DISCRETE GRÜSS TYPE MODULUS INEQUALITIES IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$. Let $a_k \in A$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$ and c, $d \in A$ with $c \neq d$. In this paper we show among others that, if either

$$\left| a_k - \frac{d+c}{2} \right|^2 \leq \frac{1}{4} |d-c|^2, k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(d^* - a_k^*\right)\left(a_k - c\right)\right] = \operatorname{Re}\left[\left(a_k^* - c^*\right)\left(d - a_k\right)\right] \geq 0, \ k \in \{1, ..., n\}$$

holds, then

$$0 \leq \sum_{k=1}^n p_k |a_k|^2 - \left|\sum_{k=1}^n p_k a_k\right|^2$$
$$\leq \operatorname{Re}\left[\left(\sum_{k=1}^n p_k a_k^* - c^*\right)\left(d - \sum_{k=1}^n p_k a_k\right)\right] \leq \frac{1}{4}|d-c|^2.$$

Some applications for power series of normal elements in A are also provided.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If $a, b \in \operatorname{Inv}(A)$, then $ab \in \operatorname{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [12] (see also [1, Theorem 41.5])

$$\text{(SF)} \qquad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [11], Tanahashi and Uchiyama [13] proved the following fundamental properties (see also [7]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;

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(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_{\gamma} f(z) (z-a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^{\alpha} := \frac{1}{2\pi i} \int_{\gamma} z^{\alpha} (z-a)^{-1} dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha} \in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^{\alpha}) = (\sigma(a))^{\alpha} = \{z^{\alpha} : z \in \sigma(a)\} \subset (0, \infty).$$

Following [7], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$ and $\left(a^2\right)^{1/2} = a$, [13, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^{\alpha} a^{\beta} = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^{\alpha})^{-1} = \left(a^{-1}\right)^{\alpha} = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^{\alpha} b^{\beta} = b^{\beta} a^{\alpha}$.

We define the following means for $\nu \in [0, 1]$, see also [7] for different notations:

$$a \nabla_{\nu} b := (1-\nu) a + \nu b, \ a, \ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a, b),

$$a !_{\nu} b := \left((1-\nu) a^{-1} + \nu b^{-1}\right)^{-1}, \ a, \ b > 0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a, b) and

$$a \sharp_{\nu} b := a^{1/2} \left(a^{-1/2} b a^{-1/2}\right)^{\upsilon} a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a, b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a \nabla b$, $a ! b$ and $a \sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [7], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;

(xiii) If $0 < a,\ b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*(a!b)\,c = (c^*ac)!\,(c^*bc) \text{ and } c^*(a\sharp b)\,c = (c^*ac)\sharp(c^*bc);$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0,1]$, then

$$(a!_\nu b)^{-1} = \left(a^{-1}\right)\nabla_\nu\left(b^{-1}\right) \text{ and } \left(a^{-1}\right)\sharp_\nu\left(b^{-1}\right) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [7] the following inequality between the weighted means introduced above:

$$\text{(HGA)}\qquad (1-\nu)\,a + \nu b \geq a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon} a^{1/2} \geq \left((1-\nu)\,a^{-1} + \nu b^{-1}\right)^{-1}$$

for any $0 < a,\ b \in A$ and $\nu \in [0,1]$.

Okayasu [11] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a,\ b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^*a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^*a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a+b| \leq |a| + |b|,\ a,\ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

Motivated by the above results, in this paper we show among others that, if either

$$\left|a_k - \frac{d+c}{2}\right|^2 \leq \frac{1}{4}|d-c|^2, k \in \{1,...,n\}$$

or, equivalently

$$\operatorname{Re}\left[(d^* - a_k^*)(a_k - c)\right] = \operatorname{Re}\left[(a_k^* - c^*)(d - a_k)\right] \geq 0,\ k \in \{1,...,n\}$$

holds, then

$$\begin{aligned} 0 &\leq \sum_{k=1}^n p_k |a_k|^2 - \left|\sum_{k=1}^n p_k a_k\right|^2 \\ &\leq \operatorname{Re}\left[\left(\sum_{k=1}^n p_k a_k^* - c^*\right)\left(d - \sum_{k=1}^n p_k a_k\right)\right] \leq \frac{1}{4}|d-c|^2. \end{aligned}$$

Some applications for power series of normal elements in A are also provided.

2. Some Inequalities

We have the following two sequences Cauchy-Bunyakowsky-Schwarz (CBS) weighted inequality:

Lemma 1. *Let* $a_k \in A$, $\alpha_k \in \mathbb{C}$ *and* $p_k \geq 0$ *for* $k \in \{1, ..., n\}$. *Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{j=1}^{n} p_j \alpha_j a_j \right|^2 = \frac{1}{2} \sum_{j,k=1}^{n} p_j p_k |\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 \geq 0. \tag{2.1}$$

In particular,

$$\sum_{k=1}^{n} |\alpha_k|^2 \sum_{k=1}^{n} |a_k|^2 - \left| \sum_{j=1}^{n} \alpha_j a_j \right|^2 = \frac{1}{2} \sum_{j,k=1}^{n} |\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 \geq 0. \tag{2.2}$$

Proof. Observe that

$$\begin{aligned}
&|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 \\
&= (\overline{\alpha_j} a_k - \overline{\alpha}_k a_j)^* (\overline{\alpha_j} a_k - \overline{\alpha}_k a_j) = \left(\alpha_j a_k^* - \alpha_k a_j^*\right)(\overline{\alpha_j} a_k - \overline{\alpha}_k a_j) \\
&= \alpha_j a_k^* \overline{\alpha_j} a_k - \alpha_j a_k^* \overline{\alpha}_k a_j - \alpha_k a_j^* \overline{\alpha_j} a_k + \alpha_k a_j^* \overline{\alpha}_k a_j \\
&= |\alpha_j|^2 |a_k|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + |\alpha_k|^2 |a_j|^2
\end{aligned}$$

for all j, $k \in \{1, ..., n\}$.

