MORE DISCRETE GRÜSS TYPE MODULUS INEQUALITIES IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS WITH APPLICATIONS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$. Assume that $b_k \in A$, $k \in \{1, ..., n\}$ are such such that either

$$\left| b_k - \frac{\gamma + \Gamma}{2} \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \text{ for } k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} - b_k^* \right) (b_k - \gamma) \right] \geq 0 \text{ for } k \in \{1, ..., n\}$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. Also assume that $\alpha_k \in \mathbb{C}$ satisfies either the condition

$$\left| \alpha_k - \frac{\delta + \Delta}{2} \right|^2 \leq \frac{1}{4} |\Delta - \delta|^2 \text{ for } k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Delta} - \overline{\alpha_k} \right) (\alpha_k - \delta) \right] \geq 0 \text{ for } k \in \{1, ..., n\}$$

for some complex constants δ, Δ with $\operatorname{Re}\left(\Delta\overline{\delta}\right) > 0$. Then

$$\left| \sum_{k=1}^n p_k \alpha_k b_k - \sum_{k=1}^n p_k \alpha_k \sum_{k=1}^n p_k b_k \right|^2$$
$$\leq \left(\frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \frac{|\delta + \Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} - 1 \right)^2 \left| \sum_{k=1}^n p_k \alpha_k \right|^2 \left| \sum_{k=1}^n p_k b_k \right|^2 .$$

Some applications for discrete Fourier transform are also provided.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If $a, b \in \operatorname{Inv}(A)$, then $ab \in \operatorname{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$.

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Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [13] (see also [1, Theorem 41.5])

$$a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [12], Tanahashi and Uchiyama [14] proved the following fundamental properties (see also [8]):

(i) If $a,\ b \in A$, then $a \geq 0,\ b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a,\ b \in A$, then $a > 0,\ b \geq 0$ imply $a + b > 0$;
(iii) If $a,\ b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z - a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [14, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha,\ \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a,\ b \in A$, $\alpha,\ \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

We define the following means for $\nu \in [0, 1]$, see also [8] for different notations:

$$a \nabla_\nu b := (1 - \nu) a + \nu b,\ \ a,\ b \in A \tag{A}$$

the *weighted arithmetic mean* of (a, b),

$$a!_\nu b := \left((1-\nu)a^{-1}+\nu b^{-1}\right)^{-1},\ a,\ b>0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a, b) and

$$a\sharp_\nu b := a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon}a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a, b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu = \frac{1}{2}$, we use the simpler notations $a\nabla b$, $a!b$ and $a\sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [8], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;

(xiii) If $0 < a,\ b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*(a!b)c = (c^*ac)!(c^*bc) \text{ and } c^*(a\sharp b)c = (c^*ac)\sharp(c^*bc);$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0, 1]$, then

$$(a!_\nu b)^{-1} = \left(a^{-1}\right)\nabla_\nu\left(b^{-1}\right) \text{ and } \left(a^{-1}\right)\sharp_\nu\left(b^{-1}\right) = (a\sharp_\nu b)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [8] the following inequality between the weighted means introduced above:

$$(1-\nu)a+\nu b \geq a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon}a^{1/2} \geq \left((1-\nu)a^{-1}+\nu b^{-1}\right)^{-1} \tag{HGA}$$

for any $0 < a,\ b \in A$ and $\nu \in [0, 1]$.

Okayasu [12] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a,\ b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^*a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^*a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a+b| \leq |a| + |b|,\ a,\ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

In the recent paper [7] we obtained the following Gruss' type inequality:

Theorem 1. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$. Then*

$$\left|\sum_{k=1}^n p_k\alpha_k a_k - \sum_{k=1}^n p_k\alpha_k\sum_{k=1}^n p_k a_k\right|^2 \leq \left(\sum_{k=1}^n p_k|\alpha_k|^2 - \left|\sum_{k=1}^n p_k\alpha_k\right|^2\right)\left(\sum_{k=1}^n p_k|a_k|^2 - \left|\sum_{k=1}^n p_k a_k\right|^2\right). \tag{1.1}$$

Moreover, if there exists c, $d \in A$ with $c \neq d$ such that either

$$\left| a_k - \frac{d+c}{2} \right|^2 \leq \frac{1}{4} |d-c|^2, k \in \{1, ..., n\} \tag{1.2}$$

or

$$\operatorname{Re}\left[(d^* - a_k^*) (a_k - c) \right] = \operatorname{Re}\left[(a_k^* - c^*) (d - a_k) \right] \geq 0, \ \ k \in \{1, ..., n\} \tag{1.3}$$

holds, then

$$\begin{aligned} &\left| \sum_{k=1}^n p_k \alpha_k a_k - \sum_{k=1}^n p_k \alpha_k \sum_{k=1}^n p_k a_k \right|^2 \\ &\leq \left(\sum_{k=1}^n p_k |\alpha_k|^2 - \left| \sum_{k=1}^n p_k \alpha_k \right|^2 \right) \operatorname{Re}\left[\left(\sum_{k=1}^n p_k a_k^* - c^* \right) \left(d - \sum_{k=1}^n p_k a_k \right) \right] \\ &\leq \frac{1}{4} \left(\sum_{k=1}^n p_k |\alpha_k|^2 - \left| \sum_{k=1}^n p_k \alpha_k \right|^2 \right) |d-c|^2 . \end{aligned} \tag{1.4}$$

Some applications for power series of normal elements in A were also provided in [7].

