REVERSES OF CBS AND GRÜSS TYPE MODULUS INEQUALITIES FOR FORWARD DIFFERENCE IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS WITH APPLICATIONS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$. Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$. In this paper we show among others that

$$\left| \sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k \right|^2 \leq \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k \right)^2 \right] \sum_{i=1}^{n-1} |\Delta \alpha_i|^2 \sum_{i=1}^{n-1} |\Delta a_i|^2 ,$$

where $\Delta \alpha_j := \alpha_{j+1} - \alpha_j$ is the forward difference. Some applications to discrete Fourier transform are also provided.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\text{Inv}(A)$. If $a, b \in \text{Inv}(A)$, then $ab \in \text{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [13] (see also [1, Theorem 41.5])

$$a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [12], Tanahashi and Uchiyama [14] proved the following fundamental properties (see also [8]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;

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(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [14, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [12] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^* a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^* a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a+b| \leq |a| + |b|, \ a, \ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$. In this paper we show among others that

$$\left|\sum_{k=1}^n p_k \alpha_k a_k - \sum_{k=1}^n p_k \alpha_k \sum_{k=1}^n p_k a_k\right|^2 \leq \left[\sum_{k=1}^n k^2 p_k - \left(\sum_{k=1}^n k p_k\right)^2\right] \sum_{i=1}^{n-1} |\Delta \alpha_i|^2 \sum_{i=1}^{n-1} |\Delta a_i|^2,$$

where $\Delta \alpha_j := \alpha_{j+1} - \alpha_j$ is the forward difference. Some applications to discrete Fourier transform are also provided.

2. Main Results

We start to the following identities of interest:

Lemma 1. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1,...,n\}$. Then*

$$\sum_{k=1}^n p_k |\alpha_k|^2 \sum_{k=1}^n p_k |a_k|^2 - \left|\sum_{j=1}^n p_j \alpha_j a_j\right|^2 = \frac{1}{2} \sum_{j,k=1}^n p_j p_k |\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 . \tag{2.1}$$

In particular,

$$\sum_{k=1}^n |\alpha_k|^2 \sum_{k=1}^n |a_k|^2 - \left|\sum_{j=1}^n \alpha_j a_j\right|^2 = \frac{1}{2} \sum_{j,k=1}^n |\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 \tag{2.2}$$

and

$$\sum_{k=1}^n p_k \sum_{k=1}^n p_k |a_k|^2 - \left|\sum_{j=1}^n p_j a_j\right|^2 = \frac{1}{2} \sum_{j,k=1}^n p_j p_k |a_k - a_j|^2 . \tag{2.3}$$

Proof. Observe that

$$\begin{aligned}
&|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j|^2 \\
&= (\overline{\alpha_j} a_k - \overline{\alpha}_k a_j)^* (\overline{\alpha_j} a_k - \overline{\alpha}_k a_j) = \left(\alpha_j a_k^* - \alpha_k a_j^*\right)(\overline{\alpha_j} a_k - \overline{\alpha}_k a_j) \\
&= \alpha_j a_k^* \overline{\alpha_j} a_k - \alpha_j a_k^* \overline{\alpha}_k a_j - \alpha_k a_j^* \overline{\alpha_j} a_k + \alpha_k a_j^* \overline{\alpha}_k a_j \\
&= |\alpha_j|^2 |a_k|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + |\alpha_k|^2 |a_j|^2
\end{aligned}$$

for all $j, k \in \{1,...,n\}$.