This implies that

$$\begin{aligned}
&\sum_{j,k=1}^{n} p_j p_k |\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 \\
&= \sum_{j,k=1}^{n} p_j p_k \left[|\alpha_j|^2 |a_k|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + |\alpha_k|^2 |a_j|^2 \right] \\
&= \sum_{j,k=1}^{n} p_j p_k |\alpha_j|^2 |a_k|^2 - \sum_{j,k=1}^{n} p_j p_k \overline{\alpha}_k a_k^* \alpha_j a_j \\
&- \sum_{j,k=1}^{n} p_j p_k \overline{\alpha_j} a_j^* \alpha_k a_k + \sum_{j,k=1}^{n} p_j p_k |\alpha_k|^2 |a_j|^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} p_j |\alpha_j|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \sum_{k=1}^{n} p_k \overline{\alpha}_k a_k^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \sum_{j=1}^{n} p_j \overline{\alpha_j} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{j=1}^{n} p_j |a_j|^2 \\
&= \sum_{j=1}^{n} p_j |\alpha_j|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \left(\sum_{j=1}^{n} p_j \alpha_j a_j \right)^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{j=1}^{n} p_j |a_j|^2 \\
&= 2 \left[\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \right],
\end{aligned}$$

which is equivalent to the desired identity (2.1). □

We also have:

Lemma 2. *Let* $a_k \in A$, $\alpha_k \in \mathbb{C}$ *and* $p_k \geq 0$ *for* $k \in \{1, ..., n\}$. *Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{j=1}^{n} p_j \alpha_j a_j \right|^2 = \sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{j=1}^{n} p_j \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \geq 0. \tag{2.3}$$

In particular,

$$\sum_{k=1}^{n} |\alpha_k|^2 \sum_{k=1}^{n} |a_k|^2 - \left| \sum_{j=1}^{n} \alpha_j a_j \right|^2 = \sum_{k=1}^{n} |\alpha_k|^2 \sum_{j=1}^{n} \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} |\alpha_k|^2} \sum_{k=1}^{n} \alpha_k a_k \right|^2 \geq 0. \tag{2.4}$$

Proof. For $j \in \{1, ..., n\}$ we have

$$\begin{aligned}
&\left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \\
&= \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right) \\
&= \left(a_j^* - \frac{\alpha_j}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \right) \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right) \\
&= |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \alpha_j a_j \\
&- \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \frac{|\alpha_j|^2}{\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 \right)^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2.
\end{aligned}$$

If we multiply this equality with $p_j \geq 0$ and sum over j from 1 to n, we derive

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \\
&= \sum_{j=1}^{n} p_j |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{j=1}^{n} p_j \overline{\alpha_j} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \frac{\sum_{j=1}^{n} p_j |\alpha_j|^2}{\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 \right)^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2
\end{aligned}$$

$$= \sum_{j=1}^{n} p_j |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2$$
$$- \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 + \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2$$
$$= \sum_{j=1}^{n} p_j |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2,$$

which is equivalent to (2.3). □

Theorem 1. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 \geq \left| \sum_{j=1}^{n} p_j \alpha_j a_j \right|^2. \tag{2.5}$$

If $p_k > 0$ for $k \in \{1, ..., n\}$, then the equality holds in (2.5) if and only if

$$a_j = \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \tag{2.6}$$

for all $j \in \{1, ..., n\}$ or if and only if

$$a_j = \overline{\alpha_j} a \tag{2.7}$$

for all $j \in \{1, ..., n\}$, where $a \in A$.

Remark 1. *Let $a_k \in A$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$. Then*

$$\sum_{k=1}^{n} p_k |a_k|^2 \geq \left| \sum_{j=1}^{n} p_j a_j \right|^2. \tag{2.8}$$

The equality holds in (2.8) if and only if $a_j = \sum_{k=1}^{n} p_k a_k$ for all $j \in \{1, ..., n\}$ or if and only if $a_j = a$ for all $j \in \{1, ..., n\}$, where $a \in A$.

If A is a Hermitian unital Banach $*$-algebra with continuous involution, then we can take the square root in (2.8) to obtain

$$\left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} p_j a_j \right|. \tag{2.9}$$

The inequality (2.5) follows by either the relation (2.1) or (2.3). The equality (2.6) follows by (2.3) while (2.7) follows by (2.1).

For $a \in A$ we define the selfadjoint element

$$\mathrm{Re}\,(a) := \frac{1}{2} (a^* + a) = \mathrm{Re}\,(a^*)$$

We have the following identity of interest:

Lemma 3. *For any a, d, $c \in A$, we have*

$$\left| a - \frac{d+c}{2} \right|^2 - \frac{1}{4} |d - c|^2 = \mathrm{Re}\,[(a^* - d^*)(a - c)] \tag{2.10}$$
$$= \mathrm{Re}\,[(a^* - c^*)(a - d)].$$

Proof. We have

$$\begin{aligned}&\left|a-\frac{d+c}{2}\right|^2-\frac{1}{4}\left|d-c\right|^2\\&=\left|a\right|^2-\frac{d^*+c^*}{2}a-a^*\frac{d+c}{2}+\frac{1}{4}\left(\left|d\right|^2+d^*c+c^*d+\left|c\right|^2\right)\\&-\frac{1}{4}\left(\left|d\right|^2-d^*c-c^*d+\left|c\right|^2\right)\\&=\left|a\right|^2-\frac{d^*+c^*}{2}a-a^*\frac{d+c}{2}+\frac{1}{2}\left(d^*c+c^*d\right)\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Re}\left[\left(a^*-d^*\right)\left(a-c\right)\right]\\&=\operatorname{Re}\left[\left|a\right|^2-d^*a-a^*c+d^*c\right]\\&=\left|a\right|^2-\operatorname{Re}\left(d^*a\right)-\operatorname{Re}\left(a^*c\right)+\operatorname{Re}\left(d^*c\right)\\&=\left|a\right|^2-\frac{1}{2}\left(d^*a+a^*d\right)-\frac{1}{2}\left(a^*c+c^*a\right)+\frac{1}{2}\left(d^*c+c^*d\right)\\&=\left|a\right|^2-\frac{1}{2}\left(d^*+c^*\right)a-\frac{1}{2}a^*\left(d+c\right)+\frac{1}{2}\left(d^*c+c^*d\right),\end{aligned}$$

which proves the desired identity (2.10). □

Corollary 1. *Let a, d, $c\in A$. The following statements are equivalent*

$$\left|a-\frac{d+c}{2}\right|^2\leq\frac{1}{4}\left|d-c\right|^2$$

and

$$\operatorname{Re}\left[\left(d^*-a^*\right)\left(a-c\right)\right]=\operatorname{Re}\left[\left(a^*-c^*\right)\left(d-a\right)\right]\geq 0.$$