2. Main Results

We have the following Cauchy-Bunyakowsky-Schwarz inequality, see also [7].:

Lemma 1. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$. Then*

$$\begin{aligned} &\sum_{k=1}^n p_k |\alpha_k|^2 \sum_{k=1}^n p_k |a_k|^2 - \left| \sum_{j=1}^n p_j \alpha_j a_j \right|^2 \\ &= \sum_{k=1}^n p_k |\alpha_k|^2 \sum_{j=1}^n p_j \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k a_k \right|^2 \geq 0. \end{aligned} \tag{2.1}$$

In particular,

$$\begin{aligned} &\sum_{k=1}^n |\alpha_k|^2 \sum_{k=1}^n |a_k|^2 - \left| \sum_{j=1}^n \alpha_j a_j \right|^2 \\ &= \sum_{k=1}^n |\alpha_k|^2 \sum_{j=1}^n \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n |\alpha_k|^2} \sum_{k=1}^n \alpha_k a_k \right|^2 \geq 0. \end{aligned} \tag{2.2}$$

Proof. For $j \in \{1, ..., n\}$ we have

$$\begin{aligned} &\left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k a_k \right|^2 \\ &= \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k a_k \right)^* \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k a_k \right) \end{aligned}$$

$$= \left(a_j^* - \frac{\alpha_j}{\sum_{k=1}^n p_k |\alpha_k|^2}\left(\sum_{k=1}^n p_k\alpha_k a_k\right)^*\right)\left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2}\sum_{k=1}^n p_k\alpha_k a_k\right)$$

$$= |a_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\left(\sum_{k=1}^n p_k\alpha_k a_k\right)^* \alpha_j a_j$$

$$- \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} a_j^* \sum_{k=1}^n p_k\alpha_k a_k + \frac{|\alpha_j|^2}{\left(\sum_{k=1}^n p_k |\alpha_k|^2\right)^2}\left|\sum_{k=1}^n p_k\alpha_k a_k\right|^2.$$

If we multiply this equality with $p_j \geq 0$ and sum over j from 1 to n, we derive

$$\sum_{j=1}^n p_j \left|a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2}\sum_{k=1}^n p_k\alpha_k a_k\right|^2$$

$$= \sum_{j=1}^n p_j |a_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\left(\sum_{k=1}^n p_k\alpha_k a_k\right)^* \sum_{j=1}^n p_j \alpha_j a_j$$

$$- \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\sum_{j=1}^n p_j \overline{\alpha_j} a_j^* \sum_{k=1}^n p_k\alpha_k a_k + \frac{\sum_{j=1}^n p_j |\alpha_j|^2}{\left(\sum_{k=1}^n p_k |\alpha_k|^2\right)^2}\left|\sum_{k=1}^n p_k\alpha_k a_k\right|^2$$

$$= \sum_{j=1}^n p_j |a_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\left|\sum_{k=1}^n p_k\alpha_k a_k\right|^2$$

$$- \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\left|\sum_{k=1}^n p_k\alpha_k a_k\right|^2 + \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\left|\sum_{k=1}^n p_k\alpha_k a_k\right|^2$$

$$= \sum_{j=1}^n p_j |a_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2}\left|\sum_{k=1}^n p_k\alpha_k a_k\right|^2,$$

which is equivalent to (2.1). □

Theorem 2. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^n p_k |\alpha_k|^2 \sum_{k=1}^n p_k |a_k|^2 \geq \left|\sum_{j=1}^n p_j \alpha_j a_j\right|^2. \tag{2.3}$$

If $p_k > 0$ for $k \in \{1, ..., n\}$, then the equality holds in (2.3) if and only if

$$a_j = \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2}\sum_{k=1}^n p_k\alpha_k a_k \tag{2.4}$$

for all $j \in \{1, ..., n\}$.

Remark 1. *Let $a_k \in A$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$. Then*

$$\sum_{k=1}^n p_k |a_k|^2 \geq \left|\sum_{j=1}^n p_j a_j\right|^2. \tag{2.5}$$

The equality holds in (2.5) if and only if $a_j = \sum_{k=1}^n p_k a_k$ for all $j \in \{1, ..., n\}$.

