A REVERSE OF THE DISCRETE CAUCHY-BUNYAKOWSKY-SCHWARZ INEQUALITY FOR MODULUS IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$. Assume that A has a continuous involution. Let u be an unitary element in A, namely $u^*u = 1$ and $a_k \in A$ and such that

$$\left|a_k u - \frac{\gamma+\Gamma}{2} u\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2 \text{ for } k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} u^* - a_k^*\right)(a_k - \gamma u)\right] \geq 0 \text{ for } k \in \{1, ..., n\}$$

for some complex constants γ, Γ with $\gamma + \Gamma \neq 0$, then

$$\left(\sum_{k=1}^n p_k |a_k u|^2\right)^{1/2} \leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|} u^* \left(\sum_{k=1}^n p_k a_k\right) u\right] + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}.$$

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If $a, b \in \operatorname{Inv}(A)$, then $ab \in \operatorname{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [13] (see also [1, Theorem 41.5])

$$a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [12], Tanahashi and Uchiyama [14] proved the following fundamental properties (see also [8]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;

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(v) If $c>0$, then $0<b<a$ if and only if $cbc<cac$, also $0<b\leq a$ if and only if $cbc\leq cac$;
(vi) If $0<a<1$, then $1<a^{-1}$;
(vii) If $0<b<a$, then $0<a^{-1}<b^{-1}$, also if $0<b\leq a$, then $0<a^{-1}\leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a):=\frac{1}{2\pi i}\int_{\gamma}f(z)(z-a)^{-1}dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a)\subset\operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a))=f(\sigma(a))$$

holds.

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re}z>0\}$, the right half open plane of the complex plane, such that $\sigma(a)\subset\operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha\in\mathbb{R}$ we define for $a\in A$ and $a>0$, the real power

$$a^{\alpha}:=\frac{1}{2\pi i}\int_{\gamma}z^{\alpha}(z-a)^{-1}dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha}\in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re}z>0\}$, then by (SMT) we have

$$\sigma(a^{\alpha})=(\sigma(a))^{\alpha}=\{z^{\alpha}:z\in\sigma(a)\}\subset(0,\infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $a^{\alpha}\in A$ with $a^{\alpha}>0$ and $\left(a^{2}\right)^{1/2}=a$, [14, Lemma 6];
(ix) If $0<a\in A$ and $\alpha,\beta\in\mathbb{R}$, then $a^{\alpha}a^{\beta}=a^{\alpha+\beta}$;
(x) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $(a^{\alpha})^{-1}=\left(a^{-1}\right)^{\alpha}=a^{-\alpha}$;
(xi) If $0<a,\ b\in A$, $\alpha,\ \beta\in\mathbb{R}$ and $ab=ba$, then $a^{\alpha}b^{\beta}=b^{\beta}a^{\alpha}$.

We define the following means for $\nu\in[0,1]$, see also [8] for different notations:

$$a\nabla_{\nu}b:=(1-\nu)a+\nu b,\ \ a,\ b\in A \tag{A}$$

the *weighted arithmetic mean* of (a,b),

$$a!_{\nu}b:=\left((1-\nu)a^{-1}+\nu b^{-1}\right)^{-1},\ \ a,\ b>0 \tag{H}$$

the *weighted harmonic mean* of positive elements (a,b) and

$$a\sharp_{\nu}b:=a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon}a^{1/2} \tag{G}$$

the *weighted geometric mean* of positive elements (a,b). Our notations above are motivated by the classical notations used in operator theory. For simplicity, if $\nu=\frac{1}{2}$, we use the simpler notations $a\nabla b$, $a!b$ and $a\sharp b$. The definition of weighted geometric mean can be extended for any real ν.