This implies that

$$\begin{aligned}
&\sum_{j,k=1}^{n} p_j p_k \left|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right|^2 \\
&= \sum_{j,k=1}^{n} p_j p_k \left[\left|\alpha_j\right|^2 \left|a_k\right|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + \left|\alpha_k\right|^2 \left|a_j\right|^2\right] \\
&= \sum_{j,k=1}^{n} p_j p_k \left|\alpha_j\right|^2 \left|a_k\right|^2 - \sum_{j,k=1}^{n} p_j p_k \overline{\alpha}_k a_k^* \alpha_j a_j \\
&- \sum_{j,k=1}^{n} p_j p_k \overline{\alpha_j} a_j^* \alpha_k a_k + \sum_{j,k=1}^{n} p_j p_k \left|\alpha_k\right|^2 \left|a_j\right|^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} p_j \left|\alpha_j\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \sum_{k=1}^{n} p_k \overline{\alpha}_k a_k^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \sum_{j=1}^{n} p_j \overline{\alpha_j} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{j=1}^{n} p_j \left|a_j\right|^2 \\
&= \sum_{j=1}^{n} p_j \left|\alpha_j\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left(\sum_{k=1}^{n} p_k \alpha_k a_k\right)^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \left(\sum_{j=1}^{n} p_j \alpha_j a_j\right)^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{j=1}^{n} p_j \left|a_j\right|^2 \\
&= 2\left[\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k a_k\right|^2\right],
\end{aligned}$$

which is equivalent to the desired identity (2.1). □

We have the following Cauchy-Bunyakowsky-Schwarz (CBS) type inequalities:

Corollary 1. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k > 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|^2. \tag{2.4}$$

In particular,

$$\sum_{k=1}^{n} \left|\alpha_k\right|^2 \sum_{k=1}^{n} \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} \alpha_j a_j\right|^2 \tag{2.5}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} p_j a_j\right|^2. \tag{2.6}$$

The equality holds in (2.6) if and only if $a_k = a$ for some $a \in A$ and all $k \in \{1, ..., n\}$.

Remark 1. *If A has a continuous involution, then we can take the square root in (2.4) to get*

$$\left(\sum_{k=1}^{n} p_k |\alpha_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|. \tag{2.7}$$

Recall that a C^*-*algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^*a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [10, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \|b\| \geq \|a\|.$$

By utilising this property, we get from (2.4) the norm inequality

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \left\|\sum_{k=1}^{n} p_k |a_k|^2\right\| \geq \left\|\sum_{j=1}^{n} p_j \alpha_j a_j\right\|^2. \tag{2.8}$$

We define the Čebyšev's functional

$$T_n(\mathbf{p}, \boldsymbol{\alpha}, \mathbf{b}) := \sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k \alpha_k b_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k b_k,$$

where $\mathbf{p} = (p_1, ..., p_n)$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$ are scalar sequences and $\mathbf{b} = (b_1, ..., b_n) \in A^n$.

Theorem 1. *Let $a_k \in A$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$. Denote*

$$s_k := \sum_{i=1}^{k-1} |\Delta a_i|^2, \ k \in \{2, ..., n\} \ \text{and } s_1 := 0. \tag{2.9}$$

Then we have

$$\begin{aligned} 0 &\leq \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2 \\ &\leq T_n(\mathbf{p}, \ell, \mathbf{s}) \leq \sum_{1 \leq j < k \leq n}^{n} p_j p_k (k - j) \sum_{i=1}^{n-1} |\Delta a_i|^2 \\ &\leq \begin{cases} \frac{1}{2}(n-1)\left(1 - \sum_{k=1}^{n} p_k^2\right) \sum\limits_{i=1}^{n-1} |\Delta a_i|^2 \\[2ex] \frac{\sqrt{2}}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{1/2} \sum\limits_{i=1}^{n-1} |\Delta a_i|^2, \end{cases} \end{aligned} \tag{2.10}$$

where $\ell = (\ell_1, ..., \ell_n)$, $\ell_k = k$, $k \in \{1, ..., n\}$ and $\mathbf{s} = (s_1, ..., s_n) \in A^n$.