If A is a Hermitian unital Banach $*$-algebra with continuous involution, then we can take the square root in (2.5) to obtain

$$\left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} p_j a_j\right|. \tag{2.6}$$

The inequality (2.3) follows by (2.1). The equality (2.4) follows by (2.1).

For $a \in A$ we define the selfadjoint element

$$\operatorname{Re}(a) := \frac{1}{2}(a^* + a) = \operatorname{Re}(a^*)$$

We have the following identity of interest:

Lemma 2. *For any a, d, $c \in A$, we have*

$$\begin{aligned} \left|a - \frac{d+c}{2}\right|^2 - \frac{1}{4}|d-c|^2 &= \operatorname{Re}[(a^* - d^*)(a - c)] \\ &= \operatorname{Re}[(a^* - c^*)(a - d)]. \end{aligned} \tag{2.7}$$

Proof. We have

$$\begin{aligned} &\left|a - \frac{d+c}{2}\right|^2 - \frac{1}{4}|d-c|^2 \\ &= |a|^2 - \frac{d^* + c^*}{2}a - a^*\frac{d+c}{2} + \frac{1}{4}\left(|d|^2 + d^*c + c^*d + |c|^2\right) \\ &- \frac{1}{4}\left(|d|^2 - d^*c - c^*d + |c|^2\right) \\ &= |a|^2 - \frac{d^* + c^*}{2}a - a^*\frac{d+c}{2} + \frac{1}{2}(d^*c + c^*d) \end{aligned}$$

and

$$\begin{aligned} &\operatorname{Re}[(a^* - d^*)(a - c)] \\ &= \operatorname{Re}\left[|a|^2 - d^*a - a^*c + d^*c\right] \\ &= |a|^2 - \operatorname{Re}(d^*a) - \operatorname{Re}(a^*c) + \operatorname{Re}(d^*c) \\ &= |a|^2 - \frac{1}{2}(d^*a + a^*d) - \frac{1}{2}(a^*c + c^*a) + \frac{1}{2}(d^*c + c^*d) \\ &= |a|^2 - \frac{1}{2}(d^* + c^*)a - \frac{1}{2}a^*(d+c) + \frac{1}{2}(d^*c + c^*d), \end{aligned}$$

which proves the desired identity (2.7). □

Corollary 1. *Let a, d, $c \in A$. The following statements are equivalent*

$$\left|a - \frac{d+c}{2}\right|^2 \leq \frac{1}{4}|d-c|^2 \tag{2.8}$$

and

$$\operatorname{Re}[(d^* - a^*)(a - c)] = \operatorname{Re}[(a^* - c^*)(d - a)] \geq 0. \tag{2.9}$$

We have the following reverse of the CBS inequality (2.5):

Theorem 3. *Assume that A has a continuous involution. Let u be an unitary element in A, namely $u^*u = 1$ and $a_k \in A$ and such that*

$$\left| a_k - \frac{\gamma + \Gamma}{2} u \right|^2 \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2 \text{ for } k \in \{1, ..., n\} \tag{2.10}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} u^* - a_k^* \right) \left(a_k - \gamma u \right) \right] \geq 0 \text{ for } k \in \{1, ..., n\} \tag{2.11}$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. Then for $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$,

$$\begin{aligned} \left(\sum_{k=1}^n p_k \left| a_k u \right|^2 \right)^{1/2} &\leq \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} u^* \left(\sum_{k=1}^n p_k a_k \right) u \right] \\ &\leq \frac{\left| \gamma + \Gamma \right|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \left| u^* \left(\sum_{k=1}^n p_k a_k \right) u \right|. \end{aligned} \tag{2.12}$$

Proof. The equivalence of the statements (2.10) and (2.11) follows by Corollary 1 for $a = a_k$, $d = \Gamma u$ and $c = \gamma u$ and taking into account that $|u|^2 = 1$.

By the properties of operator modulus, we have

$$\left| a_k u \right|^2 - 2 \operatorname{Re}\left[\left(\frac{\gamma + \Gamma}{2} u \right)^* a_k u \right] + \left| \frac{\gamma + \Gamma}{2} u \right|^2 \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2,$$

namely

$$\left| a_k u \right|^2 - 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right] + \left| \frac{\gamma + \Gamma}{2} \right|^2 \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2,$$

or

$$\left| a_k u \right|^2 + \left| \frac{\gamma + \Gamma}{2} \right|^2 - \frac{1}{4} \left| \Gamma - \gamma \right|^2 \leq 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right], \tag{2.13}$$

for $k \in \{1, ..., n\}$.

