GRÜSS TYPE MODULUS INEQUALITIES FOR FORWARD DIFFERENCE IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS WITH APPLICATIONS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$.Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$. In this paper we show among others that

$$\left|\sum_{i=1}^n p_i \alpha_i x_i - \sum_{i=1}^n p_i \alpha_i \sum_{i=1}^n p_i x_i\right|^2 \leq \left[\sum_{i=1}^n i^2 p_i - \left(\sum_{i=1}^n i p_i\right)^2\right] \max_{j \in \{1,...,n-1\}} |\Delta \alpha_j|^2 \times \left[\sum_{i=1}^n p_i |a_i|^2 - \left|\sum_{i=1}^n p_i a_i\right|^2\right],$$

where $\Delta \alpha_j := \alpha_{j+1} - \alpha_j$ is the forward difference. Some applications for discrete Fourier transform are also provided.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\text{Inv}(A)$. If $a, b \in \text{Inv}(A)$, then $ab \in \text{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [15] (see also [1, Theorem 41.5])

$$\text{(SF)} \qquad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [14], Tanahashi and Uchiyama [16] proved the following fundamental properties (see also [8]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;

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(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_{\gamma} f(z)(z-a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^{\alpha} := \frac{1}{2\pi i} \int_{\gamma} z^{\alpha}(z-a)^{-1} dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha} \in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^{\alpha}) = (\sigma(a))^{\alpha} = \{z^{\alpha} : z \in \sigma(a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$ and $(a^2)^{1/2} = a$, [16, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^{\alpha} a^{\beta} = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^{\alpha})^{-1} = (a^{-1})^{\alpha} = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^{\alpha} b^{\beta} = b^{\beta} a^{\alpha}$.

Okayasu [14] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^* a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^* a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a + b| \leq |a| + |b|, \ a, \ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$. In this paper we show among others that

$$\left| \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i \right|^2 \leq \left[\sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} i p_i \right)^2 \right] \max_{j \in \{1,...,n-1\}} |\Delta \alpha_j|^2 \times \left[\sum_{i=1}^{n} p_i |a_i|^2 - \left| \sum_{i=1}^{n} p_i a_i \right|^2 \right],$$

where $\Delta \alpha_j := \alpha_{j+1} - \alpha_j$ is the forward difference. Some applications for discrete Fourier transform are also provided.

2. Some Identities

Consider the Čebyšev functional defined for $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$ and $\mathbf{x} = (x_1, ..., x_n) \in X^n$, where X is a linear space over the real or complex number field $\mathbb{K}$:

$$T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x}) := P_n \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i, \tag{2.1}$$

where $P_n := \sum_{i=1}^{n} p_i$.

Theorem 1. *Let $\mathbf{p} = (p_1, ..., p_n)$, $\alpha = (\alpha_1, ..., \alpha_n)$ be n-tuples of complex numbers and $x = (x_1, ..., x_n)$ an n-tuple of vectors in the linear space X. If we define*

$$P_i := \sum_{k=1}^{i} p_k, \ \bar{P}_i := P_n - P_i, \ i \in \{1, ..., n-1\},$$

$$A_i(\mathbf{p}) := \sum_{k=1}^{i} p_k \alpha_k, \ \bar{A}_i(\mathbf{p}) := A_n(\mathbf{p}) - A_i(\mathbf{p}), \ i \in \{1, ..., n-1\},$$

then we have the identity

$$\begin{aligned} T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x}) &= \sum_{i=1}^{n-1} (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})) \Delta x_i \\ &= P_n \sum_{i=1}^{n-1} P_i \left(\frac{A_n(\mathbf{p})}{P_n} - \frac{A_i(\mathbf{p})}{P_i} \right) \Delta x_i \\ &\quad (\text{if } P_i \neq 0, \ i \in \{1, ..., n\}) \\ &= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i} - \frac{A_i(\mathbf{p})}{P_i} \right) \Delta x_i \\ &\quad \left(\text{if } P_i, \ \bar{P}_i \neq 0, \ i \in \{1, ..., n-1\}\right); \end{aligned} \tag{2.2}$$

where $\Delta x_i := x_{i+1} - x_i$ $(i \in \{1, ..., n-1\})$ is the forward difference.

Proof. We use the following well known summation by parts formula

$$\sum_{l=p}^{q-1} d_l \Delta v_l = d_l v_l \big|_p^q - \sum_{l=p}^{q-1} v_{l+1} \Delta d_l, \tag{2.3}$$

where d_l are real or complex numbers, and v_l are vectors in a linear space, $l = p, ..., q$ $(q > p;\ p,\ q$ are natural numbers$)$.