In [8], B. Q. Feng proved the following properties of these means in A a Hermitian unital Banach $*$-algebra:

(xii) If $0 < a,\ b \in A$, then $a!b = b!a$ and $a\sharp b = b\sharp a$;

(xiii) If $0 < a,\ b \in A$ and $c \in \operatorname{Inv}(A)$, then

$$c^*\left(a!b\right)c = \left(c^*ac\right)!\left(c^*bc\right) \text{ and } c^*\left(a\sharp b\right)c = \left(c^*ac\right)\sharp\left(c^*bc\right);$$

(xiv) If $0 < a,\ b \in A$ and $\nu \in [0,1]$, then

$$\left(a!_\nu b\right)^{-1} = \left(a^{-1}\right)\nabla_\nu\left(b^{-1}\right) \text{ and } \left(a^{-1}\right)\sharp_\nu\left(b^{-1}\right) = \left(a\sharp_\nu b\right)^{-1}.$$

Utilising the Spectral Mapping Theorem and the Bernoulli inequality for real numbers, B. Q. Feng obtained in [8] the following inequality between the weighted means introduced above:

$$\text{(HGA)} \qquad (1-\nu)\,a + \nu b \geq a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^{\upsilon}a^{1/2} \geq \left((1-\nu)\,a^{-1} + \nu b^{-1}\right)^{-1}$$

for any $0 < a,\ b \in A$ and $\nu \in [0,1]$.

Okayasu [12] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a,\ b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^*a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^*a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a+b| \leq |a| + |b|,\ a,\ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

In the recent paper [7] we obtained the follow reverse of Cauchy-Bunyakowsky-Schwarz inequality for modulus:

Theorem 1. *Assume that A has a continuous involution. Let u be an unitary element in A, namely $u^*u = 1$ and $a_k \in A$ and such that*

$$\left|a_k - \frac{\gamma+\Gamma}{2}u\right|^2 \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2 \text{ for } k \in \{1,...,n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}u^* - a_k^*\right)\left(a_k - \gamma u\right)\right] \geq 0 \text{ for } k \in \{1,...,n\}$$

for some complex constants $\gamma,\ \Gamma$ with $\operatorname{Re}\left(\Gamma\overline{\gamma}\right) > 0$. Then for $p_k \geq 0,\ k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$,

$$\begin{aligned}\left(\sum_{k=1}^n p_k\left|a_k u\right|^2\right)^{1/2} &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}u^*\left(\sum_{k=1}^n p_k a_k\right)u\right] \\ &\leq \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left|u^*\left(\sum_{k=1}^n p_k a_k\right)u\right|.\end{aligned}$$

Some applications for power series of normal elements in A were also provided in [7].

2. Main Results

We have the following Cauchy-Bunyakowsky-Schwarz inequality, see also [7]:

Lemma 1. *Let* $a_k \in A$, $\alpha_k \in \mathbb{C}$ *and* $p_k \geq 0$ *for* $k \in \{1, ..., n\}$. *Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{j=1}^{n} p_j \alpha_j a_j \right|^2 = \sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{j=1}^{n} p_j \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \geq 0. \tag{2.1}$$

In particular,

$$\sum_{k=1}^{n} |\alpha_k|^2 \sum_{k=1}^{n} |a_k|^2 - \left| \sum_{j=1}^{n} \alpha_j a_j \right|^2 = \sum_{k=1}^{n} |\alpha_k|^2 \sum_{j=1}^{n} \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} |\alpha_k|^2} \sum_{k=1}^{n} \alpha_k a_k \right|^2 \geq 0. \tag{2.2}$$

Proof. For $j \in \{1, ..., n\}$ we have

$$\begin{aligned}
&\left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \\
&= \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right) \\
&= \left(a_j^* - \frac{\alpha_j}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \right) \left(a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right) \\
&= |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \alpha_j a_j \\
&- \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \frac{|\alpha_j|^2}{\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 \right)^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2.
\end{aligned}$$

