p-NORMS DISCRETE INEQUALITIES FOR TWO SEQUENCES IN TERMS OF FORWARD DIFFERENCE

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a sequence $\{x_i\}_{i=0}^{n}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. Assume that $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers. In this paper we show among others that, if $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ are sequences of complex numbers with $x_0 = y_N = 0$, then

$$\sum_{i=1}^{N-1} |x_i y_i| \le \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i-1)(N-i) |\Delta x_i|^p \right)^{1/p} \times \left(\sum_{i=0}^{N-1} (i+1) i |\Delta y_i|^q \right)^{1/q},$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Some particular inequalities of interest are also provided.

1. Introduction

For a sequence $\{x_i\}_{i=0}^{N}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., N-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k, \tag{1.1}$$

where a_k and b_k are some sequences for which the products above exist.

In [8], Lasota provided discrete versions of Opial inequality [10] about the forward difference operator as follows:

Theorem 1. *Let $\{x_i\}_{i=0}^{N}$ be a sequence of real numbers with $x_0 = x_N = 0$. Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i \Delta x_i| \le \frac{1}{2} \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta x_i|^2, \tag{1.2}$$

where $\lfloor \cdot \rfloor$ is the integer part function. If N is even, then the inequality (1.2) is sharp.

For various Opial type inequalities, see [2]-[4] and [9].

Also, we have the following results, see [1]:

Theorem 2. *Let $\{x_i\}_{i=0}^{N}$ be a sequence of real numbers. If $x_0 = 0$, then*

$$\sum_{i=1}^{\tau-1} |x_i \Delta x_i| \le \frac{1}{2} (\tau - 1) \sum_{i=0}^{\tau-1} |\Delta x_i|^2, \ \tau \in \{2, ..., N\}. \tag{1.3}$$

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If $x_N = 0$, *then*

$$\sum_{i=\tau}^{N-1} |x_i \Delta x_i| \leq \frac{1}{2}(N-\tau+1)\sum_{i=0}^{N-1} |\Delta x_i|^2, \ \tau \in \{1, ..., N-1\}. \tag{1.4}$$

For other discrete Opial type inequalities, see [6], [7] and [11]-[14].

In the recent paper [ ] we obtained the following result:

Theorem 3. *Assume that* $\{x_i\}_{i=0}^N, \{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $y_0 = y_N = 0$, *then for* $n \in \{2, ..., N-1\}$, $p, q > 1$, $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned} \sum_{i=1}^{N-1} |\Delta x_i| |y_i| &\leq \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^p\right)^{1/p} \left(\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^q\right)^{1/q} \\ &\leq \frac{1}{p}\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^p + \frac{1}{q}\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^q, \end{aligned} \tag{1.5}$$

where

$$p_i(n) := \begin{cases} i, \ \text{if } 0 \leq i \leq n-1, \\ \\ N-i, \ \text{if } n \leq i \leq N-1 \end{cases}$$

and

$$q_i(n) := \begin{cases} n-i-1, \ \text{if } 0 \leq i \leq n-1, \\ \\ i+1-n, \ \text{if } n \leq i \leq N-1. \end{cases}$$

In particular,

$$\begin{aligned} \sum_{i=1}^{N-1} |\Delta x_i| |y_i| &\leq \left(\sum_{i=0}^{N-1} p_i(n) |\Delta x_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i(n) |\Delta y_i|^2\right)^{1/2} \\ &\leq \frac{1}{2}\sum_{i=0}^{N-1} \left[p_i(n) |\Delta x_i|^2 + q_i(n) |\Delta y_i|^2\right], \end{aligned} \tag{1.6}$$

In this paper we establish some upper bounds for the sum

$$\sum_{i=0}^{N} |x_i y_i|$$

in terms of the p-norms incorporating the forward differences Δx_i and Δy_i of the complex sequences $\{x_i\}_{i=0}^N, \{y_i\}_{i=0}^N$ under some assumptions for the end terms of these sequences.