We have the following reverse of the CBS inequality (2.8):

Theorem 2. *Let $a_k\in A$, $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^n p_k=1$ and c, $d\in A$ with $c\neq d$. If either*

$$\left|a_k-\frac{d+c}{2}\right|^2\leq\frac{1}{4}\left|d-c\right|^2, k\in\{1,...,n\}\tag{2.11}$$

or

$$\operatorname{Re}\left[\left(d^*-a_k^*\right)\left(a_k-c\right)\right]=\operatorname{Re}\left[\left(a_k^*-c^*\right)\left(d-a_k\right)\right]\geq 0,\ k\in\{1,...,n\}\tag{2.12}$$

holds, then

$$\begin{aligned}0&\leq\sum_{k=1}^n p_k\left|a_k\right|^2-\left|\sum_{k=1}^n p_ka_k\right|^2\\&\leq\operatorname{Re}\left[\left(\sum_{k=1}^n p_ka_k^*-c^*\right)\left(d-\sum_{k=1}^n p_ka_k\right)\right]\leq\frac{1}{4}\left|d-c\right|^2.\end{aligned}\tag{2.13}$$

The constant $\frac{1}{4}$ is best possible.

Proof. The equivalence of the conditions (2.11) and (2.12) follows by Corollary 1.

We have

$$\begin{aligned} K_1 &:= \operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k - c\right)^* \left(d - \sum_{k=1}^{n} p_k a_k\right)\right] \\ &= \operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k^* - c^*\right) \left(d - \sum_{k=1}^{n} p_k a_k\right)\right] \\ &= \operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k^*\right) d\right] - \left|\sum_{k=1}^{n} p_k a_k\right|^2 - \operatorname{Re}(c^* d) - \operatorname{Re}\left[c^* \sum_{k=1}^{n} p_k a_k\right] \\ &= \left(\sum_{k=1}^{n} p_k \operatorname{Re}(a_k^* d)\right) - \left|\sum_{k=1}^{n} p_k a_k\right|^2 - \operatorname{Re}(c^* d) - \sum_{k=1}^{n} p_k \operatorname{Re}(c^* a_k) \end{aligned}$$

and

$$\begin{aligned} K_2 &:= \sum_{k=1}^{n} p_k \operatorname{Re}\left[(a_k - c)^* (d - a_k)\right] \\ &= \sum_{k=1}^{n} p_k \left[\operatorname{Re}(a_k^* d) - \operatorname{Re}(c^* d) - |a_k|^2 + \operatorname{Re}(c^* a_k)\right] \\ &= \sum_{k=1}^{n} p_k \operatorname{Re}(a_k^* d) - \operatorname{Re}(c^* d) - \sum_{k=1}^{n} p_k |a_k|^2 + \sum_{k=1}^{n} p_k \operatorname{Re}(c^* a_k). \end{aligned}$$

Since

$$K_1 - K_2 = \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k a_k\right|^2,$$

hence we derive the following identity of interest

$$\begin{aligned} &\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k a_k\right|^2 \\ &= \operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k^* - c^*\right) \left(d - \sum_{k=1}^{n} p_k a_k\right)\right] - \sum_{k=1}^{n} p_k \operatorname{Re}\left[(a_k^* - c^*)(d - a_k)\right]. \end{aligned} \tag{2.14}$$

Now, if condition (2.12) holds, then

$$\sum_{k=1}^{n} p_k \operatorname{Re}\left[(a_k^* - c^*)(d - a_k)\right] \geq 0,$$

which proves the first inequality in (2.13).

Observe that we have the following inequality

$$4 \operatorname{Re}(u^* v) \leq |u + v|^2 \tag{2.15}$$

for all $u, v \in A$.

Indeed, we have

$$\begin{aligned}|u+v|^2-4\operatorname{Re}(u^*v)&=(u+v)^*(u+v)-4\frac{u^*v+v^*u}{2}\\&=(u^*+v^*)(u+v)-2(u^*v+v^*u)\\&=|u|^2+v^*u+u^*v+|v|^2-2(u^*v+v^*u)\\&=|u|^2+|v|^2-u^*v-v^*u=|u-v|^2\geq 0.\end{aligned}$$

By utilising (2.15) we get

$$\begin{aligned}&\operatorname{Re}\left[\left(\sum_{k=1}^n p_k a_k - c\right)^*\left(d-\sum_{k=1}^n p_k a_k\right)\right]\\&\leq \frac{1}{4}\left|\sum_{k=1}^n p_k a_k - c + d - \sum_{k=1}^n p_k a_k\right|^2=\frac{1}{4}|d-c|^2,\end{aligned}$$

which proves the last part of (2.13).

The best constant follows from the scalar case. □

Recall that a C^*-*algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^*a\|=\|a\|^2 \text{ for any } a\in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\|=1$.