If we multiply this equality with $p_j \geq 0$ and sum over j from 1 to n, we derive

$$\begin{aligned}
&\sum_{j=1}^{n} p_j \left| a_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \\
&= \sum_{j=1}^{n} p_j |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left(\sum_{k=1}^{n} p_k \alpha_k a_k \right)^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{j=1}^{n} p_j \overline{\alpha_j} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \frac{\sum_{j=1}^{n} p_j |\alpha_j|^2}{\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 \right)^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} p_j |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \\
&- \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 + \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 \\
&= \sum_{j=1}^{n} p_j |a_j|^2 - \frac{1}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \left| \sum_{k=1}^{n} p_k \alpha_k a_k \right|^2 ,
\end{aligned}
$$

which is equivalent to (2.1). □

Theorem 2. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 \geq \left| \sum_{j=1}^{n} p_j \alpha_j a_j \right|^2 . \tag{2.3}$$

If $p_k > 0$ for $k \in \{1, ..., n\}$, then the equality holds in (2.3) if and only if

$$a_j = \frac{\overline{\alpha_j}}{\sum_{k=1}^{n} p_k |\alpha_k|^2} \sum_{k=1}^{n} p_k \alpha_k a_k \tag{2.4}$$

for all $j \in \{1, ..., n\}$.

If A is a Hermitian unital Banach $*$-algebra with continuous involution, then also

$$\left(\sum_{k=1}^{n} p_k |\alpha_k|^2 \right)^{1/2} \left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} p_j \alpha_j a_j \right| . \tag{2.5}$$

Remark 1. *Let $a_k \in A$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$. Then*

$$\sum_{k=1}^{n} p_k |a_k|^2 \geq \left| \sum_{j=1}^{n} p_j a_j \right|^2 . \tag{2.6}$$

The equality holds in (2.6) if and only if $a_j = \sum_{k=1}^{n} p_k a_k$ for all $j \in \{1, ..., n\}$.

If A is a Hermitian unital Banach $*$-algebra with continuous involution, then we can take the square root in (2.6) to obtain

$$\left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} p_j a_j \right| . \tag{2.7}$$

The inequality (2.3) follows by (2.1). The equality (2.4) follows by (2.1).

For $a \in A$ we define the selfadjoint element

$$\operatorname{Re}(a) := \frac{1}{2}(a^* + a) = \operatorname{Re}(a^*)$$

We have the following identity of interest:

Lemma 2. *For any a, d, $c \in A$, we have*

$$
\begin{aligned}
\left| a - \frac{d+c}{2} \right|^2 - \frac{1}{4} |d - c|^2 &= \operatorname{Re}\left[(a^* - d^*)(a - c) \right] \\
&= \operatorname{Re}\left[(a^* - c^*)(a - d) \right] .
\end{aligned} \tag{2.8}
$$

Proof. We have

$$\left|a-\frac{d+c}{2}\right|^2-\frac{1}{4}\left|d-c\right|^2$$
$$=\left|a\right|^2-\frac{d^*+c^*}{2}a-a^*\frac{d+c}{2}+\frac{1}{4}\left(\left|d\right|^2+d^*c+c^*d+\left|c\right|^2\right)$$
$$-\frac{1}{4}\left(\left|d\right|^2-d^*c-c^*d+\left|c\right|^2\right)$$
$$=\left|a\right|^2-\frac{d^*+c^*}{2}a-a^*\frac{d+c}{2}+\frac{1}{2}\left(d^*c+c^*d\right)$$

and

$$\operatorname{Re}\left[\left(a^*-d^*\right)\left(a-c\right)\right]$$
$$=\operatorname{Re}\left[\left|a\right|^2-d^*a-a^*c+d^*c\right]$$
$$=\left|a\right|^2-\operatorname{Re}\left(d^*a\right)-\operatorname{Re}\left(a^*c\right)+\operatorname{Re}\left(d^*c\right)$$
$$=\left|a\right|^2-\frac{1}{2}\left(d^*a+a^*d\right)-\frac{1}{2}\left(a^*c+c^*a\right)+\frac{1}{2}\left(d^*c+c^*d\right)$$
$$=\left|a\right|^2-\frac{1}{2}\left(d^*+c^*\right)a-\frac{1}{2}a^*\left(d+c\right)+\frac{1}{2}\left(d^*c+c^*d\right),$$