2. Main Results

We have the following result for two sequences:

Theorem 4. *Assume that* $\{x_i\}_{i=0}^N$, $\{y_i\}_{i=0}^N$ *are sequences of complex numbers. If* $x_0 = y_0 = 0$, *then*

$$\sum_{i=1}^{N}|x_iy_i| \le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p\right)^{1/p} \times \left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta y_i|^q\right)^{1/q} \le \begin{cases} \frac{1}{6}N(N+1)(2N+1) \\ \times \max_{i\in\{0,\dots,N-1\}}|\Delta x_i| \max_{i\in\{0,\dots,N-1\}}|\Delta y_i|, \\ \\ \frac{1}{2}N(N+1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}|\Delta y_i|^q\right)^{1/q}, \end{cases} \tag{2.1}$$

for $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Also,

$$\sum_{i=1}^{N}|x_iy_i| \le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p\right)^{1/p} \times \left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta y_i|^q\right)^{1/q} \le \frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left(\frac{1}{p}|\Delta x_i|^p+\frac{1}{q}|\Delta y_i|^q\right). \tag{2.2}$$

Proof. Since $x_0 = y_0 = 0$, hence $x_i = \sum_{j=0}^{i-1}\Delta x_j$ and $y_i = \sum_{j=0}^{i-1}\Delta y_j$ for $i = 1, ..., N$. Then

$$\sum_{i=1}^{N}|x_i||y_i| = \sum_{i=1}^{N}\left|\sum_{j=0}^{i-1}\Delta x_j\right|\left|\sum_{j=0}^{i-1}\Delta y_j\right| = \sum_{i=1}^{N} i\frac{1}{i^{1/q}}\left|\sum_{j=0}^{i-1}\Delta x_j\right|\frac{1}{i^{1/p}}\left|\sum_{j=0}^{i-1}\Delta y_j\right| =: A.$$

By Hölder's inequality we have

$$\frac{1}{i^{1/q}}\left|\sum_{j=0}^{i-1}\Delta x_j\right| \le \left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)^{1/p}, \quad \frac{1}{i^{1/p}}\left|\sum_{j=0}^{i-1}\Delta y_j\right| \le \left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)^{1/q},$$

which gives.

$$A \le \sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)^{1/p}\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)^{1/q}.$$

By the weighted Hölder inequality we have

$$\begin{aligned}
&\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)^{1/p}\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)^{1/q} \\
&\leq\left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)^{1/p}\right]^p\right)^{1/p}\left(\sum_{i=1}^{N} i\left[\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)^{1/q}\right]^q\right)^{1/q} \\
&=\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)\right)^{1/p}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)\right)^{1/q},
\end{aligned}$$

which implies that

$$A\leq\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)\right)^{1/p}\left(\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)\right)^{1/q}=:B. \tag{2.3}$$

From the formula (1.1), we get

$$\sum_{i=1}^{N} a_i\Delta b_i=a_N b_{N+1}-a_1 b_1-\sum_{k=1}^{N-1} b_{i+1}\Delta a_i. \tag{2.4}$$

Now, if we take $a_i=\sum_{j=0}^{i-1}|\Delta x_j|^p$, $i=1,...,N$, $b_i=\frac{1}{2}i(i-1)$, then $a_N=\sum_{j=0}^{N-1}|\Delta x_j|^p$,

$$\Delta b_i=b_{i+1}-b_i=\frac{1}{2}i(i+1)-\frac{1}{2}i(i-1)=i,$$

and

$$\Delta a_i=a_{i+1}-a_i=\sum_{j=0}^{i}|\Delta x_j|^p-\sum_{j=0}^{i-1}|\Delta x_j|^p=|\Delta x_i|^p.$$

