DISCRETE INEQUALITIES FOR TWO SEQUENCES IN TERMS OF FORWARD DIFFERENCE

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Abstract. For a sequence $\{x_i\}_{i=0}^{n}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. Assume that $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers. In this paper we show among others that, if $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers with $x_0 = y_N = 0$, then

$$\sum_{i=1}^{N-1} |x_i y_i| \le \frac{1}{2} \left[\sum_{i=0}^{N-1} (N-i) |\Delta x_i|^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) |\Delta y_i|^2 \right]^{1/2}.$$

Some particular inequalities of interest are also provided.

1. Introduction

For a sequence $\{x_i\}_{i=0}^{N}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., N-1$. Recall the summation by parts formula stated as

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k, \tag{1.1}$$

where a_k and b_k are some sequences for which the products above exist.

In [8], Lasota provided discrete versions of Opial inequality [10] about the forward difference operator as follows:

Theorem 1. *Let $\{x_i\}_{i=0}^{N}$ be a sequence of real numbers with $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.2}$$

where $\lfloor \cdot \rfloor$ is the integer part function. If N is even, then the inequality (1.2) is sharp.

For various Opial type inequalities, see [2]-[4] and [9].

Also, we have the following results, see [1]:

Theorem 2. *Let $\{x_i\}_{i=0}^{N}$ be a sequence of real numbers. If $x_0 = 0$, then*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \le \frac{1}{2} (\tau - 1) \sum_{i=0}^{\tau-1} |\Delta x_i|^2, \ \tau \in \{2, ..., N\}. \tag{1.3}$$

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If $x_N = 0$*, then*

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \leq \frac{1}{2}(N-\tau+1) \sum_{i=0}^{N-1} |\Delta x_i|^2, \ \tau \in \{1,...,N-1\}. \tag{1.4}$$

For other discrete Opial type inequalities, see [6], [7] and [11]-[14].

In the recent paper [5] we obtained the following extension for two sequences:

Theorem 3. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $y_0 = y_N = 0$*, then for* $n \in \{2,...,N-1\}$,

$$\begin{aligned}\sum_{i=1}^{N-1} |\Delta x_i| |y_i| &\leq \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^2\right)^{1/2} \\ &\leq \frac{1}{2}\sum_{i=0}^{N-1}\left(p_i(n)|\Delta x_i|^2 + q_i(n)|\Delta y_i|^2\right),\end{aligned} \tag{1.5}$$

where

$$p_i(n) := \begin{cases} i, & \text{if } 0 \leq i \leq n-1, \\ N-i, & \text{if } n \leq i \leq N-1 \end{cases}$$

and

$$q_i(n) := \begin{cases} n-i-1, & \text{if } 0 \leq i \leq n-1, \\ i+1-n, & \text{if } n \leq i \leq N-1. \end{cases}$$

Corollary 1. *Assume that* $\{x_i\}_{i=0}^N$ *is a sequence of complex numbers with* $x_0 = x_N = 0$*, then*

$$\begin{aligned}\sum_{i=1}^{N-1} |\Delta x_i| |x_i| &\leq \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q(n)_i |\Delta x_i|^2\right)^{1/2} \\ &\leq \frac{1}{2}\sum_{i=0}^{N-1} s_i(n)|\Delta x_i|^2,\end{aligned} \tag{1.6}$$

where

$$s_i(n) := \begin{cases} n-1, & \text{if } 0 \leq i \leq n-1, \\ N-n+1, & \text{if } n \leq i \leq N-1. \end{cases}$$

Remark 1. *If we take in (1.6)* $n = \left\lfloor \frac{N+1}{2} \right\rfloor + 1$*, then by (1.6) we get*

$$\begin{aligned}&\sum_{i=1}^{N-1} |\Delta x_i| |x_i| \\ &\leq \left(\sum_{i=0}^{N-1} p_i\left(\left\lfloor \frac{N+1}{2}\right\rfloor + 1\right) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q\left(\left\lfloor \frac{N+1}{2}\right\rfloor + 1\right) |\Delta x_i|^2\right)^{1/2} \\ &\leq \frac{1}{2}\sum_{i=0}^{N-1} s_i\left(\left\lfloor \frac{N+1}{2}\right\rfloor + 1\right)|\Delta x_i|^2 \leq \frac{1}{2}\left\lfloor \frac{N+1}{2}\right\rfloor \sum_{i=0}^{N-1}|\Delta x_i|^2,\end{aligned} \tag{1.7}$$

where

$$s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \le i \le \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \le i \le N-1 \end{cases} \le \left\lfloor \frac{N+1}{2} \right\rfloor.$$

The inequality (1.7) is a refinement of Lasota's result (1.2).