It is well know that, if A is a C^*-algebra, then (see for instance [9, 2.2.5 Theorem])

$$b\geq a\geq 0 \text{ implies that } \|b\|\geq\|a\|.$$

Remark 2. *If A is a C^*-algebra, then under the assumptions of Theorem 2 we also have the norm inequality*

$$\begin{aligned}&\left\|\sum_{k=1}^n p_k |a_k|^2-\left|\sum_{k=1}^n p_k a_k\right|^2\right\|\\&\leq\left\|\operatorname{Re}\left[\left(\sum_{k=1}^n p_k a_k^* - c^*\right)\left(d-\sum_{k=1}^n p_k a_k\right)\right]\right\|\leq\frac{1}{4}\|d-c\|^2.\end{aligned}\tag{2.16}$$

We have the following Grüss' type inequality:

Theorem 3. *Let $a_k\in A$, $\alpha_k\in\mathbb{C}$ and $p_k\geq 0$ for $k\in\{1,...,n\}$ with $\sum_{k=1}^n p_k=1$. Then*

$$\begin{aligned}&\left|\sum_{k=1}^n p_k\alpha_k a_k-\sum_{k=1}^n p_k\alpha_k\sum_{k=1}^n p_k a_k\right|^2\\&\leq\left(\sum_{k=1}^n p_k|\alpha_k|^2-\left|\sum_{k=1}^n p_k\alpha_k\right|^2\right)\left(\sum_{k=1}^n p_k|a_k|^2-\left|\sum_{k=1}^n p_k a_k\right|^2\right).\end{aligned}\tag{2.17}$$

Moreover, if there exists c, $d \in A$ *with* $c \neq d$ *such that either (2.11) or (2.12) holds, then*

$$\left|\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k\right|^2 \leq \left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right) \operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k^* - c^*\right)\left(d - \sum_{k=1}^{n} p_k a_k\right)\right] \leq \frac{1}{4}\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right) |d-c|^2 . \tag{2.18}$$

Proof. We have the following Korkine type identity in Hermitian unital Banach $*$-algebra A,

$$\begin{aligned}
&\sum_{k,j=1}^{n} p_j p_k (\alpha_j - \alpha_k)(a_j - a_k) \\
&= \sum_{k,j=1}^{n} p_j p_k (\alpha_j a_j - \alpha_j a_k - \alpha_k a_j + \alpha_k a_k) \\
&= \sum_{k,j=1}^{n} p_j p_k \alpha_j a_j - \sum_{k,j=1}^{n} p_j p_k \alpha_j a_k - \sum_{k,j=1}^{n} p_j p_k \alpha_k a_j + \sum_{k,j=1}^{n} p_j p_k \alpha_k a_k \\
&= \sum_{j=1}^{n} p_j \alpha_j a_j - \sum_{j=1}^{n} p_j \alpha_j \sum_{k=1}^{n} p_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{j=1}^{n} p_j a_j + \sum_{k=1}^{n} p_k \alpha_k a_k \\
&= 2\left(\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{j=1}^{n} p_k a_k\right),
\end{aligned}$$

namely

$$\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{j=1}^{n} p_k a_k = \frac{1}{2} \sum_{k,j=1}^{n} p_j p_k (\alpha_j - \alpha_k)(a_j - a_k).$$

Using CBS weighted inequality for double sums, we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{j=1}^{n} p_k a_k\right|^2 \\
&= \left|\frac{1}{2} \sum_{k,j=1}^{n} p_j p_k (\alpha_j - \alpha_k)(a_j - a_k)\right|^2 \\
&\leq \frac{1}{2} \sum_{k,j=1}^{n} p_j p_k |\alpha_j - \alpha_k|^2 \frac{1}{2} \sum_{k,j=1}^{n} p_j p_k |a_j - a_k|^2 .
\end{aligned} \tag{2.19}$$

Since

$$\frac{1}{2} \sum_{k,j=1}^{n} p_j p_k |\alpha_j - \alpha_k|^2 = \sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2$$

and, by (2.1)

$$\frac{1}{2}\sum_{k,j=1}^{n} p_j p_k |a_j - a_k|^2 = \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k a_k\right|^2,$$

hence by (2.19), we derive (2.17).

The inequality (2.18) follows by Theorem 2 □

Remark 3. *If A is a Hermitian unital Banach $*$-algebra with continuous involution, then we can take the square root in (2.18) to obtain*

$$\begin{aligned}(2.20)\quad & \left|\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k\right| \\ & \leq \left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right)^{1/2} \\ & \times \left\{\operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k^* - c^*\right)\left(d - \sum_{k=1}^{n} p_k a_k\right)\right]\right\}^{1/2} \\ & \leq \frac{1}{2}\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right)^{1/2} |d-c|.\end{aligned}$$

We observe that, if A is a C^*-*algebra*, then under the assumptions of Theorem 3 we also have the norm inequality

$$\begin{aligned}(2.21)\quad & \left\|\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k\right\|^2 \\ & \leq \left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right) \left\|\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k a_k\right|^2\right\|\end{aligned}$$

and

$$\begin{aligned}(2.22)\quad & \left\|\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k\right\|^2 \\ & \leq \left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right) \\ & \times \left\|\operatorname{Re}\left[\left(\sum_{k=1}^{n} p_k a_k^* - c^*\right)\left(d - \sum_{k=1}^{n} p_k a_k\right)\right]\right\| \\ & \leq \frac{1}{4}\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right) \|d-c\|^2.\end{aligned}$$

Corollary 2. *With the assumptions of Theorem 3 and if either*

$$(2.23)\quad \left|\alpha_k - \frac{\gamma+\Gamma}{2}\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2, k \in \{1, ..., n\}$$

or

$$\operatorname{Re}\left[\left(\overline{\Gamma}-\overline{\alpha_k}\right)(\alpha_k-\gamma)\right]=\operatorname{Re}\left[\left(\overline{\alpha_k}-\overline{\gamma}\right)(\Gamma-\alpha_k)\right]\geq 0,\ \ k\in\{1,...,n\} \tag{2.24}$$

for some complex numbers γ, Γ, *then we have the Grüss' type inequality*

$$\left|\sum_{k=1}^{n}p_k\alpha_k a_k-\sum_{k=1}^{n}p_k\alpha_k\sum_{k=1}^{n}p_k a_k\right|^2\leq\frac{1}{16}\left|\Gamma-\gamma\right|^2\left|d-c\right|^2. \tag{2.25}$$

If A *is a Hermitian unital Banach* $*$*-algebra with continuous involution, then also*

$$\left|\sum_{k=1}^{n}p_k\alpha_k a_k-\sum_{k=1}^{n}p_k\alpha_k\sum_{k=1}^{n}p_k a_k\right|\leq\frac{1}{4}\left|\Gamma-\gamma\right|\left|d-c\right|. \tag{2.26}$$

3. Applications for Power Series

First we need the following facts:

Lemma 4 ([10, Lemma 2.4])**.** *Let* A *be a unital Hermitian Banach* $*$*-algebra with continuous involution, and let* $\{a_n\}$ *be a sequence of positive elements such that* $a_n\to a$ *in the norm topology. Then* a *is positive.*

Corollary 3. *Let* A *be a unital Hermitian Banach* $*$*-algebra with continuous involution, and let* $\{a_n\}$, $\{b_n\}$ *sequences of selfadjoint elements with* $a_n\geq b_n$ *for all* $n\geq 0$. *If* $a_n\to a$, $b_n\to b$ *in the norm topology, then* $a\geq b$.

Lemma 5. *Let* A *be a unital Hermitian Banach* $*$*-algebra with continuous involution. If* $a_n\to a$ *in the the norm topology, then* $|a_n|^2\to|a|^2$ *in the norm topology.*

Proof. By the continuity of the involution, we have $a_n^*\to a^*$ which implies, by the continuity of the product that $a_n^*a_n\to a^*a$, and the lemma is proved. □

Lemma 6. *Assume that* $a_n\geq 0$ *and* $a_n\to a\geq 0$ *in the norm topology, then for* $p\in[0,1]$, $a_n^p\to a^p\geq 0$ *in the norm topology.*

Proof. Assume that γ to be close rectifiable curve that goes through 0 and contain inside the spectra of a_n and a for $n\geq 0$. Then by the analytic functional calculus we have

$$\begin{aligned}
a_n^p-a^p&=\frac{1}{2\pi i}\int_\gamma z^\alpha(z-a_n)^{-1}dz-\frac{1}{2\pi i}\int_\gamma z^\alpha(z-a)^{-1}dz\\
&=\frac{1}{2\pi i}\int_\gamma z^\alpha\left[(z-a_n)^{-1}-(z-a)^{-1}\right]dz\\
&=\frac{1}{2\pi i}\int_\gamma z^\alpha\left[(z-a)^{-1}\left[(z-a)-(z-a_n)\right](z-a_n)^{-1}\right]dz\\
&=\frac{1}{2\pi i}\int_\gamma z^\alpha\left[(z-a)^{-1}(a_n-a)(z-a_n)^{-1}\right]dz.
\end{aligned}$$

By taking the norm and using the properties of the norm and integral, we get

$$\begin{aligned}
\|a_n^p-a^p\|&\leq\frac{1}{2\pi}\int_\gamma|z^\alpha|\left[\left\|(z-a)^{-1}\right\|\|a_n-a\|\left\|(z-a_n)^{-1}\right\|\right]|dz|\\
&\to 0\text{ for }n\to\infty.
\end{aligned}$$

□

Corollary 4. *If $b_n \to b$ in the the norm topology, then $|b_n| \to |b|$ in the norm topology.*

The proof follows by Lemma 5 and Lemma 6 for $p = 1/2$.

Let $a_k \in A$, β_k, $q_k \in \mathbb{C}$. By taking $p_k := |q_k|$ and $\alpha_k = \frac{q_k}{|q_k|}\beta_k$ if $q_k \neq 0$ or 0 if $q_k = 0$, then by (2.5)

$$\sum_{k=1}^{n} |q_k| \left|\beta_k\right|^2 \sum_{k=1}^{n} |q_k| \left|a_k\right|^2 = \sum_{k=1, q_k \neq 0}^{n} p_k \left|\alpha_k\right|^2 \sum_{k=1, q_k \neq 0}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{k=1, q_k \neq 0}^{n} p_k \alpha_k a_j\right|^2$$

$$= \left|\sum_{k=1, q_k \neq 0}^{n} |q_k| \frac{q_k}{|q_k|} \beta_k a_j\right|^2 = \left|\sum_{k=1}^{n} q_k \beta_k a_k\right|^2,$$

therefore we have the CBS type inequality

$$\sum_{k=1}^{n} |q_k| \left|\beta_k\right|^2 \sum_{k=1}^{n} |q_k| \left|a_k\right|^2 \geq \left|\sum_{k=1}^{n} q_k \beta_k a_k\right|^2. \tag{3.1}$$

We also have the equality

$$\sum_{k=1}^{n} q_k \sum_{k=1}^{n} q_k \alpha_k a_k - \sum_{k=1}^{n} q_k \alpha_k \sum_{j=1}^{n} q_k a_k = \frac{1}{2} \sum_{k,j=1}^{n} q_j q_k \left(\alpha_j - \alpha_k\right) \left(a_j - a_k\right).$$

for $a_k \in A$, β_k, $q_k \in \mathbb{C}$. Then by inequality (3.1) for double sums, we derive

$$\left|\sum_{k=1}^{n} q_k \sum_{k=1}^{n} q_k \alpha_k a_k - \sum_{k=1}^{n} q_k \alpha_k \sum_{j=1}^{n} q_k a_k\right|^2 = \left|\frac{1}{2} \sum_{k,j=1}^{n} q_j q_k \left(\alpha_j - \alpha_k\right) \left(a_j - a_k\right)\right|^2$$

$$\leq \frac{1}{2} \sum_{k,j=1}^{n} |q_j| |q_k| \left|\alpha_j - \alpha_k\right|^2 \frac{1}{2} \sum_{k,j=1}^{n} |q_j| |q_k| \left|a_j - a_k\right|^2.$$

Since

$$\frac{1}{2} \sum_{k,j=1}^{n} |q_j| |q_k| \left|\alpha_j - \alpha_k\right|^2 = \left(\sum_{k=1}^{n} |q_k| \sum_{k=1}^{n} |q_k| \left|\alpha_k\right|^2 - \left|\sum_{k=1}^{n} |q_k| \alpha_k\right|^2\right)$$

and

$$\frac{1}{2} \sum_{k,j=1}^{n} |q_j| |q_k| \left|a_j - a_k\right|^2 = \left(\sum_{k=1}^{n} |q_k| \sum_{k=1}^{n} |q_k| \left|a_k\right|^2 - \left|\sum_{k=1}^{n} |q_k| a_k\right|^2\right),$$

hence we obtain the inequality

$$\left|\sum_{k=1}^{n} q_k \sum_{k=1}^{n} q_k \alpha_k a_k - \sum_{k=1}^{n} q_k \alpha_k \sum_{j=1}^{n} q_k a_k\right|^2 \tag{3.2}$$
$$\leq \left(\sum_{k=1}^{n} |q_k| \sum_{k=1}^{n} |q_k| |\alpha_k|^2 - \left|\sum_{k=1}^{n} |q_k| \alpha_k\right|^2\right)$$
$$\times \left(\sum_{k=1}^{n} |q_k| \sum_{k=1}^{n} |q_k| |a_k|^2 - \left|\sum_{k=1}^{n} |q_k| a_k\right|^2\right).$$

We denote by $\mathbb{C}$ the set of all complex numbers. Let ρ_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\rho_n|^{\frac{1}{n}}}.$$

Clearly $0 \leq R \leq \infty$, but we consider only the case $0 < R \leq \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{z \in \mathbb{C} : |z| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \rho_n \lambda^n$$

and

$$\lambda \mapsto f_{Abs}(\lambda) : D(0,R) \to \mathbb{C}, f_{Abs}(\lambda) := \sum_{n=0}^{\infty} |\rho_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \tag{3.3}$$
$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C};$$
$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$
$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_{Abs}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \tag{3.4}$$

$$g_{Abs}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$

$$h_{Abs}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$

$$l_{Abs}(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{3.5}$$

$$\frac{1}{2} \ln \left(\frac{1+\lambda}{1-\lambda} \right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

We say that the element $a \in A$ is *normal*, if $a^* a = a a^*$, or equivalently $|a| = |a^*|$.

Theorem 4. *Let A be a unital Hermitian Banach $*$-algebra with continuous involution. Consider the function $\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}$, $f(\lambda) := \sum_{n=0}^{\infty} \rho_n \lambda^n$. If $a \in A$ is normal in A with $\left\| |a|^2 \right\| < R$, $\lambda \in \mathbb{C}$ with $|\lambda|^2 < R$, $\|\lambda a\| < R$, then*

$$|f(\lambda a)|^2 \le f_{Abs}(|\lambda|^2) f_{Abs}\left(|a|^2\right) \tag{3.6}$$

and

$$|f(\lambda a)| \le \left[f_{Abs}(|\lambda|^2)\right]^{1/2} \left[f_{Abs}\left(|a|^2\right)\right]^{1/2}. \tag{3.7}$$

Proof. From the inequality (3.1) we have

$$\sum_{k=0}^{m} |\rho_k| \left|\lambda^k\right|^2 \sum_{k=0}^{m} |\rho_k| \left|a^k\right|^2 \ge \left|\sum_{k=0}^{m} \rho_k \lambda^k a^k\right|^2. \tag{3.8}$$

We have that

$$\left|a^k\right|^2 = \left(a^k\right)^* a^k = (a^*)^k a^k = a^*...a^*a...a = a^*...aa^*...a$$
$$= ... = (a^*a)^k = |a|^{2k}, \ k = 0, ..., m$$

and by (3.8) we derive

$$\sum_{k=0}^{m} |\rho_k| \left(|\lambda|^2\right)^k \sum_{k=0}^{m} |\rho_k| \left(|a|^2\right)^k \geq \left|\sum_{k=0}^{m} \rho_k (\lambda a)^k\right|^2. \tag{3.9}$$

Since $|\lambda|^2 < R$, then $\sum_{k=0}^{\infty} |\rho_k| \left(|\lambda|^2\right)^k$ is convergent and

$$\sum_{k=0}^{\infty} |\rho_k| \left(|\lambda|^2\right)^k = f_{Abs}(|\lambda|^2),$$

if $\left\||a|^2\right\| < R$, then $\sum_{k=0}^{\infty} |\rho_k| \left(|a|^2\right)^k$ is convergent and

$$\sum_{k=0}^{\infty} |\rho_k| \left(|a|^2\right)^k = f_{Abs}\left(|a|^2\right).$$

Also, if $\|\lambda a\| < R$, then $\sum_{k=0}^{\infty} \rho_k (\lambda a)^k$ is convergent and

$$\sum_{k=0}^{\infty} \rho_k (\lambda a)^k = f(\lambda a).$$

If we take the limit in (3.9) and use Corollary 3 we get

$$\lim_{m\to\infty} \left[\sum_{k=0}^{m} |\rho_k| \left(|\lambda|^2\right)^k \sum_{k=0}^{m} |\rho_k| \left(|a|^2\right)^k\right] \geq \lim_{m\to\infty} \left(\left|\sum_{k=0}^{m} \rho_k (\lambda a)^k\right|^2\right). \tag{3.10}$$

We have

$$\lim_{m\to\infty} \left[\sum_{k=0}^{m} |\rho_k| \left(|\lambda|^2\right)^k \sum_{k=0}^{m} |\rho_k| \left(|a|^2\right)^k\right]$$
$$= \lim_{m\to\infty} \left(\sum_{k=0}^{m} |\rho_k| \left(|\lambda|^2\right)^k\right) \lim_{m\to\infty} \sum_{k=0}^{m} |\rho_k| \left(|a|^2\right)^k = f_{Abs}(|\lambda|^2) f_{Abs}\left(|a|^2\right).$$

Also, by Lemma 5

$$\lim_{m\to\infty} \left(\left|\sum_{k=0}^{m} \rho_k (\lambda a)^k\right|^2\right) = \left|\lim_{m\to\infty} \sum_{k=0}^{m} \rho_k (\lambda a)^k\right|^2 = |f(\lambda a)|^2$$

and by (3.10) we derive (3.6). □

We also have:

Theorem 5. *Let A be a unital Hermitian Banach $*$-algebra with continuous involution. Consider the function $\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}$, $f(\lambda) := \sum_{n=0}^{\infty} \rho_n \lambda^n$.*

If $a \in A$ is normal in A, λ, $\alpha \in \mathbb{C}$ with $|\lambda| < R$, $|\lambda\alpha|$, $|\lambda| |\alpha|^2$, $\|\lambda a\|$, $\|\lambda\alpha a\|$, $\left\||\lambda| |a|^2\right\| < R$, then

$$|f(\lambda)f(\lambda\alpha a) - f(\lambda\alpha)f(\lambda a)|^2 \leq \left[f_{Abs}(|\lambda|)f_{Abs}(|\lambda| |\alpha|^2) - |f_{Abs}(|\lambda| \alpha)|^2\right] \times \left[f_{Abs}(|\lambda|)f_{Abs}(|\lambda| |a|^2) - |f_{Abs}(|\lambda| a)|^2\right] \tag{3.11}$$

and

$$|f(\lambda)f(\lambda\alpha a) - f(\lambda\alpha)f(\lambda a)| \leq \left[f_{Abs}(|\lambda|)f_{Abs}(|\lambda| |\alpha|^2) - |f_{Abs}(|\lambda| \alpha)|^2\right]^{1/2} \times \left[f_{Abs}(|\lambda|)f_{Abs}(|\lambda| |a|^2) - |f_{Abs}(|\lambda| a)|^2\right]^{1/2}. \tag{3.12}$$

Proof. From (3.2) we get for $m \geq 1$ that

$$\begin{aligned} &\left|\sum_{k=0}^{m} \rho_k \lambda^k \sum_{k=0}^{m} \rho_k \lambda^k \alpha^k a^k - \sum_{k=0}^{m} \rho_k \lambda^k \alpha^k \sum_{k=0}^{m} \rho_k \lambda^k a^k\right|^2 \\ &\leq \left(\sum_{k=0}^{m} |\rho_k| |\lambda|^k \sum_{k=0}^{m} |\rho_k| |\lambda|^k |\alpha|^{2k} - \left|\sum_{k=0}^{m} |\rho_k| |\lambda|^k \alpha^k\right|^2\right) \\ &\times \left(\sum_{k=0}^{m} |\rho_k| |\lambda|^k \sum_{k=0}^{m} |\rho_k| |\lambda|^k \left|a^k\right|^2 - \left|\sum_{k=0}^{m} |\rho_k| |\lambda|^k a^k\right|^2\right). \end{aligned}$$

Now, by utilising a similar argument to the one in the proof of Theorem 4, we obtain the desired result. □

Remark 4. *If A is a C^*-algebra, then under the assumptions of Theorem 5 we have the norm inequality*

$$\|f(\lambda)f(\lambda\alpha a) - f(\lambda\alpha)f(\lambda a)\|^2 \leq \left[f_{Abs}(|\lambda|)f_{Abs}(|\lambda| |\alpha|^2) - |f_{Abs}(|\lambda| \alpha)|^2\right] \times \left\|f_{Abs}(|\lambda|)f_{Abs}(|\lambda| |a|^2) - |f_{Abs}(|\lambda| a)|^2\right\| \tag{3.13}$$

Now, if we write the inequalities (3.6) and (3.11) for the exponential function, then we get

$$|\exp(\lambda a)|^2 \leq \exp(|\lambda|^2)\exp\left(|a|^2\right) \tag{3.14}$$

and

$$\begin{aligned} &|\exp(\lambda)\exp(\lambda\alpha a) - \exp(\lambda\alpha)\exp(\lambda a)|^2 \\ &\leq \left[\exp(|\lambda|)\exp(|\lambda| |\alpha|^2) - |\exp(|\lambda| \alpha)|^2\right] \\ &\times \left[\exp(|\lambda|)\exp(|\lambda| |a|^2) - |\exp(|\lambda| a)|^2\right] \end{aligned} \tag{3.15}$$

for any $\lambda \in \mathbb{C}$ and any normal element $a \in A$.

By (3.6) for the functions $f(\lambda) = (1 \pm \lambda)^{-1}$ we get

$$\left|(1 \pm \lambda a)^{-1}\right|^2 \leq \left(1 - |\lambda|^2\right)^{-1}\left(1 - |a|^2\right)^{-1} \tag{3.16}$$

for $|\lambda| < 1$ and a normal element $a \in A$ with $\|a\| < 1$.

By (3.11) for the functions $f(\lambda) = (1 \pm \lambda)^{-1}$ we also get

$$\begin{aligned} (3.17) \qquad & \left|(1 \pm \lambda)^{-1}(1 \pm \lambda\alpha a)^{-1} - (1 \pm \lambda\alpha)^{-1}(1 \pm \lambda a)^{-1}\right|^2 \\ & \leq \left[(1 - |\lambda|)^{-1}(1 - |\lambda|\,|\alpha|^2)^{-1} - \left|(1 - |\lambda|\,\alpha)^{-1}\right|^2\right] \\ & \times \left[(1 - |\lambda|)^{-1}(1 - |\lambda|\,|a|^2)^{-1} - \left|(1 - |\lambda|\,a)^{-1}\right|^2\right] \end{aligned}$$

for $|\lambda|, |\alpha| < 1$ and a normal element $a \in A$ with $\|a\| < 1$.

Similar inequalities may be stated for the examples of functions listed above. We omit the details.