By (2.4) we derive

$$\begin{aligned}
\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)&=\frac{1}{2}N(N+1)\sum_{j=0}^{N-1}|\Delta x_j|^p-\sum_{k=1}^{N-1}\frac{1}{2}i(i+1)|\Delta x_i|^p \\
&=\sum_{i=0}^{N-1}\left[\frac{1}{2}N(N+1)-\frac{1}{2}i(i+1)\right]|\Delta x_i|^p \\
&=\frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p
\end{aligned} \tag{2.5}$$

and, similarly

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}|\Delta y_j|^q\right)=\frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta y_i|^q. \tag{2.6}$$

Therefore

$$B=\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta y_i|^q\right)^{1/q}$$

and by (2.3) we derive the first inequality in (2.1).

Now observe that

$$\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p \le \max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^p\sum_{i=0}^{N-1}(N-i)(N+i+1)$$
$$=\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^p,$$

which proves the first branch in (2.1).

Observe also that

$$\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p \le \max_{i\in\{0,\dots,N-1\}}[(N-i)(N+i+1)]\sum_{i=0}^{N-1}|\Delta x_i|^p$$
$$=\max_{i\in\{0,\dots,N-1\}}[N(N+1)-i(i+1)]\sum_{i=0}^{N-1}|\Delta x_i|^p$$
$$=N(N+1)\sum_{i=0}^{N-1}|\Delta x_i|^2,$$

which proves the second branch in (2.1).

The last inequality in (2.2) follows by Young's inequality

$$\alpha^{1/p}y^{1/q}\le\frac{1}{p}\alpha+\frac{1}{q}\beta,\ \alpha,\ \beta\ge 0.$$

□

Corollary 1. *Assume that* $\{x_i\}_{i=0}^N$ *is a sequence of complex numbers with* $x_0=0$, *then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\sum_{i=1}^{N}|x_i|^2 \le \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^p\right)^{1/p}$$
$$\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta x_i|^q\right)^{1/q}$$
$$\le\begin{cases}\frac{1}{6}N(N+1)(2N+1)\\ \times\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2, \\ \\ \frac{1}{2}N(N+1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}|\Delta x_i|^q\right)^{1/q}.\end{cases}\tag{2.7}$$

Also, we have:

Theorem 5. *Assume that* $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $x_N = y_N = 0$, *then*

$$\sum_{i=0}^{N-1} |x_i y_i| \tag{2.8}$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1} (i+1)(2N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1} (i+1)(2N-i)|\Delta y_i|^q\right)^{1/q}$$
$$\leq \begin{cases} \frac{1}{6}N(N+1)(2N+1) \\ \times \max_{i\in\{0,...,N-1\}} |\Delta x_i| \max_{i\in\{0,...,N-1\}} |\Delta y_i|, \\ \\ \frac{1}{2}N(N+1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}|\Delta y_i|^q\right)^{1/q}, \end{cases}$$

for $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Also, we have

$$\sum_{i=0}^{N-1} |x_i y_i| \tag{2.9}$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1} (i+1)(2N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1} (i+1)(2N-i)|\Delta y_i|^q\right)^{1/q}$$
$$\leq \frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\left(\frac{1}{p}|\Delta x_i|^p + \frac{1}{q}|\Delta y_i|^q\right).$$

Proof. If $x_N = y_N = 0$, then $x_i = -\sum_{j=i}^{N-1}\Delta x_j$ and $y_i = -\sum_{j=i}^{N-1}\Delta y_j$, $i \in \{0, ..., N-1\}$. Then

$$\sum_{i=0}^{N-1}|x_i||y_i| = \sum_{i=0}^{N-1}\left|\sum_{j=i}^{N-1}\Delta x_j\right|\left|\sum_{j=i}^{N-1}\Delta y_j\right|$$
$$= \sum_{i=0}^{N-1}(N-i)\frac{1}{(N-i)^{1/q}}\left|\sum_{j=i}^{N-1}\Delta x_j\right|\frac{1}{(N-i)^{1/p}}\left|\sum_{j=i}^{N-1}\Delta y_j\right| =: C.$$