In this paper we establish some upper bounds for the sum

$$\sum_{i=0}^{N} |x_i y_i|$$

in terms of the forward differences Δx_i and Δy_i of the complex sequences $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ under some assumptions for the end terms of these sequences.

2. Main Results

We have the following result for two sequences:

Theorem 4. *Assume that $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers. If $x_0 = y_0 = 0$, then*

$$\begin{aligned} \sum_{i=1}^{N} |x_i y_i| &\le \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta x_i|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta y_i|^2 \right)^{1/2} \\ &\le \begin{cases} \frac{1}{6} N(N+1)(2N+1) \\ \times \max_{i \in \{0,\dots,N-1\}} |\Delta x_i| \max_{i \in \{0,\dots,N-1\}} |\Delta y_i|, \\ \frac{1}{2} N(N+1) \left(\sum_{i=0}^{N-1} |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} |\Delta y_i|^2 \right)^{1/2}. \end{cases} \end{aligned} \tag{2.1}$$

Also,

$$\begin{aligned} \sum_{i=1}^{N} |x_i y_i| &\le \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta x_i|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta y_i|^2 \right)^{1/2} \\ &\le \frac{1}{4} \sum_{i=0}^{N-1} (N-i)(N+i+1) \left(|\Delta x_i|^2 + |\Delta y_i|^2 \right). \end{aligned} \tag{2.2}$$

Proof. Since $x_0 = y_0 = 0$, hence $x_i = \sum_{j=0}^{i-1} \Delta x_j$ and $y_i = \sum_{j=0}^{i-1} \Delta y_j$ for $i = 1, ..., N$. Then

$$\begin{aligned} \sum_{i=1}^{N} |x_i| |y_i| &= \sum_{i=1}^{N} \left| \sum_{j=0}^{i-1} \Delta x_j \right| \left| \sum_{j=0}^{i-1} \Delta y_j \right| \\ &= \sum_{i=1}^{N} i \frac{1}{\sqrt{i}} \left| \sum_{j=0}^{i-1} \Delta x_j \right| \frac{1}{\sqrt{i}} \left| \sum_{j=0}^{i-1} \Delta y_j \right| =: A. \end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality we have

$$\frac{1}{\sqrt{i}}\left|\sum_{j=0}^{i-1}\Delta x_j\right| \leq \left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right)^{1/2}, \ \frac{1}{\sqrt{i}}\left|\sum_{j=0}^{i-1}\Delta y_j\right| \leq \left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right)^{1/2},$$

which gives.

$$A \leq \sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right)^{1/2}\left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right)^{1/2}.$$

By the weighted (CBS) inequality we have

$$\begin{aligned}
&\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right)^{1/2}\left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right)^{1/2}\\
&\leq \left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right)^{1/2}\right]^2\right)^{1/2}\left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right)^{1/2}\right]^2\right)^{1/2}\\
&= \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right)\right)^{1/2}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right)\right)^{1/2},
\end{aligned}$$

which implies that

$$A \leq \left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right)\right)^{1/2}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right)\right)^{1/2} =: B. \tag{2.3}$$

From the formula (1.1), we get

$$\sum_{i=1}^{N} a_i \Delta b_i = a_N b_{N+1} - a_1 b_1 - \sum_{i=1}^{N-1} b_{i+1}\Delta a_i. \tag{2.4}$$