Now, if $a_k \in A$, $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$ and $c, d \in A$ with $c \neq d$ and if either (2.11) or (2.12) is satisfied, then by (3.2) and (2.1) we get

$$\begin{aligned} (3.18) \qquad & \left|\sum_{k=1}^n q_k \sum_{k=1}^n q_k \alpha_k a_k - \sum_{k=1}^n q_k \alpha_k \sum_{j=1}^n q_k a_k\right|^2 \\ & \leq \left(\sum_{k=1}^n |q_k| \sum_{k=1}^n |q_k|\,|\alpha_k|^2 - \left|\sum_{k=1}^n |q_k|\,\alpha_k\right|^2\right) \\ & \times \operatorname{Re}\left[\left(\sum_{k=1}^n |q_k|\,a_k^* - c^*\right)\left(d - \sum_{k=1}^n |q_k|\,a_k\right)\right] \\ & \leq \frac{1}{4}\left(\sum_{k=1}^n |q_k| \sum_{k=1}^n |q_k|\,|\alpha_k|^2 - \left|\sum_{k=1}^n |q_k|\,\alpha_k\right|^2\right)|d - c|^2 . \end{aligned}$$

We also have:

Theorem 6. *Let A be a unital Hermitian Banach $*$-algebra with continuous involution. Consider the function $\lambda \mapsto f(\lambda) : D(0,1) \to \mathbb{C}$, $f(\lambda) := \sum_{n=0}^\infty \rho_n \lambda^n$. If $a \in A$ is selfadjoint in A, λ, $\alpha \in \mathbb{C}$ with $|\lambda| < 1$, $|\alpha| < 1$, $0 \leq a < 1$, then*

$$\begin{aligned} (3.19) \qquad & |f(\lambda)f(\lambda\alpha a) - f(\lambda\alpha)f(\lambda a)|^2 \\ & \leq \left[f_{Abs}(|\lambda|)f_{Abs}(|\lambda|\,|\alpha|^2) - |f_{Abs}(|\lambda|\,\alpha)|^2\right] \\ & \times \operatorname{Re}\left[(f_{Abs}(|\lambda|\,a) + 1)(1 - f_{Abs}(|\lambda|\,a))\right] \\ & \leq f_{Abs}(|\lambda|)f_{Abs}(|\lambda|\,|\alpha|^2) - |f_{Abs}(|\lambda|\,\alpha)|^2 . \end{aligned}$$

Proof. From $0 \leq a < 1$ we get by multiplying both sides by $a^{1/2}$ that $0 \leq a^2 \leq a$ and in general

$$0 \leq ... \leq a^m \leq ... \leq a^2 \leq a < 1$$

for all $m \geq 1$.

Also

$$\left|a^k - \frac{1 + (-1)}{2}\right| = \left|a^k\right| = a^k \leq 1 = \frac{1}{2}(1 - (-1)),\ k = 0, ..., m.$$

By (3.18) we derive

$$\begin{aligned}
&\left|\sum_{k=0}^{m}\rho_k\lambda^k\sum_{k=0}^{m}\rho_k\lambda^k\alpha^k a^k-\sum_{k=0}^{m}\rho_k\lambda^k\alpha^k\sum_{k=0}^{m}\rho_k\lambda^k a^k\right|^2\\
&\leq\left(\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k\left|\alpha^k\right|^2-\left|\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k\alpha^k\right|^2\right)\\
&\times\operatorname{Re}\left[\left(\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k a^k+1\right)\left(1-\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k a^k\right)\right]\\
&\leq\frac{1}{4}\left(\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k|\alpha_k|^2-\left|\sum_{k=0}^{m}|\rho_k|\,|\lambda|^k\alpha^k\right|^2\right)2^2.
\end{aligned}$$

By utilising the same argument from the proof of Theorem 4, we derive the desired result (3.19). □

Under the assumptions of Theorem 6, if we write the inequality (3.19) for $f(\lambda)=(1\pm\lambda)^{-1}$, then we get

$$\begin{aligned}
&\left|(1\pm\lambda)^{-1}(1\pm\lambda\alpha a)^{-1}-(1\pm\lambda\alpha)^{-1}(1\pm\lambda a)^{-1}\right|^2\\
&\leq\left[(1-|\lambda|)^{-1}(1-|\lambda|\,|\alpha|^2)^{-1}-\left|(1-|\lambda|\,\alpha)^{-1}\right|^2\right]\\
&\times\operatorname{Re}\left[\left((1-|\lambda|\,a)^{-1}+1\right)\left(1-(1-|\lambda|\,a)^{-1}\right)\right]\\
&\leq(1-|\lambda|)^{-1}(1-|\lambda|\,|\alpha|^2)^{-1}-\left|(1-|\lambda|\,\alpha)^{-1}\right|^2,
\end{aligned}\tag{3.20}$$

where $a\in A$ is *selfadjoint in* A, $\lambda,\alpha\in\mathbb{C}$ with $|\lambda|<1,\ |\alpha|<1,\ 0\leq a<1$.

References

[1] F. F. Bonsall and J. Duncan, *Complete Normed Algebra*, Springer-Verlag, New York, 1973.
[2] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.
[3] S. S. Dragomir, Multiplicative inequalities for weighted geometric mean in Hermitian unital Banach ∗-algebras. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **112** (2018), no. 4, 1349–1365.
[4] S. S. Dragomir, Quadratic weighted geometric mean in Hermitian unital Banach ∗-algebras. *Oper. Matrices* **12** (2018), no. 4, 1009–1026.
[5] S. S. Dragomir, Inequalities of Jensen's type for positive linear functionals on Hermitian unital Banach ∗-algebras. *Bull. Aust. Math. Soc.* **102** (2020), no. 2, 308–318
[6] S. S. Dragomir, Inequalities for weighted geometric mean in Hermitian unital Banach ∗-algebras via a result of Cartwright and Field. *Oper. Matrices* **14** (2020), no. 2, 417–435.
[7] B. Q. Feng, The geometric means in Banach ∗-algebra, *J. Operator Theory* **57** (2007), No. 2, 243-250.
[8] T. Furuta, Extension of the Furuta inequality and Ando-Hiai log-majorization. *Linear Algebra Appl.* **219** (1995), 139–155.
[9] G. J. Murphy, *u^*-Algebras and Operator Theory*, Academic Press, 1990.
[10] H. Najafi, Some operator inequalities for Hermitian Banach ∗-algebra, *Math. Scand.* **126** (2020), 82–98.
[11] T. Okayasu, The Löwner-Heinz inequality in Banach ∗-algebra, *Glasgow Math. J.* **42** (2000), 243-246.
[12] S. Shirali and J. W. M. Ford, Symmetry in complex involutory Banach algebras, II. *Duke Math. J.* **37** (1970), 275-280.

[13] K. Tanahashi and A. Uchiyama, The Furuta inequality in Banach ∗-algebras, *Proc. Amer. Math. Soc.* 128 (2000), 1691-1695.

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