By Hölder's inequality we have

$$\frac{1}{(N-i)^{1/q}}\left|\sum_{j=i}^{N-1}\Delta x_j\right| \leq \left(\sum_{j=i}^{N-1}|\Delta x_j|^p\right)^{1/p}$$

and

$$\frac{1}{(N-i)^{1/p}}\left|\sum_{j=i}^{N-1}\Delta y_j\right| \leq \left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)^{1/q},$$

which gives

$$C \leq \sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^p\right)^{1/p}\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)^{1/q}.$$

By the weighted Hölder inequality we have

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^p\right)^{1/p}\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)^{1/q}$$
$$\leq\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^p\right)\right)^{1/p}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)\right)^{1/q},$$

which gives

$$C\leq\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^p\right)\right)^{1/p}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)\right)^{1/q} =: D. \tag{2.10}$$

From (1.1) we get for $m=0$ and $n=N-1$ that

$$\sum_{i=0}^{N-1}a_i\Delta b_i=a_{N-1}b_N-a_0b_0-\sum_{k=0}^{N-2}b_{i+1}\Delta a_i. \tag{2.11}$$

Take $a_i=\sum_{j=i}^{N-1}|\Delta x_j|^p$ and $b_i=-\frac{1}{2}(N-i)(N-i+1)$, then we get

$$\Delta b_i=b_{i+1}-b_i=-\frac{1}{2}(N-i-1)(N-i)+\frac{1}{2}(N-i)(N-i+1)$$
$$=\frac{1}{2}(N-i)(-N+i+1+N-i+1)=N-i$$

and

$$\Delta a_i=a_{i+1}-a_i=\sum_{j=i+1}^{N-1}|\Delta x_j|^p-\sum_{j=i}^{N-1}|\Delta x_j|^p=-|\Delta x_i|^p.$$

Then

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta x_j|^p\right)$$
$$=\frac{1}{2}N(N+1)\sum_{j=0}^{N-1}|\Delta x_j|^p-\frac{1}{2}\sum_{i=0}^{N-2}(N-i-1)(N-i)|\Delta x_i|^p$$
$$=\frac{1}{2}N(N+1)\sum_{i=0}^{N-1}|\Delta x_j|^p-\frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p$$
$$=\frac{1}{2}\sum_{i=0}^{N-1}\left[N(N+1)-(N-i-1)(N-i)\right]|\Delta x_i|^p$$
$$=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^p$$

and

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^q.$$

Therefore

$$D=\frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^q\right)^{1/q},$$

and by (2.10) we derive the first inequality in (2.8).

Now, observe that

$$\begin{aligned}\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^p &\le \max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2\sum_{i=0}^{N-1}(i+1)(2N-i)\\ &=\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^p\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta y_i|^q\le\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}|\Delta y_i|^q,$$

which proves the first branch of (2.8).

Also

$$\begin{aligned}&\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^p\\ &=\sum_{i=0}^{N-1}[N(N+1)-(N-i-1)(N-i)]|\Delta x_i|^p\\ &\le\max_{i\in\{0,\dots,N-1\}}[N(N+1)-(N-i-1)(N-i)]\sum_{i=0}^{N-1}|\Delta x_i|^p\\ &=N(N+1)\sum_{i=0}^{N-1}|\Delta x_i|^p,\end{aligned}$$

which proves the second branch of (2.8). □

Corollary 2. *Assume that $\{x_i\}_{i=0}^{N}$ is a sequence of complex numbers with $x_N=0$, then*

$$\begin{aligned}&\sum_{i=0}^{N-1}|x_i|^2 \qquad (2.12)\\ &\le\frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta x_i|^q\right)^{1/q}\\ &\le\begin{cases}\frac{1}{6}N(N+1)(2N+1)\\ \times\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2,\\ \\ \frac{1}{2}N(N+1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}|\Delta x_i|^q\right)^{1/q},\end{cases}\end{aligned}$$