Now, if we take $a_i = \sum_{j=0}^{i-1}\left|\Delta x_j\right|^2$, $i = 1, ..., N$, $b_i = \frac{1}{2}i(i-1)$, then $a_N = \sum_{j=0}^{N-1}\left|\Delta x_j\right|^2$,

$$\Delta b_i = b_{i+1} - b_i = \frac{1}{2}i(i+1) - \frac{1}{2}i(i-1) = i,$$

and

$$\Delta a_i = a_{i+1} - a_i = \sum_{j=0}^{i}\left|\Delta x_j\right|^2 - \sum_{j=0}^{i-1}\left|\Delta x_j\right|^2 = \left|\Delta x_i\right|^2.$$

By (2.4) we derive

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta x_j\right|^2\right) = \frac{1}{2}N\left(N+1\right)\sum_{j=0}^{N-1}\left|\Delta x_j\right|^2 - \sum_{k=1}^{N-1}\frac{1}{2}i\left(i+1\right)\left|\Delta x_i\right|^2 \tag{2.5}$$
$$= \sum_{i=0}^{N-1}\left[\frac{1}{2}N\left(N+1\right) - \frac{1}{2}i\left(i+1\right)\right]\left|\Delta x_i\right|^2$$
$$= \frac{1}{2}\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta x_i\right|^2$$

and, similarly

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta y_j\right|^2\right) = \frac{1}{2}\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta y_i\right|^2. \tag{2.6}$$

Therefore

$$B = \frac{1}{2}\left(\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta x_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta y_i\right|^2\right)^{1/2}$$

and by (2.3) we derive the first inequality in (2.1).

Now observe that

$$\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta x_i\right|^2 \le \max_{i\in\{0,\dots,N-1\}}\left|\Delta x_i\right|^2\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)$$
$$= \frac{1}{3}N\left(N+1\right)\left(2N+1\right)\max_{i\in\{0,\dots,N-1\}}\left|\Delta x_i\right|^2,$$

which proves the first branch in (2.1).

Observe also that

$$\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta x_i\right|^2 \le \max_{i\in\{0,\dots,N-1\}}\left[\left(N-i\right)\left(N+i+1\right)\right]\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2$$
$$= \max_{i\in\{0,\dots,N-1\}}\left[N\left(N+1\right) - i\left(i+1\right)\right]\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2$$
$$= N\left(N+1\right)\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2,$$

which proves the second branch in (2.1).

The last inequality in (2.2) follows by the A-G-means inequality

$$\sqrt{\alpha\beta} \le \frac{1}{2}\left(\alpha+\beta\right),\ \alpha,\ \beta \ge 0.$$

□

Corollary 2. *Assume that* $\{x_i\}_{i=0}^N$ *is a sequence of complex numbers with* $x_0 = 0$, *then*

$$\sum_{i=1}^{N} |x_i|^2 \le \frac{1}{2} \sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta x_i|^2 \le \begin{cases} \frac{1}{6} N(N+1)(2N+1) \max_{i\in\{0,\dots,N-1\}} |\Delta x_i|^2, \\ \\ \frac{1}{2} N(N+1) \sum_{i=0}^{N-1} |\Delta x_i|^2. \end{cases} \tag{2.7}$$

Also, we have:

Theorem 5. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $x_N = y_N = 0$, *then*

$$\begin{aligned} &\sum_{i=0}^{N-1} |x_i y_i| \\ &\le \frac{1}{2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta y_i|^2 \right)^{1/2} \\ &\le \begin{cases} \frac{1}{6} N(N+1)(2N+1) \\ \times \max_{i\in\{0,\dots,N-1\}} |\Delta x_i| \max_{i\in\{0,\dots,N-1\}} |\Delta y_i|, \\ \\ \frac{1}{2} N(N+1) \left(\sum_{i=0}^{N-1} |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} |\Delta y_i|^2 \right)^{1/2}. \end{cases} \end{aligned} \tag{2.8}$$

Also, we have

$$\begin{aligned} &\sum_{i=0}^{N-1} |x_i y_i| \\ &\le \frac{1}{2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta x_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta y_i|^2 \right)^{1/2} \\ &\le \frac{1}{4} \sum_{i=0}^{N-1} (i+1)(2N-i) \left(|\Delta x_i|^2 + |\Delta y_i|^2 \right). \end{aligned} \tag{2.9}$$