which proves the desired identity (2.8). □

Corollary 1. *Let a, d, $c\in A$. The following statements are equivalent*

$$\left|a-\frac{d+c}{2}\right|^2\leq\frac{1}{4}\left|d-c\right|^2 \tag{2.9}$$

and

$$\operatorname{Re}\left[\left(d^*-a^*\right)\left(a-c\right)\right]=\operatorname{Re}\left[\left(a^*-c^*\right)\left(d-a\right)\right]\geq 0. \tag{2.10}$$

We have the following reverse of the CBS inequality (2.6):

Theorem 3. *Assume that A has a continuous involution. Let u be an unitary element in A, namely $u^*u=1$ and $a_k\in A$ and such that*

$$\left|a_ku-\frac{\gamma+\Gamma}{2}u\right|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2 \text{ for } k\in\{1,...,n\} \tag{2.11}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}u^*-a_k^*\right)\left(a_k-\gamma u\right)\right]\geq 0 \text{ for } k\in\{1,...,n\} \tag{2.12}$$

for some complex constants γ, Γ with $\gamma+\Gamma\neq 0$. Then for $p_k\geq 0$ with $\sum_{k=1}^n p_k=1$,

$$\left(\sum_{k=1}^n p_k\left|a_ku\right|^2\right)^{1/2}\leq\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{\left|\gamma+\Gamma\right|}u^*\left(\sum_{k=1}^n p_ka_k\right)u\right]+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|}. \tag{2.13}$$

Proof. The equivalence of the statements (2.11) and (2.12) follows by Corollary 1 for $a=a_k$, $d=\Gamma u$ and $c=\gamma u$ and taking into account that $\left|u\right|^2=1$.

By the properties of operator modulus, we have

$$\left|a_ku\right|^2-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}u\right)^*a_ku\right]+\left|\frac{\gamma+\Gamma}{2}u\right|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2,$$

namely

$$|a_k u|^2 - 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}u^* a_k u\right] + \left|\frac{\gamma+\Gamma}{2}\right|^2 \le \frac{1}{4}|\Gamma-\gamma|^2,$$

or

$$|a_k u|^2 + \left|\frac{\gamma+\Gamma}{2}\right|^2 \le 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}u^* a_k u\right] + \frac{1}{4}|\Gamma-\gamma|^2, \tag{2.14}$$

for $k \in \{1,...,n\}$.

If we multiply (2.14) by $p_k \ge 0$ and sum, then we get

$$\begin{aligned}\sum_{k=1}^n p_k |a_k u|^2 + \left|\frac{\gamma+\Gamma}{2}\right|^2 &\le 2\sum_{k=1}^n p_k \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}u^* a_k u\right] + \frac{1}{4}|\Gamma-\gamma|^2 \\ &= 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}u^*\left(\sum_{k=1}^n p_k a_k\right)u\right] + \frac{1}{4}|\Gamma-\gamma|^2.\end{aligned} \tag{2.15}$$

Using the elementary operator inequality

$$2\alpha a \le a^2 + \alpha^2, \tag{2.16}$$

where $a \ge 0$ in the operator order and $\alpha \ge 0$, then we also have

$$2\left|\frac{\gamma+\Gamma}{2}\right|\left(\sum_{k=1}^n p_k |a_k u|^2\right)^{1/2} \le \sum_{k=1}^n p_k |a_k u|^2 + \left|\frac{\gamma+\Gamma}{2}\right|^2. \tag{2.17}$$

By utilising (2.15) and (2.17) we deduce

$$\begin{aligned}2\left|\frac{\gamma+\Gamma}{2}\right|\left(\sum_{k=1}^n p_k |a_k u|^2\right)^{1/2} \le 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}u^*\left(\sum_{k=1}^n p_k a_k\right)u\right] + \frac{1}{4}|\Gamma-\gamma|^2.\end{aligned}$$