We also have:

Theorem 6. *Assume that* $\{x_i\}_{i=0}^{N}$, $\{y_i\}_{i=0}^{N}$ *are sequences of complex numbers. If* $x_0 = y_N = 0$, *then*

$$\sum_{i=1}^{N-1} |x_i y_i| \tag{2.13}$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^q\right)^{1/q}$$
$$\leq \begin{cases} \frac{1}{6}(N-1)N(N+1) \\ \times \max_{i\in\{0,\dots,N-1\}}|\Delta x_i| \max_{i\in\{0,\dots,N-1\}}|\Delta y_i|, \\ \\ \frac{1}{2}N(N-1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}|\Delta y_i|^q\right)^{1/q}. \end{cases}$$

Also,

$$\sum_{i=1}^{N-1} |x_i y_i| \tag{2.14}$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^q\right)^{1/q}$$
$$\leq \frac{1}{2}\sum_{i=0}^{N-1}\left[\frac{1}{p}(N-i-1)(N-i)|\Delta x_i|^p + \frac{1}{q}(i+1)\,i\,|\Delta y_i|^q\right].$$

Proof. Since $x_0 = 0$, and $y_N = 0$, hence $x_i = \sum_{j=0}^{i-1}\Delta x_j$ and $y_i = -\sum_{j=i}^{N-1}\Delta y_j$. Then by Hölder's inequality

$$\sum_{i=1}^{N-1}|x_i y_i| = \sum_{i=1}^{N-1}|x_i|\,|y_i| = \sum_{i=1}^{N-1}\left|\sum_{j=0}^{i-1}\Delta x_j\right|\left|\sum_{j=i}^{N-1}\Delta y_j\right| \tag{2.15}$$
$$\leq \sum_{i=1}^{N-1} i^{1/q}\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)^{1/p}(N-i)^{1/p}\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)^{1/q}$$

$$\leq \left(\sum_{i=1}^{N-1}\left[(N-i)^{1/p}\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)^{1/p}\right]^p\right)^{1/p}$$
$$\times\left(\sum_{i=1}^{N-1}\left[i^{1/q}\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)^{1/q}\right]^q\right)^{1/q}$$

$$= \left(\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}|\Delta x_j|^p\right)\right)^{1/p}\left(\sum_{i=1}^{N-1} i\left(\sum_{j=i}^{N-1}|\Delta y_j|^q\right)\right)^{1/q} =: E.$$

We use the identity

$$\sum_{i=1}^{N} a_i \Delta b_i = a_N b_{N+1} - a_1 b_1 - \sum_{i=1}^{N-1} b_{i+1} \Delta a_i. \tag{2.16}$$

Take $a_i = \sum_{j=0}^{i-1} |\Delta x_j|^p$ and $b_i = -\frac{1}{2}(N-i)(N-i+1)$, then

$$\begin{aligned} \Delta b_i = b_{i+1} - b_i &= -\frac{1}{2}(N-i-1)(N-i) + \frac{1}{2}(N-i)(N-i+1) \\ &= \frac{1}{2}(N-i)[N-i+1-(N-i-1)] = (N-i), \end{aligned}$$

$$\Delta a_i = a_{i+1} - a_i = \sum_{j=0}^{i} |\Delta x_j|^p - \sum_{j=0}^{i-1} |\Delta x_j|^p = |\Delta x_i|^p$$

and by (2.16)

$$\begin{aligned} &\sum_{i=1}^{N-1} (N-i) \left(\sum_{j=0}^{i-1} |\Delta x_j|^p \right) \\ &= \frac{1}{2} N (N-1) |\Delta x_0|^p + \frac{1}{2} \sum_{i=1}^{N-1} (N-i-1)(N-i) |\Delta x_i|^p \\ &= \frac{1}{2} \sum_{i=0}^{N-1} (N-i-1)(N-i) |\Delta x_i|^p . \end{aligned} \tag{2.17}$$