Observe that

$$\begin{aligned} \frac{1}{4} \left| \Gamma + \gamma \right|^2 - \frac{1}{4} \left| \Gamma - \gamma \right|^2 &= \frac{1}{4} \left(\left| \Gamma \right|^2 + 2 \operatorname{Re}(\Gamma\overline{\gamma}) + \left| \gamma \right|^2 \right) \\ &\quad - \frac{1}{4} \left(\left| \Gamma \right|^2 - 2 \operatorname{Re}(\Gamma\overline{\gamma}) + \left| \gamma \right|^2 \right) = \operatorname{Re}(\Gamma\overline{\gamma}) > 0, \end{aligned}$$

then by (2.13) we get

$$\left| a_k u \right|^2 + \operatorname{Re}(\Gamma\overline{\gamma}) \leq 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right], \tag{2.14}$$

for $k \in \{1, ..., n\}$.

If we multiply (2.14) by $p_k \geq 0$ and sum, then we get

$$\begin{aligned} \sum_{k=1}^n p_k \left| a_k u \right|^2 + \operatorname{Re}(\Gamma\overline{\gamma}) &\leq 2 \sum_{k=1}^n p_k \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right] \\ &= 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* \left(\sum_{k=1}^n p_k a_k \right) u \right]. \end{aligned} \tag{2.15}$$

Using the elementary operator inequality

$$2\alpha b \leq b^2 + \alpha^2,$$

where $b \geq 0$ in the operator order and real number $\alpha \geq 0$, then we also have

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\left(\sum_{k=1}^{n} p_k |a_k u|^2\right)^{1/2} \leq \sum_{k=1}^{n} p_k |a_k u|^2 + \operatorname{Re}(\Gamma\overline{\gamma}), \tag{2.16}$$

and by (2.15) and (2.16) we derive the first part of (2.12).

For an element a we consider the selfadjoint elements

$$\operatorname{Re}(a) := \frac{a^* + a}{2}, \quad \operatorname{Im}(a) := \frac{a - a^*}{2i}.$$

Then $a = \operatorname{Re}(a) + i\operatorname{Im}(a)$, $|a|^2 = (\operatorname{Re}(a))^2 + (\operatorname{Im}(a))^2$. We have $|a|^2 \geq (\operatorname{Re}(a))^2$ which implies, by taking the square root, that $|a| \geq |\operatorname{Re}(a)|$.

Therefore

$$\begin{aligned} 0 &\leq \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right] \\ &\leq \left|\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right| \\ &= \left|\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\right| \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right| \\ &= \frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right|, \end{aligned}$$

and the last part of (2.12) is thus proved. □

Remark 2. *Observe that for $z = \alpha + i\beta$ and $a \in A$, we have*

$$\begin{aligned} \operatorname{Re}(\overline{z}a) &= \operatorname{Re}[(\alpha - i\beta)(\operatorname{Re} a + i\operatorname{Im} a)] \\ &= \operatorname{Re}[\alpha\operatorname{Re} a + \beta\operatorname{Im} a - i(\beta\operatorname{Re} a - \alpha\operatorname{Im} a)] \\ &= \alpha\operatorname{Re} a + \beta\operatorname{Im} a = \operatorname{Re} z\operatorname{Re} a + \operatorname{Im} z\operatorname{Im} a \end{aligned}$$

and then

$$\begin{aligned} &\operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right] \\ &= \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \operatorname{Re}(\gamma + \Gamma)\operatorname{Re}\left[u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right] \\ &+ \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \operatorname{Im}(\gamma + \Gamma)\operatorname{Im}\left[u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right] \\ &= \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \operatorname{Re}(\gamma + \Gamma) u^* \left(\sum_{k=1}^{n} p_k \operatorname{Re}(a_k)\right) u \\ &+ \frac{1}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \operatorname{Im}(\gamma + \Gamma) u^* \left(\sum_{k=1}^{n} p_k \operatorname{Im}(a_k)\right) u. \end{aligned}$$

Therefore by (2.12) we have the unpacked inequality

$$\begin{aligned} &\left(\sum_{k=1}^{n} p_k \left|a_k u\right|^2\right)^{1/2} \\ &\le \frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \left[\operatorname{Re}\left(\gamma+\Gamma\right) u^* \left(\sum_{k=1}^{n} p_k \operatorname{Re}\left(a_k\right)\right) u \right. \\ &\left. + \operatorname{Im}\left(\gamma+\Gamma\right) u^* \left(\sum_{k=1}^{n} p_k \operatorname{Im}\left(a_k\right)\right) u\right] \\ &\le \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right|. \end{aligned} \tag{2.17}$$

From (2.17) we also derive the additive reverse of Cauchy-Bunyakowsky-Schwarz inequality

$$\begin{aligned} 0 &\le \left(\sum_{k=1}^{n} p_k \left|a_k u\right|^2\right)^{1/2} - \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right| \\ &\le \frac{\left|\gamma+\Gamma\right| - 2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right|, \end{aligned} \tag{2.18}$$

provided that $a_k,\ k \in \{1, ..., n\}$ *satisfies the conditions from Theorem 3.*

Recall that a C^*-*algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^* a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [10, 2.2.5 Theorem])

$$b \ge a \ge 0 \text{ implies that } \|b\| \ge \|a\|.$$

Since for $a \ge 0$ we have $\left\|a^{1/2}\right\| = \|a\|^{1/2}$ then under the assumptions of Theorem 3, we have

$$\begin{aligned} \left\|\sum_{k=1}^{n} p_k \left|a_k u\right|^2\right\|^{1/2} &\le \left\|\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right]\right\| \\ &\le \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \left\|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right\|. \end{aligned} \tag{2.19}$$

Corollary 2. *Let* u *be an unitary element in* A *and* $a_k \in A$ *and such that*

$$\left|a_k u - \frac{m+M}{2} u\right|^2 \le \frac{1}{4}\left(M-m\right)^2 \text{ for } k \in \{1, ..., n\} \tag{2.20}$$

or, equivalently

$$\operatorname{Re}\left[\left(M u^* - a_k^*\right)\left(a_k - m u\right)\right] \ge 0 \text{ for } k \in \{1, ..., n\} \tag{2.21}$$

for some real numbers $M > m > 0$. Then

$$\left(\sum_{k=1}^{n} p_k |a_k u|^2\right)^{1/2} \leq \frac{m+M}{2\sqrt{mM}} \operatorname{Re}\left[u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right] \leq \frac{m+M}{2\sqrt{Mm}} \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right|. \tag{2.22}$$

Also, we have the additive inequality

$$0 \leq \left(\sum_{k=1}^{n} p_k |a_k u|^2\right)^{1/2} - \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right| \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right|. \tag{2.23}$$

If A is a C^*-algebra, then under the assumptions of Corollary 2 we have

$$\left\|\sum_{k=1}^{n} p_k |a_k u|^2\right\|^{1/2} \leq \frac{m+M}{2\sqrt{mM}} \left\|\operatorname{Re}\left[u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right]\right\| \leq \frac{m+M}{2\sqrt{Mm}} \left\|u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right\| \tag{2.24}$$

and

$$\left\|\left(\sum_{k=1}^{n} p_k |a_k u|^2\right)^{1/2} - \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right|\right\| \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{2\sqrt{Mm}} \left\|u^*\left(\sum_{k=1}^{n} p_k a_k\right) u\right\|. \tag{2.25}$$

3. Grüss Type Inequalities

We have the following result that is of interest in itself:

Lemma 3. *Assume that $\alpha_k \in \mathbb{C}$, $a_k \in A$, and $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$, then*

$$\begin{aligned} &\left|\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k\right|^2 \\ &\leq \left(\sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2\right)\left(\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k a_k\right|^2\right) \\ &\leq \left[\left(\sum_{k=1}^{n} p_k |\alpha_k|^2\right)^{1/2}\left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{1/2} - \left|\sum_{k=1}^{n} p_k \alpha_k\right|\left|\sum_{k=1}^{n} p_k a_k\right|\right]^2. \end{aligned} \tag{3.1}$$

Proof. We use the following Sonin type identity that can be proved by performing the calculations in the right side

$$\begin{aligned}&\sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k \\ &= \sum_{k=1}^{n} p_k \left(\alpha_k - \sum_{j=1}^{n} p_j \alpha_j\right)\left(a_k - \sum_{j=1}^{n} p_j a_j\right).\end{aligned}$$

By using CBS inequality (2.3) we have

$$\begin{aligned}&\left|\sum_{k=1}^{n} p_k \left(\alpha_k - \sum_{j=1}^{n} p_j \alpha_j\right)\left(a_k - \sum_{j=1}^{n} p_j a_j\right)\right|^2 \\ &\le \left[\sum_{k=1}^{n} p_k \left|\alpha_k - \sum_{j=1}^{n} p_j \alpha_j\right|^2\right]\left[\sum_{k=1}^{n} p_k \left|a_k - \sum_{j=1}^{n} p_j a_j\right|^2\right].\end{aligned} \tag{3.2}$$

Since

$$\sum_{k=1}^{n} p_k \left|\alpha_k - \sum_{j=1}^{n} p_j \alpha_j\right|^2 = \sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2 \tag{3.3}$$

and

$$\sum_{k=1}^{n} p_k \left|a_k - \sum_{j=1}^{n} p_j a_j\right|^2 = \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left|\sum_{k=1}^{n} p_k a_k\right|^2, \tag{3.4}$$

hence by (3.2), (3.3) and (3.4) we derive the first part of (3.1).

Now, observe that for real numbers α, β and selfadjoint elements a, $b \in A$ we have the inequality

$$\left(\beta^2 - \alpha^2\right)\left(b^2 - a^2\right) \le \left(\beta b - \alpha a\right)^2.$$

Indeed, we have

$$\begin{aligned}&\left(\beta b - \alpha a\right)^2 - \left(\beta^2 - \alpha^2\right)\left(b^2 - a^2\right) \\ &= \beta^2 b^2 - \beta\alpha\left(ab + ba\right) + \alpha^2 a^2 - \beta^2 b^2 + \alpha^2 b^2 + \beta^2 a^2 - \alpha^2 a^2 \\ &= \alpha^2 b^2 + \beta^2 a^2 - \beta\alpha\left(ab + ba\right) = \left(\alpha b - \beta a\right)^2 \ge 0.\end{aligned}$$

Therefore by taking

$$\beta = \left(\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2\right)^{1/2}, \ \alpha = \left|\sum_{k=1}^{n} p_k \alpha_k\right|$$

and

$$b = \left(\sum_{k=1}^{n} p_k \left|a_k\right|^2\right)^{1/2}, \ a = \left|\sum_{k=1}^{n} p_k a_k\right|$$

we deduce the second part of (3.1). □

By taking the square root in (3.1) we also have:

Corollary 3. *With the assumptions of Lemma 3 and if A has a continuous involution, then we have*

$$\left|\sum_{k=1}^{n}p_k\alpha_k a_k-\sum_{k=1}^{n}p_k\alpha_k\sum_{k=1}^{n}p_k a_k\right| \leq\left(\sum_{k=1}^{n}p_k\left|\alpha_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{n}p_k\left|a_k\right|^2\right)^{1/2}-\left|\sum_{k=1}^{n}p_k\alpha_k\right|\left|\sum_{k=1}^{n}p_k a_k\right|. \tag{3.5}$$

We have the following Grüss' type inequality:

Theorem 4. *Assume that $b_k\in A$, $k\in\{1,...,n\}$ are such such that either*

$$\left|b_k-\frac{\gamma+\Gamma}{2}\right|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2 \text{ for } k\in\{1,...,n\} \tag{3.6}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}-b_k^*\right)\left(b_k-\gamma\right)\right]\geq 0 \text{ for } k\in\{1,...,n\} \tag{3.7}$$

for some complex constants γ,Γ with $\operatorname{Re}(\Gamma\overline{\gamma})>0$. Also assume that $\alpha_k\in\mathbb{C}$ satisfies either the condition

$$\left|\alpha_k-\frac{\delta+\Delta}{2}\right|^2\leq\frac{1}{4}\left|\Delta-\delta\right|^2 \text{ for } k\in\{1,...,n\} \tag{3.8}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Delta}-\overline{\alpha_k}\right)\left(\alpha_k-\delta\right)\right]\geq 0 \text{ for } k\in\{1,...,n\} \tag{3.9}$$

for some complex constants δ, Δ with $\operatorname{Re}\left(\Delta\overline{\delta}\right)>0$. Then

$$\left|\sum_{k=1}^{n}p_k\alpha_k b_k-\sum_{k=1}^{n}p_k\alpha_k\sum_{k=1}^{n}p_k b_k\right|^2 \leq\left(\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}}-1\right)^2\left|\sum_{k=1}^{n}p_k\alpha_k\right|^2\left|\sum_{k=1}^{n}p_k b_k\right|^2. \tag{3.10}$$

Proof. From (3.5) we get

$$\left|\sum_{k=1}^{n}p_k\alpha_k b_k-\sum_{k=1}^{n}p_k\alpha_k\sum_{k=1}^{n}p_k b_k\right|^2 \leq\left[\left(\sum_{k=1}^{n}p_k\left|\alpha_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{n}p_k\left|b_k\right|^2\right)^{1/2}-\left|\sum_{k=1}^{n}p_k\alpha_k\right|\left|\sum_{k=1}^{n}p_k b_k\right|\right]^2 =:K. \tag{3.11}$$

From (2.12) we have for $u=1$ that

$$\left(\sum_{k=1}^{n}p_k\left|b_k\right|^2\right)^{1/2}\leq\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|\sum_{k=1}^{n}p_k b_k\right|. \tag{3.12}$$

Also, we have the scalar inequality

$$\left(\sum_{k=1}^{n} p_k |\alpha_k|^2\right)^{1/2} \le \frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} \left|\sum_{k=1}^{n} p_k \alpha_k\right|. \tag{3.13}$$

Therefore

$$\begin{aligned} K &\le \left[\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} \left|\sum_{k=1}^{n} p_k \alpha_k\right| \left|\sum_{k=1}^{n} p_k b_k\right| \right. \\ &\quad \left. - \left|\sum_{k=1}^{n} p_k \alpha_k\right| \left|\sum_{k=1}^{n} p_k b_k\right|\right]^2 \\ &= \left(\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} - 1\right)^2 \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2 \left|\sum_{k=1}^{n} p_k b_k\right|^2, \end{aligned}$$

which proves the desired result (3.10). □

Remark 3. *If A is a C^*-algebra, then under the assumptions of Theorem 4 we have*

$$\begin{aligned} &\left\|\sum_{k=1}^{n} p_k \alpha_k b_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k b_k\right\| \\ &\le \left(\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} - 1\right) \left|\sum_{k=1}^{n} p_k \alpha_k\right| \left\|\sum_{k=1}^{n} p_k b_k\right\|. \end{aligned} \tag{3.14}$$

Remark 4. *If the conditions (3.6)-(3.9) hold for $\gamma = \delta = \phi$ and $\Gamma = \Delta = \Phi$, then we have the simpler bounds*

$$\begin{aligned} &\left|\sum_{k=1}^{n} p_k \alpha_k b_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k b_k\right|^2 \\ &\le \left(\frac{|\Phi-\phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)}\right)^2 \left|\sum_{k=1}^{n} p_k \alpha_k\right|^2 \left|\sum_{k=1}^{n} p_k b_k\right|^2. \end{aligned} \tag{3.15}$$

Observe that for $\gamma = \delta = \phi$ and $\Gamma = \Delta = \Phi$,

$$\begin{aligned} \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} \frac{|\delta+\Delta|}{2\sqrt{\operatorname{Re}\left(\Delta\overline{\delta}\right)}} - 1 &= \frac{|\phi+\Phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} - 1 = \frac{|\phi+\Phi|^2 - 4\operatorname{Re}\left(\Phi\overline{\phi}\right)}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} \\ &= \frac{|\Phi-\phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)}, \end{aligned}$$

where $\operatorname{Re}\left(\Phi\overline{\phi}\right) > 0$ and by (3.10) we derive (3.15).

If A has a continuous involution, then by taking the square root in (3.15), we get

$$\left|\sum_{k=1}^{n} p_k \alpha_k b_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k b_k\right| \le \frac{|\Phi-\phi|^2}{4\operatorname{Re}\left(\Phi\overline{\phi}\right)} \left|\sum_{k=1}^{n} p_k \alpha_k\right| \left|\sum_{k=1}^{n} p_k b_k\right|. \tag{3.16}$$

4. Applications to the Discrete Fourier Transform

Let A be a Hermitian unital Banach $*$-algebra and $a = (a_1, \ldots, a_n)$ be a sequence of vectors in A.

For a given $w \in \mathbb{R}$, define the *discrete Fourier transform* as

$$\mathcal{F}_w(\mathbf{a})(m) := \sum_{k=1}^{n} \exp(2wimk) a_k, \; m = 1, \ldots, n. \tag{4.1}$$

The following approximation result holds:

Theorem 5. *Let $a = (a_1, \ldots, a_n)$ be a sequence of vectors in A. If there exists the elements $b, c \in A$ such that either (1.2) or (1.3) is valid, then we have the inequality*

$$\begin{aligned} &\left| \mathcal{F}_w(\mathbf{a})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{1}{4} n^2 \left[1 - \frac{\sin^2(wmn)}{n^2 \sin^2(wm)} \right] |d-c|^2, \end{aligned} \tag{4.2}$$

for all $m \in \{1, \ldots, n\}$ and $w \in \mathbb{R}$, $w \neq \frac{l}{m}\pi$, $l \in \mathbb{Z}$.

Proof. From the inequality (1.1), we can state that

$$\begin{aligned} &\left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k a_k - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{1}{4} \left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^2 - \left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k \right|^2 \right) |d-c|^2 \end{aligned} \tag{4.3}$$

for all $\alpha_k \in \mathbb{C}$, $a_k \in A$ $(k = 1, \ldots, n)$.

We now choose in (**??**), $\alpha_k = \exp(2wimk)$ to obtain

$$\begin{aligned} &\left| \mathcal{F}_w(\mathbf{a})(m) - \sum_{k=1}^{n} \exp(2wimk) \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{1}{4} \left(n \sum_{k=1}^{n} |\exp(2wimk)|^2 - \left| \sum_{k=1}^{n} \exp(2wimk) \right|^2 \right) |d-c|^2 \end{aligned} \tag{4.4}$$

for all $m \in \{1, \ldots, n\}$.

As a simple calculation reveals that

$$\begin{aligned}\sum_{k=1}^{n}\exp\left(2wimk\right) &= \exp\left(2wim\right)\times\left[\frac{\exp\left(2wimn\right)-1}{\exp\left(2wim\right)-1}\right]\\ &= \exp\left(2wim\right)\times\left[\frac{\cos\left(2wmn\right)+i\sin\left(2wmn\right)-1}{\cos\left(2wm\right)+i\sin\left(2wm\right)-1}\right]\\ &= \exp\left(2wim\right)\times\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\left[\frac{\cos\left(wmn\right)+i\sin\left(wmn\right)}{\cos\left(wm\right)+i\sin\left(wm\right)}\right]\\ &= \frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\times\exp\left(2wim\right)\left[\frac{\exp\left(iwmn\right)}{\exp\left(iwm\right)}\right]\\ &= \frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\times\exp\left[w\left(n+1\right)im\right],\end{aligned}$$

$$\sum_{k=1}^{n}\left|\exp\left(2wimk\right)\right|^{2}=n$$

and

$$\left|\sum_{k=1}^{n}\exp\left(2wimk\right)\right|^{2}=\frac{\sin^{2}\left(wmn\right)}{\sin^{2}\left(wm\right)},\ \text{for } w\neq\frac{l}{m}\pi,\ l\in\mathbb{Z},$$

thus, from (4.4), we deduce the desired inequality (4.2). □

Remark 5. *With the assumptions of Theorem* **??** *and if A has a continuous involution, then by taking the square root in (4.2) then we get*

$$\begin{aligned}&\left|\mathcal{F}_{w}\left(\mathbf{a}\right)\left(m\right)-\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\exp\left[w\left(n+1\right)im\right]\frac{1}{n}\sum_{k=1}^{n}a_{k}\right|\\ &\leq\frac{1}{2}n\left[1-\frac{\sin^{2}\left(wmn\right)}{n^{2}\sin^{2}\left(wm\right)}\right]^{1/2}\left|d-c\right|,\end{aligned}\tag{4.5}$$

If A is a C^{*}-algebra, then under the assumptions of Theorem **??** we have

$$\begin{aligned}&\left\|\mathcal{F}_{w}\left(\mathbf{a}\right)\left(m\right)-\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\exp\left[w\left(n+1\right)im\right]\frac{1}{n}\sum_{k=1}^{n}a_{k}\right\|\\ &\leq\frac{1}{2}n\left[1-\frac{\sin^{2}\left(wmn\right)}{n^{2}\sin^{2}\left(wm\right)}\right]^{1/2}\left\|d-c\right\|.\end{aligned}\tag{4.6}$$

Assume that A has a continuous involution and $a_k\in A$, $k\in\{1,\ldots,n\}$, such that either (2.10) or (2.11) for $u=1$ is valid. From (2.12) we get

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left|a_{k}\right|^{2}\right)^{1/2}\leq\frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left|\frac{1}{n}\sum_{k=1}^{n}a_{k}\right|.$$

By (3.5) we derive

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k a_k-\frac{1}{n}\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{k=1}^{n}a_k\right| \tag{4.7}$$
$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}|\alpha_k|^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n}|a_k|^2\right)^{1/2}-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|$$
$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}|\alpha_k|^2\right)^{1/2}\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|$$
$$=\left[\left(\frac{1}{n}\sum_{k=1}^{n}|\alpha_k|^2\right)^{1/2}\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|\right]\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|.$$

Therefore

$$\left|\sum_{k=1}^{n}\alpha_k a_k-\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{k=1}^{n}a_k\right| \tag{4.8}$$
$$\leq\left[\left(n\sum_{k=1}^{n}|\alpha_k|^2\right)^{1/2}\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}-\left|\sum_{k=1}^{n}\alpha_k\right|\right]\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|$$

for all $\alpha_k\in\mathbb{C}$ and either

$$\left|a_k-\frac{\gamma+\Gamma}{2}\right|^2\leq\frac{1}{4}|\Gamma-\gamma|^2\text{ for }k\in\{1,...,n\} \tag{4.9}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}-a_k^*\right)(a_k-\gamma)\right]\geq 0\text{ for }k\in\{1,...,n\} \tag{4.10}$$

holds for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma})>0$.

We can state the following result:

Theorem 6. *Let A be a Hermitian unital Banach $*$-algebra with continuous involution and $\mathbf{a}\in A^n$. If there exists the complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma})>0$ and such that either (4.9) or (4.10) is valid, then*

$$\left|\mathcal{F}_w(\mathbf{a})(m)-\frac{\sin(wmn)}{\sin(wm)}\exp[w(n+1)im]\frac{1}{n}\sum_{k=1}^{n}a_k\right| \tag{4.11}$$
$$\leq n\left[\frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}-\left|\frac{\sin(wmn)}{n\sin(wm)}\right|\right]\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|$$

for all $m\in\{1,\dots,n\}$ and $w\in\mathbb{R},\ w\neq\frac{l}{m}\pi,\ l\in\mathbb{Z}$.

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