Since $\gamma+\Gamma \neq 0$, hence by dividing with $|\gamma+\Gamma| \neq 0$, we get

$$\left(\sum_{k=1}^n p_k |a_k u|^2\right)^{1/2} \le \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^n p_k a_k\right)u\right] + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|},$$

which proves the first inequality in (2.13). □

Corollary 2. *Let u be an unitary element in A, namely $u^*u = 1$ and $a_k \in A$ such that*

$$\left|a_k u - \frac{m+M}{2}u\right|^2 \le \frac{1}{4}(M-m)^2 \text{ for } k \in \{1,...,n\} \tag{2.18}$$

or, equivalently

$$\operatorname{Re}\left[(Mu^* - a_k^*)(a_k - mu)\right] \ge 0 \text{ for } k \in \{1,...,n\} \tag{2.19}$$

for some real numbers $M > m > 0$. Then

$$\left(\sum_{k=1}^{n} p_k |a_k u|^2\right)^{1/2} \le \operatorname{Re}\left[u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right] + \frac{1}{4}\frac{(M-m)^2}{m+M}. \tag{2.20}$$

Remark 2. *Observe that for $z = \alpha + i\beta$ and $a \in A$, we have*

$$\begin{aligned}\operatorname{Re}(\overline{z}a) &= \operatorname{Re}[(\alpha - i\beta)(\operatorname{Re} a + i \operatorname{Im} a)] \\ &= \operatorname{Re}[\alpha \operatorname{Re} a + \beta \operatorname{Im} a - i(\beta \operatorname{Re} a - \alpha \operatorname{Im} a)] \\ &= \alpha \operatorname{Re} a + \beta \operatorname{Im} a = \operatorname{Re} z \operatorname{Re} a + \operatorname{Im} z \operatorname{Im} a\end{aligned}$$

and then

$$\begin{aligned}&\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right] \\ &= \frac{1}{|\gamma+\Gamma|}\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\left[u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right] \\ &+ \frac{1}{|\gamma+\Gamma|}\operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\left[u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right] \\ &= \frac{1}{|\gamma+\Gamma|}\operatorname{Re}(\gamma+\Gamma)u^*\left(\sum_{k=1}^{n} p_k \operatorname{Re}(a_k)\right)u \\ &+ \frac{1}{|\gamma+\Gamma|}\operatorname{Im}(\gamma+\Gamma)u^*\left(\sum_{k=1}^{n} p_k \operatorname{Im}(a_k)\right)u.\end{aligned}$$

Therefore, (2.13) can be written as

$$\begin{aligned}&\left(\sum_{k=1}^{n} p_k |a_k u|^2\right)^{1/2} \\ &\le \frac{1}{|\gamma+\Gamma|} \\ &\times\left[\operatorname{Re}(\gamma+\Gamma)u^*\left(\sum_{k=1}^{n} p_k \operatorname{Re}(a_k)\right)u + \operatorname{Im}(\gamma+\Gamma)u^*\left(\sum_{k=1}^{n} p_k \operatorname{Im}(a_k)\right)u\right] \\ &+ \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|}.\end{aligned} \tag{2.21}$$

Remark 3. *If a is selfadjoint, then $|a| - a$ is selfadjoint and $\sigma(|a| - a) \subseteq [0, \infty)$ which gives that $|a| \ge a$. Then we have*

$$\begin{aligned}\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right] &\le \left|\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right]\right| \\ &\le \left|\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right| \\ &= \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right|.\end{aligned}$$

Therefore, under the assumptions of Theorem 3 we can also state the inequality

$$\left(\sum_{k=1}^{n} p_k \left|a_k u\right|^2\right)^{1/2} \leq \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right| + \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|}. \tag{2.22}$$

Assume that A has a continuous involution. Let u be an unitary element in A and $a_k \in A$ such that

$$\left|a_k - \frac{\gamma+\Gamma}{2}\right|^2 \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2 \text{ for } k \in \{1,...,n\} \tag{2.23}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}-a_k^*\right)\left(a_k-\gamma\right)\right] \geq 0 \text{ for } k \in \{1,...,n\} \tag{2.24}$$

for some complex constants γ, Γ with $\gamma+\Gamma \neq 0$. Then

$$\begin{aligned}\left(\sum_{k=1}^{n} p_k \left|a_k\right|^2\right)^{1/2} &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{\left|\gamma+\Gamma\right|}\left(\sum_{k=1}^{n} p_k a_k\right)\right] + \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|} \\ &\leq \left|\sum_{k=1}^{n} p_k a_k\right| + \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|},\end{aligned} \tag{2.25}$$

and

$$0 \leq \left(\sum_{k=1}^{n} p_k \left|a_k\right|^2\right)^{1/2} - \left|\sum_{k=1}^{n} p_k a_k\right| \leq \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|}$$

which provide reverses of the inequality (2.7).

Recall that a C^**-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\left\|a^*a\right\| = \left\|a\right\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [10, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \left\|b\right\| \geq \left\|a\right\|.$$

If $a_k \in A$ is such that either (2.23) or (2.24) is satisfied, then

$$\left\|\left(\sum_{k=1}^{n} p_k \left|a_k\right|^2\right)^{1/2} - \left|\sum_{k=1}^{n} p_k a_k\right|\right\| \leq \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|}. \tag{2.26}$$

3. A Generalization

We can insert a sequence of scalars as follows:

Theorem 4. *Assume that A has a continuous involution. Let u be an unitary element in A, namely $u^*u = 1$ and $a_k \in A$, $\alpha_k \in \mathbb{C}\setminus\{0\}$ and such that*

$$\left|\frac{a_k}{\alpha_k}u - \frac{\gamma+\Gamma}{2}u\right|^2 \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2 \textit{ for } k \in \{1,...,n\} \tag{3.1}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}u^* - \frac{a_k^*}{\overline{\alpha_k}}\right)\left(\frac{a_k}{\alpha_k} - \gamma u\right)\right] \geq 0 \textit{ for } k \in \{1,...,n\} \tag{3.2}$$

for some complex constants γ, Γ with $\gamma+\Gamma\neq 0$. Then for $w_k\geq 0$, $k\in\{1,...,n\}$ we have

$$\begin{aligned}(3.3)\quad &\left(\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{n} w_k\left|a_k u\right|^2\right)^{1/2}\\ &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{\left|\gamma+\Gamma\right|}u^*\left(\sum_{k=1}^{n} w_k\alpha_k a_k\right)u\right]+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|}\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2.\\ &\leq \left|u^*\left(\sum_{k=1}^{n} w_k\alpha_k a_k\right)u\right|+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\gamma+\Gamma\right|}\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2.\end{aligned}$$

Proof. If we multiply (3.1) with $\left|\alpha_k\right|^2>0$, we get

$$\left|a_k u-\frac{\gamma+\Gamma}{2}\overline{\alpha_k}u\right|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\left|\alpha_k\right|^2$$

for $k\in\{1,...,n\}$.

This is equivalent to

$$\left|a_k u\right|^2-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}u\right)^*\alpha_k a_k u\right]+\left|\frac{\gamma+\Gamma}{2}\right|^2\left|\alpha_k\right|^2\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\left|\alpha_k\right|^2,$$

for $k\in\{1,...,n\}$.

If we multiply with $w_k\geq 0$ and sum, then we get

$$\begin{aligned}&\sum_{k=1}^{n} w_k\left|a_k u\right|^2-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}u\right)^*\left(\sum_{k=1}^{n} w_k\alpha_k a_k\right)u\right]+\left|\frac{\gamma+\Gamma}{2}\right|^2\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2\\ &\leq\frac{1}{4}\left|\Gamma-\gamma\right|^2\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2,\end{aligned}$$

or, equivalently

$$\begin{aligned}(3.4)\quad &\sum_{k=1}^{n} w_k\left|a_k u\right|^2+\left|\frac{\gamma+\Gamma}{2}\right|^2\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2\\ &\leq 2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}u\right)^*\left(\sum_{k=1}^{n} w_k\alpha_k a_k\right)u\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^2\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2.\end{aligned}$$

Since, from (2.16) we have

$$\begin{aligned}(3.5)\quad &2\left|\frac{\gamma+\Gamma}{2}\right|\left(\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{n} w_k\left|a_k u\right|^2\right)^{1/2}\\ &\leq\sum_{k=1}^{n} w_k\left|a_k u\right|^2+\left|\frac{\gamma+\Gamma}{2}\right|^2\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2,\end{aligned}$$

hence by (3.4) and (3.5) we derive

$$\begin{aligned} & 2\left|\frac{\gamma+\Gamma}{2}\right|\left(\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{n} w_k\left|a_k u\right|^2\right)^{1/2} \\ & \leq 2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}u\right)^*\left(\sum_{k=1}^{n} w_k \alpha_k a_k\right)u\right]+\frac{1}{4}|\Gamma-\gamma|^2 \sum_{k=1}^{n} w_k\left|\alpha_k\right|^2, \end{aligned}$$

which is equivalent to the first inequality in (3.3).

The second part follows by the inequality $\operatorname{Re} b \leq |\operatorname{Re} b| \leq |b|$ for $b \in A$. □

Corollary 3. *Assume that A has a continuous involution and $a_k \in A$, $\alpha_k \in \mathbb{C}\setminus\{0\}$ and such that*

$$\left|\frac{a_k}{\overline{\alpha_k}}-\frac{\gamma+\Gamma}{2}\right|^2 \leq \frac{1}{4}|\Gamma-\gamma|^2 \text{ for } k \in\{1, \ldots, n\} \tag{3.6}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}-\frac{a_k^*}{\alpha_k}\right)\left(\frac{a_k}{\overline{\alpha_k}}-\gamma\right)\right] \geq 0 \text{ for } k \in\{1, \ldots, n\} \tag{3.7}$$

for some complex constants γ, Γ with $\gamma+\Gamma \neq 0$. Then for $w_k \geq 0$, $k \in\{1, \ldots, n\}$ we have

$$\begin{aligned} & \left(\sum_{k=1}^{n} w_k\left|\alpha_k\right|^2\right)^{1/2}\left(\sum_{k=1}^{n} w_k\left|a_k\right|^2\right)^{1/2} \\ & \leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}\left(\sum_{k=1}^{n} w_k \alpha_k a_k\right)\right]+\frac{1}{4} \frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|} \sum_{k=1}^{n} w_k\left|\alpha_k\right|^2 . \\ & \leq\left|\sum_{k=1}^{n} w_k \alpha_k a_k\right|+\frac{1}{4} \frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|} \sum_{k=1}^{n} w_k\left|\alpha_k\right|^2 . \end{aligned} \tag{3.8}$$

Remark 4. *If $|\alpha_k|=1$, $k \in\{1, \ldots, n\}$ in (3.1) or (3.2) and $w_k=p_k$, $k \in\{1, \ldots, n\}$ with $\sum_{k=1}^{n} p_k=1$, then we have the inequalities*

$$\begin{aligned} & \left(\sum_{k=1}^{n} p_k\left|a_k u\right|^2\right)^{1/2} \\ & \leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|} u^*\left(\sum_{k=1}^{n} p_k \alpha_k a_k\right) u\right]+\frac{1}{4} \frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|} \\ & \leq\left|u^*\left(\sum_{k=1}^{n} p_k \alpha_k a_k\right) u\right|+\frac{1}{4} \frac{|\Gamma-\gamma|^2}{|\gamma+\Gamma|} . \end{aligned} \tag{3.9}$$

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