Proof. If $x_N = y_N = 0$, then $x_i = -\sum_{j=i}^{N-1} \Delta x_j$ and $y_i = -\sum_{j=i}^{N-1} \Delta y_j$, $i \in \{0, ..., N-1\}$. Then

$$\begin{aligned} \sum_{i=0}^{N-1} |x_i| |y_i| &= \sum_{i=0}^{N-1} \left| \sum_{j=i}^{N-1} \Delta x_j \right| \left| \sum_{j=i}^{N-1} \Delta y_j \right| \\ &= \sum_{i=0}^{N-1} (N-i) \frac{1}{\sqrt{N-i}} \left| \sum_{j=i}^{N-1} \Delta x_j \right| \frac{1}{\sqrt{N-i}} \left| \sum_{j=i}^{N-1} \Delta y_j \right| =: C. \end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality we have

$$\frac{1}{\sqrt{N-i}} \left| \sum_{j=i}^{N-1} \Delta x_j \right| \le \left(\sum_{j=i}^{N-1} |\Delta x_j|^2 \right)^{1/2}$$

and

$$\frac{1}{\sqrt{N-i}}\left|\sum_{j=i}^{N-1}\Delta y_j\right| \leq \left(\sum_{j=i}^{N-1}\left|\Delta y_j\right|^2\right)^{1/2},$$

which gives

$$C \leq \sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta x_j\right|^2\right)^{1/2}\left(\sum_{j=i}^{N-1}\left|\Delta y_j\right|^2\right)^{1/2}.$$

By the weighted (CBS) inequality we have

$$\begin{aligned}&\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta x_j\right|^2\right)^{1/2}\left(\sum_{j=i}^{N-1}\left|\Delta y_j\right|^2\right)^{1/2}\\ &\leq \left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta x_j\right|^2\right)\right)^{1/2}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta y_j\right|^2\right)\right)^{1/2},\end{aligned}$$

which gives

$$\begin{aligned}C &\leq \left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta x_j\right|^2\right)\right)^{1/2}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta y_j\right|^2\right)\right)^{1/2}\\ &=: D.\end{aligned} \tag{2.10}$$

From (1.1) we get for $m=0$ and $n=N-1$ that

$$\sum_{i=0}^{N-1}a_i\Delta b_i = a_{N-1}b_N - a_0b_0 - \sum_{i=0}^{N-2}b_{i+1}\Delta a_i. \tag{2.11}$$

Take $a_i = \sum_{j=i}^{N-1}\left|\Delta x_j\right|^2$ and $b_i = -\frac{1}{2}(N-i)(N-i+1)$, then we get

$$\begin{aligned}\Delta b_i = b_{i+1}-b_i &= -\frac{1}{2}(N-i-1)(N-i)+\frac{1}{2}(N-i)(N-i+1)\\ &= \frac{1}{2}(N-i)(-N+i+1+N-i+1) = N-i\end{aligned}$$

and

$$\Delta a_i = a_{i+1}-a_i = \sum_{j=i+1}^{N-1}\left|\Delta x_j\right|^2 - \sum_{j=i}^{N-1}\left|\Delta x_j\right|^2 = -\left|\Delta x_i\right|^2.$$

Then

$$\begin{aligned}
&\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^2\right) \\
&= \frac{1}{2}N(N+1)\sum_{j=0}^{N-1}|\Delta x_j|^2 - \frac{1}{2}\sum_{i=0}^{N-2}(N-i-1)(N-i)|\Delta x_i|^2 \\
&= \frac{1}{2}N(N+1)\sum_{i=0}^{N-1}|\Delta x_j|^2 - \frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2 \\
&= \frac{1}{2}\sum_{i=0}^{N-1}\left[N(N+1)-(N-i-1)(N-i)\right]|\Delta x_i|^2 \\
&= \frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^2
\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta y_j|^2\right) = \frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2 .$$

Therefore

$$D = \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2\right)^{1/2},$$

and by (2.10) we derive the first inequality in (2.8).

Now, observe that

$$\begin{aligned}
\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^2 &\le \max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2\sum_{i=0}^{N-1}(i+1)(2N-i) \\
&= \frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2
\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2 \le \frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta y_i|^2 ,$$

which proves the first branch of (2.8).

Also

$$\begin{aligned}
&\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta x_i\right|^2 \\
&=\sum_{i=0}^{N-1}\left[N(N+1)-(N-i-1)(N-i)\right]\left|\Delta x_i\right|^2 \\
&\leq \max_{i\in\{0,\dots,N-1\}}\left[N(N+1)-(N-i-1)(N-i)\right]\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2 \\
&=N(N+1)\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2,
\end{aligned}$$

which proves the second branch of (2.8). □

Corollary 3. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $y_N = 0$, *then*

$$\sum_{i=0}^{N-1}\left|x_i\right|^2 \leq \frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta x_i\right|^2 \leq \begin{cases} \frac{1}{6}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}\left|\Delta x_i\right|^2, \\ \frac{1}{2}N(N+1)\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2. \end{cases} \tag{2.12}$$

We also have:

Theorem 6. *Assume that* $\{x_i\}_{i=0}^{N}, \{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $x_0 = y_N = 0$, *then*

$$\begin{aligned}
\sum_{i=1}^{N-1}\left|x_i y_i\right| &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta x_i\right|^2\right)^{1/2} \\
&\quad\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta y_i\right|^2\right)^{1/2} \\
&\leq \begin{cases} \frac{1}{6}N(N+1)(2N+1) \\ \times\max_{i\in\{0,\dots,N-1\}}\left|\Delta x_i\right|\max_{i\in\{0,\dots,N-1\}}\left|\Delta y_i\right|, \\ \frac{1}{2}N(N+1)\left(\sum_{i=0}^{N-1}\left|\Delta x_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\left|\Delta y_i\right|^2\right)^{1/2}. \end{cases}
\end{aligned} \tag{2.13}$$

Also,

$$\sum_{i=1}^{N-1}|x_iy_i| \tag{2.14}$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2\right)^{1/2}$$
$$\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2\right)^{1/2}$$
$$\leq \frac{1}{4}\left[\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2+\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2\right].$$

Proof. Since $x_0=0$, and $y_N=0$, hence $x_i=\sum_{j=0}^{i-1}\Delta x_j$ and $y_i=-\sum_{j=i}^{N-1}\Delta y_j$. Then by (CBS) inequality

$$\sum_{i=1}^{N-1}|x_iy_i|=\sum_{i=1}^{N-1}|x_i||y_i|=\sum_{i=1}^{N-1}\left|\sum_{j=0}^{i-1}\Delta x_j\right|\left|\sum_{j=i}^{N-1}\Delta y_j\right| \tag{2.15}$$
$$\leq\sum_{i=1}^{N-1}\sqrt{i}\left(\sum_{j=0}^{i-1}|\Delta x_j|^2\right)^{1/2}\sqrt{N-i}\left(\sum_{j=i}^{N-1}|\Delta y_j|^2\right)^{1/2}$$
$$\leq\left(\sum_{i=1}^{N-1}\left[\sqrt{i}\left(\sum_{j=0}^{i-1}|\Delta x_j|^2\right)^{1/2}\right]^2\right)^{1/2}$$
$$\times\left(\sum_{i=1}^{N-1}\left[\sqrt{N-i}\left(\sum_{j=i}^{N-1}|\Delta y_j|^2\right)^{1/2}\right]^2\right)^{1/2}$$
$$=\left(\sum_{i=1}^{N-1}i\left(\sum_{j=0}^{i-1}|\Delta x_j|^2\right)\right)^{1/2}\left(\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta y_j|^2\right)\right)^{1/2}=:E.$$

Since

$$\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}|\Delta x_j|^2\right)=\frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2$$

and

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta y_j|^2\right)=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2,$$

hence

$$E=\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2\right)^{1/2}.$$

By employing (2.15) we get the first part of (2.13).

The rest follows as above. □

Corollary 4. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $x_0 = x_N = 0$, *then*

$$\begin{aligned}
(2.16)\qquad \sum_{i=1}^{N-1} |x_i|^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2\right)^{1/2} \\
&\quad\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^2\right)^{1/2} \\
&\le \begin{cases} \frac{1}{6}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2, \\ \frac{1}{2}N(N+1)\sum_{i=0}^{N-1}|\Delta x_i|^2. \end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
(2.17)\qquad \sum_{i=1}^{N-1} |x_i|^2 &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2\right)^{1/2} \\
&\quad\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^2\right)^{1/2} \\
&\le \frac{1}{4}\left[\sum_{i=0}^{N-1}\left((N-i)(N+i+1)+(i+1)(2N-i)\right)|\Delta x_i|^2\right].
\end{aligned}$$

Remark 2. *If we use the inequality (2.15) for* $y_i = x_i$, *we get for* $\{x_i\}_{i=0}^{N}$, *a sequence of complex numbers with* $x_0 = x_N = 0$,

$$\begin{aligned}
(2.18)\qquad \sum_{i=1}^{N-1} |x_i|^2 &\le \left(\sum_{i=1}^{N-1} i\left(\sum_{j=0}^{i-1}|\Delta x_j|^2\right)\right)^{1/2}\left(\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^2\right)\right)^{1/2} \\
&\le \frac{1}{2}\sum_{i=1}^{N-1}\left[i\left(\sum_{j=0}^{i-1}|\Delta x_j|^2\right)+(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^2\right)\right] \\
&\le \frac{1}{2}N\sum_{i=1}^{N-1}\max\left\{\sum_{j=0}^{i-1}|\Delta x_j|^2, \sum_{j=i}^{N-1}|\Delta x_j|^2\right\}.
\end{aligned}$$

3. Related Results

From a different perspective, we also have:

Theorem 7. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $x_0 = y_N = 0$, *then*

$$\begin{aligned}
(3.1)\qquad & \sum_{i=1}^{N-1} |x_i y_i| \\
& \leq \frac{1}{2}\left[\sum_{i=0}^{N-1} (N-i-1)(N-i)|\Delta x_i|^2\right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1)\, i\, |\Delta y_i|^2\right]^{1/2} \\
& \leq \begin{cases} \frac{1}{6}(N-1)N(N+1)\max_{i\in\{0,\dots,N-1\}} |\Delta x_i| \max_{i\in\{0,\dots,N-1\}} |\Delta y_i|, \\ \\ \frac{1}{2}N(N-1)\left(\sum_{i=0}^{N-1} |\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1} |\Delta y_i|^2\right)^{1/2}. \end{cases}
\end{aligned}$$

Also,

$$\begin{aligned}
(3.2)\qquad & \sum_{i=1}^{N-1} |x_i y_i| \\
& \leq \frac{1}{2}\left[\sum_{i=0}^{N-1} (N-i-1)(N-i)|\Delta x_i|^2\right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1)\, i\, |\Delta y_i|^2\right]^{1/2} \\
& \leq \frac{1}{4}\sum_{i=0}^{N-1} \left[(N-i-1)(N-i)|\Delta x_i|^2 + (i+1)\, i\, |\Delta y_i|^2\right].
\end{aligned}$$

Proof. From (2.15) we also have

$$\begin{aligned}
(3.3)\qquad & \sum_{i=1}^{N-1} |x_i y_i| \\
& = \sum_{i=1}^{N-1} |x_i|\,|y_i| = \sum_{i=1}^{N-1} \left|\sum_{j=0}^{i-1} \Delta x_j\right| \left|\sum_{j=i}^{N-1} \Delta y_j\right| \\
& \leq \sum_{i=1}^{N-1} \sqrt{i}\left(\sum_{j=0}^{i-1} |\Delta x_j|^2\right)^{1/2} \sqrt{N-i}\left(\sum_{j=i}^{N-1} |\Delta y_j|^2\right)^{1/2} \\
& = \sum_{i=1}^{N-1} \sqrt{N-i}\left(\sum_{j=0}^{i-1} |\Delta x_j|^2\right)^{1/2} \sqrt{i}\left(\sum_{j=i}^{N-1} |\Delta y_j|^2\right)^{1/2} \\
& \leq \left[\sum_{i=1}^{N-1} (N-i)\left(\sum_{j=0}^{i-1} |\Delta x_j|^2\right)\right]^{1/2} \left[\sum_{i=1}^{N-1} i\left(\sum_{j=i}^{N-1} |\Delta y_j|^2\right)\right]^{1/2} =: F.
\end{aligned}$$

We use the identity

$$(3.4)\qquad \sum_{i=1}^{N} a_i \Delta b_i = a_N b_{N+1} - a_1 b_1 - \sum_{i=1}^{N-1} b_{i+1}\Delta a_i.$$

Take $a_i = \sum_{j=0}^{i-1} |\Delta x_j|^2$ and $b_i = -\frac{1}{2}(N-i)(N-i+1)$, then

$$\Delta b_i = b_{i+1} - b_i = -\frac{1}{2}(N-i-1)(N-i) + \frac{1}{2}(N-i)(N-i+1)$$
$$= \frac{1}{2}(N-i)[N-i+1-(N-i-1)] = (N-i),$$

$$\Delta a_i = a_{i+1} - a_i = \sum_{j=0}^{i} |\Delta x_j|^2 - \sum_{j=0}^{i-1} |\Delta x_j|^2 = |\Delta x_i|^2$$

and by (3.4)

$$\sum_{i=1}^{N-1} (N-i) \left(\sum_{j=0}^{i-1} |\Delta x_j|^2 \right) \tag{3.5}$$
$$= 0 \sum_{j=0}^{N-1} |\Delta x_j|^2 + \frac{1}{2} N(N-1) |\Delta x_0|^2 + \frac{1}{2} \sum_{i=1}^{N-1} (N-i-1)(N-i) |\Delta x_i|^2$$
$$= \frac{1}{2} \sum_{i=0}^{N-1} (N-i-1)(N-i) |\Delta x_i|^2 .$$

Take $a_i = \sum_{j=i}^{N-1} |\Delta y_j|^2$ and $b_i = \frac{1}{2} i(i-1)$, then

$$\Delta b_i = b_{i+1} - b_i = \frac{1}{2}(i+1)i - \frac{1}{2}i(i-1) = \frac{1}{2}i(i+1-i+1) = i,$$

$$\Delta a_i = a_{i+1} - a_i = \sum_{j=i+1}^{N-1} |\Delta y_j|^2 - \sum_{j=i}^{N-1} |\Delta y_j|^2 = -|\Delta y_i|^2$$

and by the identity

$$\sum_{i=1}^{N-1} a_i \Delta b_i = a_{N-1} b_N - a_1 b_1 - \sum_{i=1}^{N-2} b_{i+1} \Delta a_i .$$

we get

$$\sum_{i=1}^{N-1} i \left(\sum_{j=i}^{N-1} |\Delta y_j|^2 \right) = \frac{1}{2} N(N-1) |\Delta y_{N-1}|^2 + \frac{1}{2} \sum_{i=1}^{N-2} (i+1) i |\Delta y_i|^2 \tag{3.6}$$
$$= \frac{1}{2} \sum_{i=1}^{N-1} (i+1) i |\Delta y_i|^2 = \frac{1}{2} \sum_{i=0}^{N-1} (i+1) i |\Delta y_i|^2 .$$

Therefore

$$F = \left[\frac{1}{2} \sum_{i=0}^{N-1} (N-i-1)(N-i) |\Delta x_i|^2 \right]^{1/2} \left[\frac{1}{2} \sum_{i=0}^{N-1} (i+1) i |\Delta y_i|^2 \right]^{1/2}$$
$$= \frac{1}{2} \left[\sum_{i=0}^{N-1} (N-i-1)(N-i) |\Delta x_i|^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) i |\Delta y_i|^2 \right]^{1/2},$$

which proves the first inequality in (3.1).

Now, observe that

$$\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2 \le \max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2\sum_{i=0}^{N-1}(N-i-1)(N-i)$$
$$= \max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2\frac{1}{3}(N-1)N(N+1)$$

and

$$\sum_{i=0}^{N-1}(i+1)i|\Delta y_i|^2 \le \max_{i\in\{0,\dots,N-1\}}|\Delta y_i|^2\sum_{i=0}^{N-1}(i+1)i$$
$$= \max_{i\in\{0,\dots,N-1\}}|\Delta y_i|^2\frac{1}{3}(N-1)N(N+1),$$

which proves the first branch in (3.1).

Observe also that

$$\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2 \le \max_{i\in\{0,\dots,N-1\}}[(N-i-1)(N-i)]\sum_{i=0}^{N-1}|\Delta x_i|^2$$
$$= N(N-1)\sum_{i=0}^{N-1}|\Delta x_i|^2$$

and

$$\sum_{i=0}^{N-1}(i+1)i|\Delta y_i|^2 \le \max_{i\in\{0,\dots,N-1\}}[i(i+1)]\sum_{i=0}^{N-1}|\Delta y_i|^2 \le N(N-1)\sum_{i=0}^{N-1}|\Delta y_i|^2,$$

which proves the second branch in (3.1). □

Corollary 5. *Assume that* $\{x_i\}_{i=0}^{N}$ *is a sequence of complex numbers with* $x_0 = x_N = 0$, *then*

$$\sum_{i=1}^{N-1}|x_i|^2 \tag{3.7}$$
$$\le \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)i|\Delta x_i|^2\right]^{1/2}$$
$$\le \begin{cases} \frac{1}{6}(N-1)N(N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2 \\ \\ \frac{1}{2}N(N-1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^2\right). \end{cases}$$

Also,

$$\begin{aligned}\sum_{i=1}^{N-1}|x_iy_i| & \\ &\leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta x_i|^2\right]^{1/2} \\ &\leq \frac{1}{4}\left[\sum_{i=0}^{N-1}[(N-i-1)(N-i)+(i+1)\,i]\,|\Delta x_i|^2\right].\end{aligned} \tag{3.8}$$

We also have the following simpler result:

Theorem 8. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $x_0=y_N=0$, *then*

$$\begin{aligned}\sum_{i=1}^{N-1}|x_iy_i| &\leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i)|\Delta x_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)|\Delta y_i|^2\right]^{1/2} \\ &\leq \frac{1}{4}\sum_{i=0}^{N-1}\left[(N-i)|\Delta x_i|^2+(i+1)|\Delta y_i|^2\right].\end{aligned} \tag{3.9}$$

Proof. From (2.13) we have

$$\begin{aligned}\sum_{i=1}^{N-1}|x_iy_i| &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2\right)^{1/2} \\ &\quad\times\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2\right)^{1/2}\end{aligned}$$

and from (3.1)

$$\begin{aligned}\sum_{i=1}^{N-1}|x_iy_i| & \\ &\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^2\right)^{1/2}.\end{aligned}$$

If we add these two inequalities, then we get by the elementary inequality

$$ab+cd\leq\left(a^2+c^2\right)^{1/2}\left(b^2+d^2\right)^{1/2}$$

that

$$
\begin{aligned}
& 2\sum_{i=1}^{N-1}|x_iy_i| \\
& \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2\right)^{1/2} \\
& +\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)i|\Delta y_i|^2\right)^{1/2} \\
& \leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^2+\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^2\right]^{1/2} \\
& \times\left[\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^2+\sum_{i=0}^{N-1}(i+1)i|\Delta y_i|^2\right]^{1/2} \\
& = N\left[\sum_{i=0}^{N-1}(N-i)|\Delta x_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)|\Delta y_i|^2\right]^{1/2},
\end{aligned}
$$

which proves (3.9). □

Corollary 6. *Assume that* $\{x_i\}_{i=0}^{N}$, *is a sequences of complex numbers with* $x_0 = x_N = 0$, *then*

$$
\begin{aligned}
\sum_{i=1}^{N-1}|x_i|^2 & \leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i)|\Delta x_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)|\Delta x_i|^2\right]^{1/2} \\
& \leq \frac{1}{4}(N+1)\sum_{i=0}^{N-1}|\Delta x_i|^2 .
\end{aligned} \tag{3.10}
$$

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