Moreover, if there exists an element $a \in A$ such that

$$(0 \leq) \ |\Delta a_i|^2 \leq a \text{ for all } i \in \{1, ..., n-1\},$$

then also

$$0 \le \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2 \le T_n(\mathbf{p}, \ell, \mathbf{s}) \le \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right] a. \tag{2.11}$$

Proof. Due to symmetry we have

$$\frac{1}{2}\sum_{j,k=1}^{n} p_j p_k |a_k - a_j|^2 = \sum_{1\le j<k\le n} p_j p_k |a_k - a_j|^2,$$

which gives the equality

$$\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2 = \sum_{1\le j<k\le n} p_j p_k |a_k - a_j|^2. \tag{2.12}$$

As $1 \le j < k \le n$ we can write that

$$a_k - a_j = \sum_{i=j}^{k-1} (a_{i+1} - a_i) = \sum_{i=j}^{k-1} \Delta a_i.$$

Using (2.6) we have

$$\begin{aligned} |a_k - a_j|^2 &= \left|\sum_{i=j}^{k-1} \Delta a_i\right|^2 \le (k-j)\sum_{i=j}^{k-1} |\Delta a_i|^2 \\ &= (k-j)\left(\sum_{i=1}^{k-1} |\Delta a_i|^2 - \sum_{i=1}^{j-1} |\Delta a_i|^2\right) = (k-j)(s_k - s_j) \end{aligned} \tag{2.13}$$

for $2 \le j < k \le n$. Also, if $j = 1$, then (2.13) also holds for $k > 1$.

By (2.13) and (2.12) we get

$$\begin{aligned} &\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2 \\ &\le \sum_{1\le j<k\le n} p_j p_k (k-j)(s_k - s_j) = \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k (k-j)(s_k - s_j) \\ &= \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k (\ell_k - \ell_j)(s_k - s_j) \\ &= \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k (\ell_k s_k + \ell_j s_j - \ell_k s_j - \ell_j s_k) = \sum_{k=1}^{n} p_k \ell_k s_k - \sum_{k=1}^{n} p_k \ell_k \sum_{k=1}^{n} p_k b_k \end{aligned}$$

and the first part of inequality (2.10) is proved.

Now, since

$$T_n(\mathbf{p}, \ell, \mathbf{s}) = \sum_{1\le j<k\le n} p_j p_k (k-j) \sum_{i=j}^{k-1} |\Delta a_i|^2$$

and

$$\sum_{i=j}^{k-1} |\Delta a_i|^2 \le \sum_{i=1}^{n-1} |\Delta a_i|^2,$$

hence

$$T_n(\mathbf{p},\ell,\mathbf{s}) \le \sum_{1\le j<k\le n}^{n} p_j p_k (k-j) \sum_{i=1}^{n-1} |\Delta a_i|^2,$$

which proves the second inequality in (2.10).

Now, since $k-j \le n-1$, hence

$$\begin{aligned}
&\sum_{1\le j<k\le n}^{n} p_j p_k (k-j) \\
&\le (n-1)\sum_{1\le j<k\le n}^{n} p_j p_k = \frac{1}{2}(n-1)\left(\sum_{j,k=1}^{n} p_j p_k - \sum_{k=1}^{n} p_k^2\right) \\
&= \frac{1}{2}(n-1)\left[\left(\sum_{k=1}^{n} p_k\right)^2 - \sum_{k=1}^{n} p_k^2\right] = \frac{1}{2}(n-1)\left(1-\sum_{k=1}^{n} p_k^2\right)
\end{aligned}$$

and the first branch in the last inequality is thus proved.

Also

$$\begin{aligned}
\sum_{1\le j<k\le n}^{n} p_j p_k (k-j) &= \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k |k-j| \\
&\le \frac{1}{2}\left[\sum_{j,k=1}^{n} p_j p_k \sum_{j,k=1}^{n} p_j p_k (k-j)^2\right]^{1/2} \\
&= \frac{1}{2}\left(2\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]\right)^{1/2} \\
&= \frac{\sqrt{2}}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{1/2},
\end{aligned}$$

which proves the second branch in the last inequality in (2.10).