Take $a_i = \sum_{j=i}^{N-1} |\Delta y_j|^q$ and $b_i = \frac{1}{2} i (i-1)$, then

$$\Delta b_i = b_{i+1} - b_i = \frac{1}{2}(i+1) i - \frac{1}{2} i (i-1) = \frac{1}{2} i (i+1-i+1) = i,$$

$$\Delta a_i = a_{i+1} - a_i = \sum_{j=i+1}^{N-1} |\Delta y_j|^q - \sum_{j=i}^{N-1} |\Delta y_j|^q = -|\Delta y_i|^q$$

and by the identity

$$\sum_{i=1}^{N-1} a_i \Delta b_i = a_{N-1} b_N - a_1 b_1 - \sum_{i=1}^{N-2} b_{i+1} \Delta a_i.$$

we get

$$\begin{aligned} \sum_{i=1}^{N-1} i \left(\sum_{j=i}^{N-1} |\Delta y_j|^q \right) &= \frac{1}{2} N (N-1) |\Delta y_{N-1}|^q + \frac{1}{2} \sum_{i=1}^{N-2} (i+1) i |\Delta y_i|^q \\ &= \frac{1}{2} \sum_{i=1}^{N-1} (i+1) i |\Delta y_i|^q = \frac{1}{2} \sum_{i=0}^{N-1} (i+1) i |\Delta y_i|^q . \end{aligned} \tag{2.18}$$

Therefore

$$E=\left(\frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\right)^{1/p}\left(\frac{1}{2}\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^q\right)^{1/q}$$
$$=\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^q\right)^{1/q},$$

which proves the first inequality in (2.13)

Now, observe that

$$\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\le\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^p\sum_{i=0}^{N-1}(N-i-1)(N-i)$$
$$=\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^p\frac{1}{3}(N-1)N(N+1)$$

and

$$\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^q\le\max_{i\in\{0,\dots,N-1\}}|\Delta y_i|^q\sum_{i=0}^{N-1}(i+1)\,i$$
$$=\max_{i\in\{0,\dots,N-1\}}|\Delta y_i|^q\frac{1}{3}(N-1)N(N+1),$$

which proves the first branch in (2.13).

Observe also that

$$\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\le\max_{i\in\{0,\dots,N-1\}}[(N-i-1)(N-i)]\sum_{i=0}^{N-1}|\Delta x_i|^p$$
$$=N(N-1)\sum_{i=0}^{N-1}|\Delta x_i|^p$$

and

$$\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta y_i|^q\le\max_{i\in\{0,\dots,N-1\}}[i(i+1)]\sum_{i=0}^{N-1}|\Delta y_i|^q\le N(N-1)\sum_{i=0}^{N-1}|\Delta y_i|^q,$$

which proves the second branch in (2.13). □

Corollary 3. *Assume that $\{x_i\}_{i=0}^{N}$ is a sequence of complex numbers with $x_N=0$, then*

$$\sum_{i=1}^{N-1}|x_i|^2 \tag{2.19}$$
$$\le\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta x_i|^q\right)^{1/q}$$
$$\le\begin{cases}\frac{1}{6}(N-1)N(N+1)\\ \times\max_{i\in\{0,\dots,N-1\}}|\Delta x_i|^2,\\ \\ \frac{1}{2}N(N-1)\left(\sum_{i=0}^{N-1}|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}|\Delta x_i|^q\right)^{1/q}.\end{cases}$$

Also,

$$\sum_{i=1}^{N-1}|x_i|^2 \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta x_i|^p\right)^{1/p}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta x_i|^q\right)^{1/q} \leq \frac{1}{2}\sum_{i=0}^{N-1}\left[\frac{1}{p}(N-i-1)(N-i)|\Delta x_i|^p+\frac{1}{q}(i+1)\,i\,|\Delta x_i|^q\right]. \tag{2.20}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa