=n}^{N-1} \left(\sum_{j=i}^{N-1} |\Delta y_i|^q \right)$$

$$= N |\Delta y_{N-1}|^q - n \sum_{j=n}^{N-1} |\Delta y_i|^q - \sum_{i=n}^{N-2} (i+1) \left(\sum_{j=i+1}^{N-1} |\Delta y_i|^q - \sum_{j=i}^{N-1} |\Delta y_j|^q \right)$$

$$= N |\Delta y_{N-1}|^q + \sum_{i=n}^{N-2} (i+1) |\Delta y_i|^q - n \sum_{j=n}^{N-1} |\Delta y_i|^q .$$

$$= N |\Delta y_{N-1}|^q + \sum_{i=n}^{N-2} (i+1) |\Delta y_i|^q - n \sum_{j=n}^{N-2} |\Delta y_i|^q - n |\Delta y_{N-1}|^q$$

$$= \sum_{i=n}^{N-2} (i+1) |\Delta y_i|^q - n \sum_{j=n}^{N-2} |\Delta y_i|^q$$

$$= \sum_{i=n}^{N-2} (i+1-n) |\Delta y_i|^q + (N-n) |\Delta y_{N-1}|^q = \sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^q .$$

Then

$$D = \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^q \right)^{1/q},$$

which, by (2.10), proves the first inequality in (2.8).

The second part follows by Young's inequality (2.4). The last part is obvious. □

Corollary 2. *With the assumptions of Theorem 5 we have*

$$\begin{aligned}(2.11)\qquad &\sum_{i=n}^{N-1}|\Delta x_i|\,|y_i| \\ &\leq \left(\sum_{i=n}^{N-1}(N-i)\,|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=n}^{N-1}(i+1-n)\,|\Delta y_i|^q\right)^{1/q} \\ &\leq \begin{cases}\frac{(N-n)(N-n+1)}{2}\max_{i\in\{n,\ldots,N-1\}}|\Delta x_i|\max_{i\in\{n,\ldots,N-1\}}|\Delta y_i|, \\ (N-n)\left(\sum_{i=n}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=n}^{N-1}|\Delta y_i|^q\right)^{1/q},\end{cases}\end{aligned}$$

where $p,\, q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

In particular,

$$(2.12)\qquad \sum_{i=n}^{N-1}|\Delta x_i|\,|y_i| \leq (N-n)\left(\sum_{i=n}^{N-1}|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}|\Delta y_i|^2\right)^{1/2}.$$

Proof. We have

$$\begin{aligned}\sum_{i=n}^{N-1}(N-i)\,|\Delta x_i|^p &\leq \max_{i\in\{n,\ldots,N-1\}}|\Delta x_i|^p\sum_{i=n}^{N-1}(N-i) \\ &= \frac{(N-n)(N-n+1)}{2}\max_{i\in\{n,\ldots,N-1\}}|\Delta x_i|^p\end{aligned}$$

and

$$\begin{aligned}\sum_{i=n}^{N-1}(i+1-n)\,|\Delta y_i|^q &\leq \max_{i\in\{n,\ldots,N-1\}}|\Delta y_i|^q\sum_{i=n}^{N-1}(i+1-n) \\ &= \frac{(N-n)(N-n+1)}{2}\max_{i\in\{n,\ldots,N-1\}}|\Delta y_i|^q.\end{aligned}$$

Also

$$\begin{aligned}\sum_{i=n}^{N-1}(N-i)\,|\Delta x_i|^p \leq (N-n)\sum_{i=n}^{N-1}|\Delta x_i|^p,\ \ &\sum_{i=n}^{N-1}(i+1-n)\,|\Delta y_i|^q \\ &\leq (N-n)\sum_{i=n}^{N-1}|\Delta y_i|^q\end{aligned}$$

and the inequality is proved. □

Remark 2. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $x_N=0$, *then for* $n\in\{1,\ldots,N-1\}$,

$$\begin{aligned}(2.13)\qquad \sum_{i=n}^{N-1}|\Delta x_i|\,|x_i| &\leq \left(\sum_{i=n}^{N-1}(N-i)\,|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\,|\Delta x_i|^2\right)^{1/2} \\ &\leq \frac{1}{2}(N-n)\sum_{i=n}^{N-1}|\Delta x_i|^2.\end{aligned}$$

We also have the following result that incorporates both cases:

Theorem 6. *Assume that* $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $y_0 = y_N = 0$, *then for* $n \in \{2, ..., N-1\}$, $p,\ q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{i=1}^{N-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^q \right)^{1/q} \tag{2.14}$$
$$\le \frac{1}{p} \sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^p + \frac{1}{q} \sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^q ,$$

where

$$p_i(n) := \begin{cases} i, & \text{if } 0 \le i \le n-1, \\ N-i, & \text{if } n \le i \le N-1 \end{cases}$$

and

$$q_i(n) := \begin{cases} n-i-1, & \text{if } 0 \le i \le n-1, \\ i+1-n, & \text{if } n \le i \le N-1. \end{cases}$$

In particular,

$$\sum_{i=1}^{N-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^2 \right)^{1/2} \tag{2.15}$$
$$\le \frac{1}{2} \sum_{i=0}^{N-1} \left[p_i(n) |\Delta x_i|^2 + q_i(n) |\Delta y_i|^2 \right],$$

Proof. We have for $n \in \{2, ..., N-1\}$ that

$$\sum_{i=1}^{n-1} |\Delta x_i| |y_i| \le \left(\sum_{i=0}^{n-1} i |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta y_i|^q \right)^{1/q}$$

and

$$\sum_{i=n}^{N-1} |\Delta x_i| |y_i| \le \left(\sum_{i=n}^{N-1} (N-i) |\Delta x_i|^p \right)^{1/p} \left(\sum_{i=n}^{N-1} (i+1-n) |\Delta y_i|^q \right)^{1/q}.$$

If we add these inequalities, then we get, by the elementary inequality

$$ab + cd \le (a^p + c^p)^{1/p} (b^q + d^q)^{1/q}, \ a,\ b,\ c,\ d \ge 0$$

that

$$\begin{aligned}\sum_{i=1}^{N-1}|\Delta x_i||y_i| &\leq \left(\sum_{i=0}^{n-1} i|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{n-1}(n-i-1)|\Delta y_i|^q\right)^{1/q} \\ &\quad + \left(\sum_{i=n}^{N-1}(N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=n}^{N-1}(i+1-n)|\Delta y_i|^2\right)^{1/2} \\ &\leq \left(\sum_{i=0}^{n-1} i|\Delta x_i|^p + \sum_{i=n}^{N-1}(N-i)|\Delta x_i|^p\right)^{1/p} \\ &\quad \times \left(\sum_{i=0}^{n-1}(n-i-1)|\Delta y_i|^q + \sum_{i=n}^{N-1}(i+1-n)|\Delta y_i|^q\right)^{1/q} \\ &= \left(\sum_{i=0}^{N-1} p_i|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1} q_i|\Delta y_i|^q\right)^{1/q},\end{aligned}$$

which proves the first inequality in (2.14). □

Corollary 3. *With the assumptions of Theorem 6, we have*

$$\begin{aligned}&\sum_{i=1}^{N-1}|\Delta x_i||y_i| \\ &\leq \left(\sum_{i=0}^{N-1} p_i(n)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1} q_i(n)|\Delta y_i|^q\right)^{1/q} \\ &\leq \left[\left(n-\frac{N+1}{2}\right)^2+\frac{1}{4}\left(N^2-1\right)\right]\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|\max_{i\in\{0,\dots,N-1\}}|\Delta y_i|\end{aligned} \tag{2.16}$$

for $n\in\{2,\dots,N-1\}$, $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

For $n=\left\lfloor\frac{N+1}{2}\right\rfloor$, *where* $\lfloor\cdot\rfloor$ *is the integer part function, we get*

$$\begin{aligned}&\sum_{i=1}^{N-1}|\Delta x_i||y_i| \\ &\leq \left(\sum_{i=0}^{N-1} p_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor\right)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1} q_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor\right)|\Delta y_i|^q\right)^{1/q} \\ &\leq \left[\left(\left\lfloor\frac{N+1}{2}\right\rfloor-\frac{N+1}{2}\right)^2+\frac{1}{4}\left(N^2-1\right)\right] \\ &\times \max_{i\in\{0,\dots,N-1\}}|\Delta x_i|\max_{i\in\{0,\dots,N-1\}}|\Delta y_i|.\end{aligned} \tag{2.17}$$

Proof. We have

$$\begin{aligned}
&\sum_{i=0}^{N-1} p_i(n)|\Delta x_i|^p \\
&\leq \max_{i\in\{0,\dots,N-1\}} |\Delta x_i|^p \sum_{i=0}^{N-1} p_i(n) = \max_{i\in\{0,\dots,N-1\}} |\Delta x_i|^p \left[\sum_{i=0}^{n-1} i + \sum_{i=n}^{N-1}(N-i)\right] \\
&= \max_{i\in\{0,\dots,N-1\}} |\Delta x_i|^p \left[\frac{(n-1)n}{2} + \frac{(N-n)(N-n+1)}{2}\right] \\
&= \max_{i\in\{0,\dots,N-1\}} |\Delta x_i|^p \left[\left(n-\frac{N+1}{2}\right)^2 + \frac{1}{4}\left(N^2-1\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{N-1} q_i(n)|\Delta y_i|^q &\leq \max_{i\in\{0,\dots,N-1\}} |\Delta y_i|^p \sum_{i=0}^{N-1} q_i(n) \\
&= \max_{i\in\{0,\dots,N-1\}} |\Delta y_i|^p \left[\sum_{i=0}^{n-1}(n-i-1) + \sum_{i=n}^{N-1}(i+1-n)\right] \\
&= \max_{i\in\{0,\dots,N-1\}} |\Delta y_i|^p \left[\frac{(n-1)n}{2} + \frac{(N-n)(N-n+1)}{2}\right] \\
&= \max_{i\in\{0,\dots,N-1\}} |\Delta y_i|^p \left[\left(n-\frac{N+1}{2}\right)^2 + \frac{1}{4}\left(N^2-1\right)\right],
\end{aligned}$$

which proves □

Remark 3. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $x_0 = x_N = 0$, *then*

$$\begin{aligned}
\sum_{i=1}^{N-1} |\Delta x_i||x_i| &\leq \left(\sum_{i=0}^{N-1} p_i(n)|\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n)|\Delta x_i|^2\right)^{1/2} \\
&\leq \frac{1}{2}\sum_{i=0}^{N-1} s_i(n)|\Delta x_i|^2
\end{aligned} \tag{2.18}$$

where

$$s_i(n) := \begin{cases} n-1, & \text{if } 0 \leq i \leq n-1, \\ N-n+1, & \text{if } n \leq i \leq N-1. \end{cases}$$

If we take in (2.18) $n=\left\lfloor\frac{N+1}{2}\right\rfloor+1$, *then by (2.18) we get*

$$\sum_{i=1}^{N-1}|\Delta x_i|\,|x_i| \le \left(\sum_{i=0}^{N-1}p_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor+1\right)|\Delta x_i|^2\right)^{1/2} \times\left(\sum_{i=0}^{N-1}q\left(\left\lfloor\frac{N+1}{2}\right\rfloor+1\right)|\Delta x_i|^2\right)^{1/2} \le \frac{1}{2}\sum_{i=0}^{N-1}s_i\left(\left\lfloor\frac{N+1}{2}\right\rfloor+1\right)|\Delta x_i|^2 \le \frac{1}{2}\left\lfloor\frac{N+1}{2}\right\rfloor\sum_{i=0}^{N-1}|\Delta x_i|^2, \tag{2.19}$$

where

$$s_i(n):=\begin{cases}\left\lfloor\frac{N+1}{2}\right\rfloor, & \text{if } 0\le i\le\left\lfloor\frac{N+1}{2}\right\rfloor,\\ N-\left\lfloor\frac{N+1}{2}\right\rfloor, & \text{if } \left\lfloor\frac{N+1}{2}\right\rfloor+1\le i\le N-1\end{cases}\le\left\lfloor\frac{N+1}{2}\right\rfloor.$$

The inequality (2.19) improves Lasota's result (1.4).

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