Now, since $(0 \le)\, |\Delta a_i|^2 \le a$ for all $i \in \{1,...,n-1\}$, hence

$$\sum_{i=j}^{k-1} |\Delta a_i|^2 \le (k-j)\,a$$

and

$$\begin{aligned}
T_n(\mathbf{p},\ell,\mathbf{s}) &= \sum_{1\le j<k\le n}^{n} p_j p_k (k-j) \sum_{i=j}^{k-1} |\Delta a_i|^2 \le \left[\sum_{1\le j<k\le n}^{n} p_j p_k (k-j)^2\right] a \\
&= \left[\frac{1}{2}\sum_{j,k=1}^{n} p_j p_k (k-j)^2\right] a = \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right] a,
\end{aligned}$$

and the inequality (2.11) is thus proved. □

Remark 2. *If* $|a_k|^2 \le b$ *for all* $k \in \{1, ..., n\}$, *then*

$$|\Delta a_k|^2 = |a_{k+1} - a_k|^2 \le 2\left(|a_{k+1}|^2 + |a_k|^2\right) \le 4b$$

for $k \in \{1, ..., n-1\}$. *Then by (2.11) we derive*

$$0 \le \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2 \le T_n(\mathbf{p}, \ell, \mathbf{s}) \le 4\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right] b. \tag{2.14}$$

Remark 3. *If* A *is a* C^**-algebra, then by (2.10) we have the norm inequalities*

$$\begin{aligned} &\left\|\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2\right\| \\ &\le \|T_n(\mathbf{p}, \ell, \mathbf{s})\| \le \sum_{1\le j<k\le n}^{n} p_j p_k (k-j) \left\|\sum_{i=1}^{n-1} |\Delta a_i|^2\right\| \\ &\le \begin{cases} \frac{1}{2}(n-1)\left(1 - \sum_{k=1}^{n} p_k^2\right)\left\|\sum\limits_{i=1}^{n-1} |\Delta a_i|^2\right\| \\ \\ \frac{\sqrt{2}}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{1/2} \left\|\sum\limits_{i=1}^{n-1} |\Delta a_i|^2\right\|, \end{cases} \end{aligned} \tag{2.15}$$

while from (2.11),

$$\begin{aligned} \left\|\sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2\right\| &\le \|T_n(\mathbf{p}, \ell, \mathbf{s})\| \\ &\le \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right] \|a\|. \end{aligned} \tag{2.16}$$

Theorem 2. *Let* $a_k \in A$, $\alpha_k \in \mathbb{C}$ *and* $p_k \ge 0$ *for* $k \in \{1, ..., n\}$. *Then*

$$\begin{aligned} |T_n(\mathbf{p}, \boldsymbol{\alpha}, \mathbf{a})|^2 &\le \left(\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{j=1}^{n} p_j \alpha_j\right|^2\right) \\ &\quad \times \left(\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2\right). \end{aligned} \tag{2.17}$$

If A *has a continuous involution, then*

$$\begin{aligned} |T_n(\mathbf{p}, \boldsymbol{\alpha}, \mathbf{a})| &\le \left(\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{j=1}^{n} p_j \alpha_j\right|^2\right)^{1/2} \\ &\quad \times \left(\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2\right)^{1/2}. \end{aligned} \tag{2.18}$$

Proof. We have by Cauchy-Bunyakowsky-Schwarz for double sums

$$\begin{aligned} |T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})|^2 &= \left|\sum_{k=1}^n p_k \sum_{k=1}^n p_k\alpha_k a_k - \sum_{k=1}^n p_k\alpha_k \sum_{k=1}^n p_k a_k\right|^2 \\ &= \frac{1}{4}\left|\sum_{j,k=1}^n p_j p_k (\alpha_k-\alpha_j)(a_k-a_j)\right|^2 \\ &\le \frac{1}{4}\sum_{j,k=1}^n p_j p_k |\alpha_k-\alpha_j|^2 \sum_{j,k=1}^n p_j p_k |a_k-a_j|^2, \end{aligned} \tag{2.19}$$

Since

$$\frac{1}{2}\sum_{j,k=1}^n p_j p_k |\alpha_k-\alpha_j|^2 = \sum_{k=1}^n p_k \sum_{k=1}^n p_k |\alpha_k|^2 - \left|\sum_{j=1}^n p_j\alpha_j\right|^2$$

and

$$\frac{1}{2}\sum_{j,k=1}^n p_j p_k |a_k-a_j|^2 = \sum_{k=1}^n p_k \sum_{k=1}^n p_k |a_k|^2 - \left|\sum_{j=1}^n p_j a_j\right|^2$$

hence by (2.19) we derive (2.17). □

Remark 4. *If A is a C^*-algebra, then by (2.17)*

$$\begin{aligned} \|T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})\|^2 &\le \left(\sum_{k=1}^n p_k \sum_{k=1}^n p_k |\alpha_k|^2 - \left|\sum_{j=1}^n p_j\alpha_j\right|^2\right) \\ &\quad\times \left\|\sum_{k=1}^n p_k \sum_{k=1}^n p_k |a_k|^2 - \left|\sum_{j=1}^n p_j a_j\right|^2\right\|. \end{aligned} \tag{2.20}$$

Corollary 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} |T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})|^2 &\le \left(\sum_{k=1}^n p_k \sum_{k=1}^n p_k |\alpha_k|^2 - \left|\sum_{j=1}^n p_j\alpha_j\right|^2\right) T_n(\mathbf{p},\ell,\mathbf{s}) \\ &\le \left(\sum_{k=1}^n p_k \sum_{k=1}^n p_k |\alpha_k|^2 - \left|\sum_{j=1}^n p_j\alpha_j\right|^2\right) \\ &\quad\times \sum_{1\le j<k\le n} p_j p_k (k-j) \sum_{i=1}^{n-1} |\Delta a_i|^2 \end{aligned} \tag{2.21}$$

$$\leq \left(\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{j=1}^{n} p_j \alpha_j\right|^2\right)$$

$$\times \begin{cases} \frac{1}{2}(n-1)\left(1-\sum_{k=1}^{n} p_k^2\right) \sum_{i=1}^{n-1} |\Delta a_i|^2, \\ \\ \frac{\sqrt{2}}{2}\left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{1/2} \sum_{i=1}^{n-1} |\Delta a_i|^2 \end{cases},$$

and

$$|T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})|^2 \leq \left(\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |\alpha_k|^2 - \left|\sum_{j=1}^{n} p_j \alpha_j\right|^2\right) T_n(\mathbf{p},\ell,\mathbf{s}) \tag{2.22}$$

$$\leq \left[\sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right] a.$$

We also have:

Theorem 3. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1,...,n\}$. Then*

$$|T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})|^2 \leq \left[\sum_{k=1}^{n} p_k \sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right] \sum_{i=1}^{n-1} |\Delta \alpha_i|^2 \sum_{i=1}^{n-1} |\Delta a_i|^2. \tag{2.23}$$

If A has a continuous involution, then

$$|T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})| \leq \left[\sum_{k=1}^{n} p_k \sum_{k=1}^{n} k^2 p_k - \left(\sum_{k=1}^{n} k p_k\right)^2\right]^{1/2} \tag{2.24}$$

$$\times \left(\sum_{i=1}^{n-1} |\Delta \alpha_i|^2\right)^{1/2} \left(\sum_{i=1}^{n-1} |\Delta a_i|^2\right)^{1/2}.$$

Proof. We have by Cauchy-Bunyakowsky-Schwarz for double sums

$$|T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})|^2 = \left|\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k \alpha_k a_k - \sum_{k=1}^{n} p_k \alpha_k \sum_{k=1}^{n} p_k a_k\right|^2$$

$$= \frac{1}{4}\left|\sum_{j,k=1}^{n} p_j p_k (\alpha_k - \alpha_j)(a_k - a_j)\right|^2$$

$$\leq \frac{1}{4} \sum_{j,k=1}^{n} p_j p_k \sum_{j,k=1}^{n} p_j p_k |(\alpha_k - \alpha_j)(a_k - a_j)|^2$$

$$= \frac{1}{4} \sum_{j,k=1}^{n} p_j p_k |\alpha_k - \alpha_j|^2 |a_k - a_j|^2$$

$$= \frac{1}{2} \sum_{1 \leq j < k \leq n} p_j p_k \left|\sum_{i=j}^{k-1} \Delta \alpha_i\right|^2 \left|\sum_{i=j}^{k-1} \Delta a_i\right|^2 =: K.$$

Since

$$\left|\sum_{i=j}^{k-1}\Delta\alpha_i\right|^2\le(k-j)\sum_{i=j}^{k-1}|\Delta\alpha_i|^2$$

and

$$\left|\sum_{i=j}^{k-1}\Delta a_i\right|^2\le(k-j)\sum_{i=j}^{k-1}|\Delta a_i|^2,$$

hence

$$\begin{aligned}K&\le\frac{1}{2}\sum_{1\le j<k\le n}^{n}p_jp_k(k-j)^2\sum_{i=j}^{k-1}|\Delta\alpha_i|^2\sum_{i=j}^{k-1}|\Delta a_i|^2\\&\le\frac{1}{2}\sum_{1\le j<k\le n}^{n}p_jp_k(k-j)^2\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\sum_{i=1}^{n-1}|\Delta a_i|^2\\&=\sum_{j,k=1}^{n}p_jp_k(k-j)^2\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\sum_{i=1}^{n-1}|\Delta a_i|^2\\&=\left[\sum_{k=1}^{n}p_k\sum_{k=1}^{n}k^2p_k-\left(\sum_{k=1}^{n}kp_k\right)^2\right]\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\sum_{i=1}^{n-1}|\Delta a_i|^2,\end{aligned}$$

which proves the desired result (2.23). □

Remark 5. *If A is a C^*-algebra, then by (2.23),*

$$\|T_n(\mathbf{p},\boldsymbol{\alpha},\mathbf{a})\|^2\le\left[\sum_{k=1}^{n}p_k\sum_{k=1}^{n}k^2p_k-\left(\sum_{k=1}^{n}kp_k\right)^2\right]\times\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\left\|\sum_{i=1}^{n-1}|\Delta a_i|^2\right\|.\tag{2.25}$$

For $\mathbf{p}=(1,...,1)$ we put

$$T_n(\boldsymbol{\alpha},\mathbf{a}):=n\sum_{k=1}^{n}\alpha_ka_k-\sum_{k=1}^{n}\alpha_k\sum_{k=1}^{n}a_k.$$

By (2.23) we get

$$|T_n(\boldsymbol{\alpha},\mathbf{a})|^2\le\left[n\sum_{k=1}^{n}k^2-\left(\sum_{k=1}^{n}k\right)^2\right]\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\sum_{i=1}^{n-1}|\Delta a_i|^2.$$

Since

$$\begin{aligned}n\sum_{k=1}^{n}k^2-\left(\sum_{k=1}^{n}k\right)^2&=\frac{n^2(n+1)(2n+1)}{6}-\left[\frac{n(n+1)}{2}\right]^2\\&=\frac{(n-1)n^2(n+1)}{12},\end{aligned}$$

hence we obtain the following inequality of interest for the modulus

$$|T_n(\boldsymbol{\alpha},\mathbf{a})|^2 \le \frac{(n-1)\,n^2\,(n+1)}{12}\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\sum_{i=1}^{n-1}|\Delta a_i|^2. \tag{2.26}$$

If A is a C^*-algebra, then we also have the norm inequality

$$\|T_n(\boldsymbol{\alpha},\mathbf{a})\|^2 \le \frac{(n-1)\,n^2\,(n+1)}{12}\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\left\|\sum_{i=1}^{n-1}|\Delta a_i|^2\right\|. \tag{2.27}$$

3. Applications to the Discrete Fourier Transform

Let A be a Hermitian unital Banach $*$-algebra and $a=(a_1,\dots,a_n)$ be a sequence of vectors in A.

For a given $w\in\mathbb{R}$, define the *discrete Fourier transform* as

$$\mathcal{F}_w(\mathbf{a})(m) := \sum_{k=1}^{n}\exp(2wimk)\,a_k,\ m=1,\dots,n. \tag{3.1}$$

The following approximation result holds:

Theorem 4. *Let $a=(a_1,\dots,a_n)$ be a sequence of elements in A. Then*

$$\begin{aligned}&\left|\mathcal{F}_w(\mathbf{a})(m)-\sum_{k=1}^{n}\exp(2wimk)\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2\\ &\le \frac{(n-1)^2\,n\,(n+1)}{3}\sin^2(wm)\sum_{i=1}^{n-1}|\Delta a_i|^2\end{aligned} \tag{3.2}$$

for all $m\in\{1,\dots,n\}$ and $w\in\mathbb{R}$, $w\neq\frac{l}{m}\pi$, $l\in\mathbb{Z}$.

Proof. From the inequality (2.26), we can state that

$$\left|\sum_{k=1}^{n}\alpha_k a_k-\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2 \le \frac{(n-1)\,n\,(n+1)}{12}\sum_{i=1}^{n-1}|\Delta\alpha_i|^2\sum_{i=1}^{n-1}|\Delta a_i|^2 \tag{3.3}$$

for all $\alpha_k\in\mathbb{C},\ a_k\in A\ (k=1,\dots,n)$.

We now choose in (3.3), $\alpha_k=\exp(2wimk)$ to obtain

$$\begin{aligned}&\left|\mathcal{F}_w(\mathbf{a})(m)-\sum_{k=1}^{n}\exp(2wimk)\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2\\ &\le \frac{(n-1)\,n\,(n+1)}{12}\sum_{k=1}^{n-1}|\exp(2wim(k+1))-\exp(2wimk)|^2\sum_{i=1}^{n-1}|\Delta a_i|^2\end{aligned} \tag{3.4}$$

for all $m\in\{1,\dots,n\}$.

However,

$$\begin{aligned}
&\sum_{k=1}^{n} \exp(2wimk) \\
&= \exp(2wim) \times \left[\frac{\exp(2wimn) - 1}{\exp(2wim) - 1}\right] \\
&= \exp(2wim) \times \left[\frac{\cos(2wmn) + i\sin(2wmn) - 1}{\cos(2wm) + i\sin(2wm) - 1}\right] \\
&= \exp(2wim) \times \left[\frac{-2\sin^2(wmn) + 2i\sin(wmn)\cos(wmn)}{-2\sin^2(wm) + 2i\sin(wm)\cos(wm)}\right] \\
&= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)}\left[\frac{\sin(wmn) - i\cos(wmn)}{\sin(wm) - i\cos(wm)}\right] \\
&= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)}\left[\frac{\cos(wmn) + i\sin(wmn)}{\cos(wm) + i\sin(wm)}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp(2wim)\left[\frac{\exp(iwmn)}{\exp(iwm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp[2wim + iwmn - iwm] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp[(n+1)wmi].
\end{aligned}$$

We observe that

$$\begin{aligned}
&\exp(2wim(k+1)) - \exp(2wimk) \\
&= \cos(2wm(k+1)) + i\sin(2wm(k+1)) - \cos(2wmk) - i\sin(2wmk) \\
&= \cos(2wm(k+1)) - \cos(2wmk) + i[\sin(2wm(k+1)) - \sin(2wmk)] \\
&= -2\sin\left[\frac{2wm(k+1) + 2wmk}{2}\right]\sin\left[\frac{2wm(k+1) - 2wmk}{2}\right] \\
&+ i2\cos\left[\frac{2wm(k+1) + 2wmk}{2}\right]\sin\left[\frac{2wm(k+1) - 2wmk}{2}\right] \\
&= -2\sin((2k+1)wm)\sin(wm) + 2i\cos((2k+1)wm)\sin(wm) \\
&= 2i\sin(wm)[\cos[(2k+1)mw] + i\sin[(2k+1)mw]] \\
&= 2i\sin(wm)\exp[(2k+1)mwi],
\end{aligned}$$

and then

$$|\exp(2wim(k+1)) - \exp(2wimk)| = 2|\sin(wm)|$$

for all $k = 1, ..., n-1$.

Consequently,

$$\sum_{k=1}^{n-1} |\exp(2wim(k+1)) - \exp(2wimk)|^2 = 4\sin^2(wm)(n-1)$$

and by (3.4) we get the desired inequality (3.2). □

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