If we choose in (2.3), $p = 1$, $q = n$, $d_i = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})$ and $v_i = x_i$ $(i \in \{1, ..., n-1\})$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1} (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}))\,\Delta x_i \\
&= [P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p})]\, x_i\big|_1^n - \sum_{i=1}^{n-1} \Delta (P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}))\, x_{i+1} \\
&= [P_n A_n(\mathbf{p}) - P_n A_n(\mathbf{p})]\, x_n - [P_1 A_n(\mathbf{p}) - P_n A_1(\mathbf{p})]\, x_1 \\
&- \sum_{i=1}^{n-1} [P_{i+1} A_n(\mathbf{p}) - P_n A_{i+1}(\mathbf{p}) - P_i A_n(\mathbf{p}) + P_n A_i(\mathbf{p})]\, x_{i+1} \\
&= P_n p_1 \alpha_1 x_1 - p_1 A_n(\mathbf{p})\, x_1 - \sum_{i=1}^{n-1} (p_{i+1} A_n(\mathbf{p}) - P_n p_{i+1} \alpha_{i+1})\, x_{i+1} \\
&= P_n p_1 \alpha_1 x_1 - p_1 A_n(\mathbf{p})\, x_1 - A_n(\mathbf{p}) \sum_{i=1}^{n-1} p_{i+1} x_{i+1} + P_n \sum_{i=1}^{n-1} p_{i+1} \alpha_{i+1} x_{i+1} \\
&= P_n \sum_{i=1}^{n} p_i \alpha_i x_i - \sum_{i=1}^{n} p_i \alpha_i \sum_{i=1}^{n} p_i x_i = T_n(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{x}),
\end{aligned}$$

which produce the first identity in (2.2).

The second and the third identities are obvious and we omit the details. □

Remark 1. *If* $\mathbf{p} = (1, ..., 1)$, *then*

$$T_n(\boldsymbol{\alpha}, \mathbf{x}) := n \sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} \alpha_i \sum_{i=1}^{n} x_i,$$

$P_i = i,\ \bar{P}_i = n - i,\ A_i := \sum_{k=1}^{i} \alpha_k,\ \bar{A}_i := A_n - A_i = \sum_{k=i+1}^{n} \alpha_k$ *and by (2.2),*

$$\begin{aligned}
T_n(\boldsymbol{\alpha}, \mathbf{x}) &= \sum_{i=1}^{n-1} (i A_n - n A_i)\,\Delta x_i = n \sum_{i=1}^{n-1} i \left(\frac{A_n}{n} - \frac{A_i}{i}\right) \Delta x_i \\
&= \sum_{i=1}^{n-1} i\,(n - i) \left(\frac{\bar{A}_i}{n - i} - \frac{A_i}{i}\right) \Delta x_i.
\end{aligned} \tag{2.4}$$

If $\alpha = \ell = (1, ..., n)$ *, then* $A_i(\mathbf{p}) := \sum_{k=1}^{i} kp_k$, $\bar{A}_i(\mathbf{p}) := \sum_{k=1}^{n} kp_k - \sum_{k=1}^{i} kp_k = \sum_{k=i+1}^{n} kp_k$, $i \in \{1, ..., n-1\}$ *and by (2.2),*

$$T_n(\mathbf{p}; \ell, \mathbf{x}) = P_n \sum_{i=1}^{n} ip_i x_i - \sum_{i=1}^{n} ip_i \sum_{i=1}^{n} p_i x_i \tag{2.5}$$
$$= \sum_{i=1}^{n-1} \left(P_i \sum_{k=1}^{n} kp_k - P_n \sum_{k=1}^{i} kp_k \right) \Delta x_i$$
$$= P_n \sum_{i=1}^{n-1} P_i \left(\frac{\sum_{k=1}^{n} kp_k}{P_n} - \frac{\sum_{k=1}^{i} kp_k}{P_i} \right) \Delta x_i$$
$$= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\sum_{k=i+1}^{n} kp_k}{\bar{P}_i} - \frac{\sum_{k=1}^{i} kp_k}{P_i} \right) \Delta x_i.$$

If $x = \ell \cdot \mathbf{1} = (1 \cdot 1, 2 \cdot 1, ..., n \cdot 1)$ *, then* $\Delta x_i = 1$ *and by (2.5) we get*

$$T_n(\mathbf{p}; \ell, \ell) = P_n \sum_{i=1}^{n} i^2 p_i - \left(\sum_{i=1}^{n} ip_i \right)^2 \tag{2.6}$$
$$= \sum_{i=1}^{n-1} \left(P_i \sum_{k=1}^{n} kp_k - P_n \sum_{k=1}^{i} kp_k \right)$$
$$= P_n \sum_{i=1}^{n-1} P_i \left(\frac{\sum_{k=1}^{n} kp_k}{P_n} - \frac{\sum_{k=1}^{i} kp_k}{P_i} \right)$$
$$= \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\sum_{k=i+1}^{n} kp_k}{\bar{P}_i} - \frac{\sum_{k=1}^{i} kp_k}{P_i} \right).$$

Before we prove the second result, we need the following lemma providing an identity that is interesting in itself as well.

Lemma 1. *Let* $\mathbf{p} = (p_1, ..., p_n)$ *and* $\alpha = (\alpha_1, ..., \alpha_n)$ *be* n*-tuples of complex numbers. Then we have the equality*

$$P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p}) = \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta\alpha_j, \tag{2.7}$$

for each $i \in \{1, ..., n-1\}$ *.*

Proof. Define, for $i \in \{1, ..., n-1\}$,

$$K(i) := \sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta\alpha_j.$$

We have

$$K(i) = \sum_{j=1}^{i} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta\alpha_j + \sum_{j=i+1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta\alpha_j \tag{2.8}$$
$$= \sum_{j=1}^{i} P_j \bar{P}_i \Delta\alpha_j + \sum_{j=i+1}^{n-1} P_i \bar{P}_j \Delta\alpha_j = \bar{P}_i \sum_{j=1}^{i} P_j \Delta\alpha_j + P_i \sum_{j=i+1}^{n-1} \bar{P}_j \Delta\alpha_j.$$

Using the summation by parts formula, we have

$$\sum_{j=1}^{i} P_j \Delta\alpha_j = P_j \alpha_j\big|_1^{i+1} - \sum_{j=1}^{i} (P_{j+1} - P_j)\,\alpha_{j+1} \tag{2.9}$$
$$= P_{i+1}\alpha_{i+1} - p_1\alpha_1 - \sum_{j=1}^{i} p_{j+1}\alpha_{j+1} = P_{i+1}\alpha_{i+1} - \sum_{j=1}^{i+1} p_j\alpha_j$$

and

$$\sum_{j=i+1}^{n-1} \bar{P}_j \Delta\alpha_j = \bar{P}_j \alpha_j\big|_{i+1}^{n} - \sum_{j=i+1}^{n-1} \left(\bar{P}_{j+1} - \bar{P}_j\right)\alpha_{j+1} \tag{2.10}$$
$$= \bar{P}_n\alpha_n - \bar{P}_{i+1}\alpha_{i+1} - \sum_{j=i+1}^{n-1} (P_n - P_{j+1} - P_n + P_j)\,\alpha_{j+1}$$
$$= -\bar{P}_{i+1}\alpha_{i+1} + \sum_{j=i+1}^{n-1} p_{j+1}\alpha_{j+1}.$$

Using (2.9) and (2.10) we have

$$K(i) = \bar{P}_i \left(P_{i+1}\alpha_{i+1} - \sum_{j=1}^{i+1} p_j\alpha_j \right) + P_i \left(\sum_{j=i+1}^{n-1} p_{j+1}\alpha_{j+1} - \bar{P}_{i+1}\alpha_{i+1} \right)$$
$$= \bar{P}_i P_{i+1}\alpha_{i+1} - P_i\bar{P}_{i+1}\alpha_{i+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j\alpha_j + P_i \sum_{j=i+1}^{n-1} p_{j+1}\alpha_{j+1}$$
$$= \left[(P_n - P_i)\,P_{i+1} - P_i\,(P_n - P_{i+1})\right]\alpha_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1}\alpha_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j\alpha_j$$

$$= P_n p_{i+1}\alpha_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1}\alpha_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j\alpha_j$$
$$= \left(P_i + \bar{P}_i\right) p_{i+1}\alpha_{i+1} + P_i \sum_{j=i+1}^{n-1} p_{j+1}\alpha_{j+1} - \bar{P}_i \sum_{j=1}^{i+1} p_j\alpha_j$$
$$= P_i \sum_{j=i+1}^{n-1} p_j\alpha_j - \bar{P}_i \sum_{j=1}^{i} p_j\alpha_j = P_i\bar{A}_i(\mathbf{p}) - \bar{P}_i A_i(\mathbf{p}) = P_i A_n(\mathbf{p}) - P_n A_i(\mathbf{p});$$

and the identity is proved. □

We are able now to state and prove the second identity for the Čebyšev functional:

Theorem 2. *With the assumptions of Theorem 1, we have the equality*

$$T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{x}) = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta\alpha_j\Delta x_i. \tag{2.11}$$

The proof is obvious by Theorem 1 and Lemma 1

Remark 2. *The identity* (2.11)*, for n-tuples of real numbers, was stated without a proof in paper* [10]*. It also may be found for the same sequences in* [11, p. 281]*, again without a proof. In the second place mentioned above there is a misprint for the index of* $\bar{P}$ *which, instead of* $\max\{i,j\}+1$*, should be* $\max\{i,j\}$*.*

Remark 3. *If* $\mathbf{p}=(1,...,1)$ *, then*

$$T_n(\boldsymbol{\alpha},\mathbf{x}) = n\sum_{i=1}^{n}\alpha_i x_i - \sum_{i=1}^{n}\alpha_i\sum_{i=1}^{n}x_i = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\min\{i,j\}(n-\max\{i,j\})\Delta\alpha_j\Delta x_i. \tag{2.12}$$

Also, we have

$$T_n(\mathbf{p};\ell,\mathbf{x}) = P_n\sum_{i=1}^{n}ip_i x_i - \sum_{i=1}^{n}ip_i\sum_{i=1}^{n}p_i x_i = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}\Delta x_i \tag{2.13}$$

and

$$T_n(\mathbf{p};\ell,\ell) = P_n\sum_{i=1}^{n}i^2 p_i - \left(\sum_{i=1}^{n}ip_i\right)^2 = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}. \tag{2.14}$$

3. Modulus Inequalities

We start to the following identities of interest:

Lemma 2. *Let* $a_k\in A$, $\alpha_k\in\mathbb{C}$ *and* $p_k\geq 0$ *for* $k\in\{1,...,n\}$ *. Then*

$$\sum_{k=1}^{n}p_k|\alpha_k|^2\sum_{k=1}^{n}p_k|a_k|^2 - \left|\sum_{j=1}^{n}p_j\alpha_j a_j\right|^2 = \frac{1}{2}\sum_{j,k=1}^{n}p_j p_k\left|\overline{\alpha_j}a_k - \overline{\alpha}_k a_j\right|^2. \tag{3.1}$$

In particular,

$$\sum_{k=1}^{n}|\alpha_k|^2\sum_{k=1}^{n}|a_k|^2 - \left|\sum_{j=1}^{n}\alpha_j a_j\right|^2 = \frac{1}{2}\sum_{j,k=1}^{n}\left|\overline{\alpha_j}a_k - \overline{\alpha}_k a_j\right|^2 \tag{3.2}$$

and

$$\sum_{k=1}^{n}p_k\sum_{k=1}^{n}p_k|a_k|^2 - \left|\sum_{j=1}^{n}p_j a_j\right|^2 = \frac{1}{2}\sum_{j,k=1}^{n}p_j p_k|a_k - a_j|^2. \tag{3.3}$$

Proof. Observe that

$$\begin{aligned}&\left|\overline{\alpha_j}a_k - \overline{\alpha}_k a_j\right|^2\\ &= \left(\overline{\alpha_j}a_k - \overline{\alpha}_k a_j\right)^*\left(\overline{\alpha_j}a_k - \overline{\alpha}_k a_j\right) = \left(\alpha_j a_k^* - \alpha_k a_j^*\right)\left(\overline{\alpha_j}a_k - \overline{\alpha}_k a_j\right)\\ &= \alpha_j a_k^*\overline{\alpha_j}a_k - \alpha_j a_k^*\overline{\alpha}_k a_j - \alpha_k a_j^*\overline{\alpha_j}a_k + \alpha_k a_j^*\overline{\alpha}_k a_j\\ &= |\alpha_j|^2|a_k|^2 - \overline{\alpha}_k a_k^*\alpha_j a_j - \overline{\alpha_j}a_j^*\alpha_k a_k + |\alpha_k|^2|a_j|^2\end{aligned}$$

for all j, $k\in\{1,...,n\}$.

This implies that

$$\begin{aligned}
&\sum_{j,k=1}^{n} p_j p_k \left|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right|^2 \\
&= \sum_{j,k=1}^{n} p_j p_k \left[\left|\alpha_j\right|^2 \left|a_k\right|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + \left|\alpha_k\right|^2 \left|a_j\right|^2\right] \\
&= \sum_{j,k=1}^{n} p_j p_k \left|\alpha_j\right|^2 \left|a_k\right|^2 - \sum_{j,k=1}^{n} p_j p_k \overline{\alpha}_k a_k^* \alpha_j a_j \\
&- \sum_{j,k=1}^{n} p_j p_k \overline{\alpha_j} a_j^* \alpha_k a_k + \sum_{j,k=1}^{n} p_j p_k \left|\alpha_k\right|^2 \left|a_j\right|^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} p_j \left|\alpha_j\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \sum_{k=1}^{n} p_k \overline{\alpha}_k a_k^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \sum_{j=1}^{n} p_j \overline{\alpha_j} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{j=1}^{n} p_j \left|a_j\right|^2 \\
&= \sum_{j=1}^{n} p_j \left|\alpha_j\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left(\sum_{k=1}^{n} p_k \alpha_k a_k\right)^* \sum_{j=1}^{n} p_j \alpha_j a_j \\
&- \left(\sum_{j=1}^{n} p_j \alpha_j a_j\right)^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{j=1}^{n} p_j \left|a_j\right|^2 \\
&= 2\left[\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k a_k\right|^2\right],
\end{aligned}$$

which is equivalent to the desired identity (3.1). □

Corollary 1. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k > 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|^2. \tag{3.4}$$

In particular,

$$\sum_{k=1}^{n} \left|\alpha_k\right|^2 \sum_{k=1}^{n} \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} \alpha_j a_j\right|^2 \tag{3.5}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} p_j a_j\right|^2. \tag{3.6}$$

The equality holds in (3.6) if and only if $a_k = a$ for some $a \in A$ and all $k \in \{1, ..., n\}$.

Remark 4. *If A has a continuous involution, then we can take the square root in (3.4) to get*

$$\left(\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2\right)^{1/2} \left(\sum_{k=1}^{n} p_k \left|a_k\right|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|. \tag{3.7}$$

Recall that a C^*-*algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\left\|a^* a\right\| = \left\|a\right\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\left\|1\right\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [12, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \left\|b\right\| \geq \left\|a\right\|.$$

Then by (3.4) we get the norm inequality

$$\sum_{k=1}^{n} p_k \left|\alpha_k\right|^2 \left\|\sum_{k=1}^{n} p_k \left|a_k\right|^2\right\| \geq \left\|\sum_{j=1}^{n} p_j \alpha_j a_j\right\|^2. \tag{3.8}$$

Theorem 3. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k \in \mathbb{R}$ for $k \in \{1, ..., n\}$. Then*

$$\begin{aligned} &\left|T_n\left(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{a}\right)\right|^2 \\ &\leq \left[\left|A_n\left(\mathbf{p}\right)\right|^2 \sum_{i=1}^{n-1} P_i^2 - 2P_n \operatorname{Re}\left(\overline{A_n\left(\mathbf{p}\right)} \sum_{i=1}^{n-1} P_i A_i\left(\mathbf{p}\right)\right)\right. \\ &\left. + P_n^2 \sum_{i=1}^{n-1} \left|A_i\left(\mathbf{p}\right)\right|^2\right] \sum_{i=1}^{n-1} \left|\Delta a_i\right|^2. \end{aligned} \tag{3.9}$$

If $P_i > 0$, for $i \in \{1, ..., n\}$, then

$$\begin{aligned} &\left|T_n\left(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{a}\right)\right|^2 \\ &\leq \left[\left|A_n\left(\mathbf{p}\right)\right|^2 \sum_{i=1}^{n-1} P_i - 2P_n \operatorname{Re}\left(A_n\left(\mathbf{p}\right) \sum_{i=1}^{n-1} \overline{A_i\left(\mathbf{p}\right)}\right) + P_n^2 \sum_{i=1}^{n-1} \frac{\left|A_i\left(\mathbf{p}\right)\right|^2}{P_i}\right] \\ &\times \sum_{i=1}^{n-1} P_i \left|\Delta a_i\right|^2. \end{aligned} \tag{3.10}$$

If P_i, $\bar{P}_i > 0$, for $i \in \{1, ..., n\}$, then

$$\begin{aligned} &\left|T_n\left(\mathbf{p}; \boldsymbol{\alpha}, \mathbf{a}\right)\right|^2 \\ &\leq \left[\sum_{i=1}^{n-1} \frac{P_i}{\bar{P}_i} \left|\bar{A}_i\left(\mathbf{p}\right)\right|^2 - 2 \operatorname{Re}\left(\sum_{i=1}^{n-1} \bar{A}_i\left(\mathbf{p}\right) \overline{A_i\left(\mathbf{p}\right)}\right) + \sum_{i=1}^{n-1} \frac{\bar{P}_i}{P_i} \left|A_i\left(\mathbf{p}\right)\right|^2\right] \\ &\times \sum_{i=1}^{n-1} P_i \bar{P}_i \left|\Delta x_i\right|^2. \end{aligned} \tag{3.11}$$

Proof. By utilising the inequality (3.5) we have by the first identity in (2.2) that

$$
\begin{aligned}
&|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})|^2 \\
&= \left|\sum_{i=1}^{n-1}(P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p}))\Delta a_i\right|^2 \\
&\leq \sum_{i=1}^{n-1}|P_iA_n(\mathbf{p})-P_nA_i(\mathbf{p})|^2\sum_{i=1}^{n-1}|\Delta a_i|^2 \\
&= \sum_{i=1}^{n-1}\left[P_i^2|A_n(\mathbf{p})|^2-2\operatorname{Re}\left(P_iA_i(\mathbf{p})\overline{A_n(\mathbf{p})}P_n\right)+P_n^2|A_i(\mathbf{p})|^2\right]\sum_{i=1}^{n-1}|\Delta a_i|^2 \\
&= \left[|A_n(\mathbf{p})|^2\sum_{i=1}^{n-1}P_i^2-2P_n\operatorname{Re}\left[\left(\sum_{i=1}^{n-1}P_iA_i(\mathbf{p})\right)\overline{A_n(\mathbf{p})}\right]+P_n^2\sum_{i=1}^{n-1}|A_i(\mathbf{p})|^2\right] \\
&\times\sum_{i=1}^{n-1}|\Delta a_i|^2,
\end{aligned}
$$

which proves (3.9).

If we use the second identity in (2.2) and the inequality (3.4), then we get

$$
\begin{aligned}
&|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})|^2 \\
&= P_n^2\left|\sum_{i=1}^{n-1}P_i\left(\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{P_i}\right)\Delta x_i\right|^2 \\
&\leq P_n^2\sum_{i=1}^{n-1}P_i\left|\frac{A_n(\mathbf{p})}{P_n}-\frac{A_i(\mathbf{p})}{P_i}\right|^2\sum_{i=1}^{n-1}P_i|\Delta x_i|^2 \\
&= P_n^2\sum_{i=1}^{n-1}P_i\left[\frac{|A_n(\mathbf{p})|^2}{P_n^2}-2\operatorname{Re}\left(\frac{A_n(\mathbf{p})}{P_n}\frac{\overline{A_i(\mathbf{p})}}{P_i}\right)+\frac{|A_i(\mathbf{p})|^2}{P_i^2}\right]\sum_{i=1}^{n-1}P_i|\Delta x_i|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left[P_n^2\frac{|A_n(\mathbf{p})|^2}{P_n^2}\sum_{i=1}^{n-1}P_i-2P_n^2\operatorname{Re}\left[\frac{A_n(\mathbf{p})}{P_n}\sum_{i=1}^{n-1}P_i\frac{\overline{A_i(\mathbf{p})}}{P_i}\right]\right. \\
&\left.+P_n^2\sum_{i=1}^{n-1}P_i\frac{|A_i(\mathbf{p})|^2}{P_i^2}\right]\times\sum_{i=1}^{n-1}P_i|\Delta x_i|^2 \\
&= \left[|A_n(\mathbf{p})|^2\sum_{i=1}^{n-1}P_i-2P_n\operatorname{Re}\left[A_n(\mathbf{p})\sum_{i=1}^{n-1}\overline{A_i(\mathbf{p})}\right]+P_n^2\sum_{i=1}^{n-1}\frac{|A_i(\mathbf{p})|^2}{P_i}\right] \\
&\times\sum_{i=1}^{n-1}P_i|\Delta x_i|^2,
\end{aligned}
$$

which proves (3.10).

By the third equality in (2.2) we get

$$
\begin{aligned}
&\left|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})\right|^2\\
&=\left|\sum_{i=1}^{n-1}P_i\bar{P}_i\left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i}-\frac{A_i(\mathbf{p})}{P_i}\right)\Delta x_i\right|^2\\
&\le\sum_{i=1}^{n-1}P_i\bar{P}_i\left(\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i}-\frac{A_i(\mathbf{p})}{P_i}\right)^2\sum_{i=1}^{n-1}P_i\bar{P}_i\left|\Delta x_i\right|^2\\
&=\sum_{i=1}^{n-1}P_i\bar{P}_i\left(\frac{\left|\bar{A}_i(\mathbf{p})\right|^2}{\left(\bar{P}_i\right)^2}-2\operatorname{Re}\left[\frac{\bar{A}_i(\mathbf{p})}{\bar{P}_i}\frac{\overline{A_i(\mathbf{p})}}{P_i}\right]+\frac{\left|A_i(\mathbf{p})\right|^2}{P_i^2}\right)\\
&\times\sum_{i=1}^{n-1}P_i\bar{P}_i\left|\Delta x_i\right|^2\\
&=\left[\sum_{i=1}^{n-1}\frac{P_i}{\bar{P}_i}\left|\bar{A}_i(\mathbf{p})\right|^2-2\operatorname{Re}\left(\sum_{i=1}^{n-1}\bar{A}_i(\mathbf{p})\,\overline{A_i(\mathbf{p})}\right)+\sum_{i=1}^{n-1}\frac{\bar{P}_i}{P_i}\left|A_i(\mathbf{p})\right|^2\right]\\
&\times\sum_{i=1}^{n-1}P_i\bar{P}_i\left|\Delta x_i\right|^2,
\end{aligned}
$$

which proves (3.11). □

Corollary 2. *Let $a_k\in A$ and $\alpha_k\in\mathbb{C}$ for $k\in\{1,...,n\}$. Then*

$$
\begin{aligned}
&\left|T_n(\boldsymbol{\alpha},\mathbf{a})\right|^2\\
&\le n\left[\frac{1}{6}(n-1)(2n-1)\left|A_n\right|^2-2\operatorname{Re}\left(\overline{A_n}\sum_{i=1}^{n-1}iA_i\right)+n\sum_{i=1}^{n-1}\left|A_i\right|^2\right]\\
&\times\sum_{i=1}^{n-1}\left|\Delta a_i\right|^2,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&\left|T_n(\boldsymbol{\alpha},\mathbf{a})\right|^2\\
&\le n\left[\frac{(n-1)}{2}\left|A_n\right|^2-2\operatorname{Re}\left(A_n\sum_{i=1}^{n-1}\overline{A_i}\right)+n\sum_{i=1}^{n-1}\frac{\left|A_i\right|^2}{i}\right]\\
&\times\sum_{i=1}^{n-1}i\left|\Delta a_i\right|^2.
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
&\left|T_n(\boldsymbol{\alpha},\mathbf{a})\right|^2\\
&\le\left[\sum_{i=1}^{n-1}\frac{i}{n-i}\left|\bar{A}_i\right|^2-2\operatorname{Re}\left(\sum_{i=1}^{n-1}\bar{A}_i\overline{A_i}\right)+\sum_{i=1}^{n-1}\frac{n-i}{i}\left|A_i\right|^2\right]\\
&\times\sum_{i=1}^{n-1}i(n-i)\left|\Delta x_i\right|^2.
\end{aligned}
\tag{3.14}
$$

Theorem 4. *Let* $a_k \in A$, $\alpha_k \in \mathbb{C}$ *and* $p_k \in \mathbb{R}$ *for* $k \in \{1, ..., n\}$. *Then*

$$|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})|^2 \leq \max_{j\in\{1,\dots,n-1\}} |\Delta\alpha_j|^2 T_n(\mathbf{p};\ell,\ell) T_n(\mathbf{p};\mathbf{a}^*,\mathbf{a}), \tag{3.15}$$

where

$$T_n(\mathbf{p};\mathbf{a}^*,\mathbf{a}) := P_n \sum_{i=1}^n p_i |a_i|^2 - \left|\sum_{i=1}^n p_i a_i\right|^2.$$

Also, if $|\Delta a_i|^2 \leq b$ *for* $i \in \{1, ..., n-1\}$, *then*

$$|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})|^2 \leq T_n(\mathbf{p};\ell,\ell) T_n(\mathbf{p};\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha}) b, \tag{3.16}$$

where

$$T_n(\mathbf{p};\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha}) = P_n \sum_{i=1}^n p_i |\alpha_i|^2 - \left|\sum_{i=1}^n p_i \alpha_i\right|^2.$$

Proof. By the CBS inequality for double sums, we have

$$\begin{aligned}
&|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})|^2 \\
&= \left|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta\alpha_j \Delta a_i\right|^2 \\
&\leq \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} |\Delta\alpha_j \Delta a_i|^2 \\
&= T_n(\mathbf{p};\ell,\ell) \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} |\Delta\alpha_j|^2 |\Delta a_i|^2 \\
&\leq \max_{j\in\{1,\dots,n-1\}} |\Delta\alpha_j|^2 T_n(\mathbf{p};\ell,\ell) \\
&\times \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} (a_{i+1}-a_i)^* (a_{i+1}-a_i) \\
&= \max_{j\in\{1,\dots,n-1\}} |\Delta\alpha_j|^2 T_n(\mathbf{p};\ell,\ell) \\
&\times \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \left(a_{i+1}^* - a_i^*\right)(a_{i+1}-a_i) \\
&= \max_{j\in\{1,\dots,n-1\}} |\Delta\alpha_j|^2 T_n(\mathbf{p};\ell,\ell) \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta a_i^* \Delta a_i.
\end{aligned} \tag{3.17}$$

Now, by a similar argument as in the previous section and by replacing α_i with a_i^* and x_i with a_i we derive the following identity of interest

$$\begin{aligned}
&\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}} \bar{P}_{\max\{i,j\}} \Delta a_i^* \Delta a_i \\
&= P_n \sum_{i=1}^n p_i a_i^* a_i - \sum_{i=1}^n p_i a_i^* \sum_{i=1}^n p_i a_i = P_n \sum_{i=1}^n p_i |a_i|^2 - \left|\sum_{i=1}^n p_i a_i\right|^2.
\end{aligned} \tag{3.18}$$

By utilising (3.17) we derive (3.15).

From (3.17) we get

$$\begin{aligned}
|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})|^2 &\leq T_n(\mathbf{p};\ell,\ell)\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}|\Delta\alpha_j|^2|\Delta a_i|^2 \\
&\leq T_n(\mathbf{p};\ell,\ell)\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}|\Delta\alpha_j|^2 b \\
&= T_n(\mathbf{p};\ell,\ell)\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} P_{\min\{i,j\}}\bar{P}_{\max\{i,j\}}(\Delta\overline{\alpha_j})(\Delta\alpha_j)\, b \\
&= T_n(\mathbf{p};\ell,\ell)\,T_n(\mathbf{p};\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha})\, b,
\end{aligned}$$

which proves (3.16). □

Remark 5. *If A is a C^*-algebra, then by (3.15) we get*

$$\|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})\|^2 \leq \max_{j\in\{1,\dots,n-1\}}|\Delta\alpha_j|^2\, T_n(\mathbf{p};\ell,\ell)\,\|T_n(\mathbf{p};\mathbf{a}^*,\mathbf{a})\|, \tag{3.19}$$

while from (3.16)

$$\|T_n(\mathbf{p};\boldsymbol{\alpha},\mathbf{a})\|^2 \leq T_n(\mathbf{p};\ell,\ell)\,T_n(\mathbf{p};\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha})\,\|b\|. \tag{3.20}$$

For $\mathbf{p}=(1,\dots,1)$ we have

$$\begin{aligned}
T_n(\mathbf{p};\ell,\ell) &= n\sum_{i=1}^{n} i^2 - \left(\sum_{i=1}^{n} i\right)^2 = \frac{n^2(n+1)(2n+1)}{6} - \left[\frac{n(n+1)}{2}\right]^2 \\
&= \frac{(n-1)\,n^2(n+1)}{12},
\end{aligned}$$

$$T_n(\mathbf{a}^*,\mathbf{a}) = n\sum_{i=1}^{n}|a_i|^2 - \left|\sum_{i=1}^{n} a_i\right|^2$$

and

$$T_n(\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha}) = n\sum_{i=1}^{n}|\alpha_i|^2 - \left|\sum_{i=1}^{n}\alpha_i\right|^2.$$

We can state the following unweighted inequalities as well:

Corollary 3. *Let $a_k\in A$, $\alpha_k\in\mathbb{C}$ for $k\in\{1,\dots,n\}$. Then*

$$|T_n(\boldsymbol{\alpha},\mathbf{a})|^2 \leq \frac{(n-1)\,n^2(n+1)}{12}\max_{j\in\{1,\dots,n-1\}}|\Delta\alpha_j|^2\, T_n(\mathbf{a}^*,\mathbf{a}). \tag{3.21}$$

Also, if $|\Delta a_i|^2\leq b$ for $i\in\{1,\dots,n-1\}$ and some $b>0$, then

$$|T_n(\boldsymbol{\alpha},\mathbf{a})|^2 \leq \frac{(n-1)\,n^2(n+1)}{12}T_n(\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha})\, b. \tag{3.22}$$

f A is a C^*-algebra, then by (3.21) we get

$$\|T_n(\boldsymbol{\alpha},\mathbf{a})\|^2 \leq \frac{(n-1)\,n^2(n+1)}{12}\max_{j\in\{1,\dots,n-1\}}|\Delta\alpha_j|^2\,\|T_n(\mathbf{a}^*,\mathbf{a})\|$$

while from (3.22) we get

$$\|T_n(\boldsymbol{\alpha},\mathbf{a})\|^2 \leq \frac{(n-1)\,n^2(n+1)}{12}T_n(\overline{\boldsymbol{\alpha}},\boldsymbol{\alpha})\,\|b\|.$$

4. Applications to the Discrete Fourier Transform

Let A be a Hermitian unital Banach $*$-algebra and $a= (a_1, \dots, a_n)$ be a sequence of vectors in A.

For a given $w \in \mathbb{R}$, define the *discrete Fourier transform* as

$$\mathcal{F}_w(\mathbf{a})(m) := \sum_{k=1}^{n} \exp(2wimk)\, a_k,\ m = 1, \dots, n. \tag{4.1}$$

The following approximation result holds:

Theorem 5. *Let $a= (a_1, \dots, a_n)$ be a sequence of elements in A. Then*

$$\begin{aligned} &\left| \mathcal{F}_w(\mathbf{a})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \frac{1}{n}\sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{(n-1)n^3(n+1)}{3} \sin^2(wm) \left(\frac{1}{n}\sum_{i=1}^{n} |a_i|^2 - \left| \frac{1}{n}\sum_{i=1}^{n} a_i \right|^2 \right) \end{aligned} \tag{4.2}$$

for all $m \in \{1, \dots, n\}$ and $w \in \mathbb{R},\ w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z}$.

Proof. From the inequality (3.21), we can state that

$$\begin{aligned} &\left| \frac{1}{n}\sum_{k=1}^{n} \alpha_k a_k - \frac{1}{n}\sum_{k=1}^{n} \alpha_k \frac{1}{n}\sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{(n-1)n^2(n+1)}{12} \\ &\times \max_{j\in\{1,\dots,n-1\}} |\Delta\alpha_j|^2 \left(\frac{1}{n}\sum_{i=1}^{n} |a_i|^2 - \left| \frac{1}{n}\sum_{i=1}^{n} a_i \right|^2 \right) \end{aligned} \tag{4.3}$$

for all $\alpha_k \in \mathbb{C},\ a_k \in A\ (k = 1, \dots, n)$.

We now choose in (4.3), $\alpha_k = \exp(2wimk)$ to obtain

$$\begin{aligned} &\left| \mathcal{F}_w(\mathbf{a})(m) - \sum_{k=1}^{n} \exp(2wimk) \frac{1}{n}\sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{(n-1)n^3(n+1)}{12} \max_{k\in\{1,\dots,n-1\}} |\exp(2wim(k+1)) - \exp(2wimk)|^2 \\ &\times \left(\frac{1}{n}\sum_{i=1}^{n} |a_i|^2 - \left| \frac{1}{n}\sum_{i=1}^{n} a_i \right|^2 \right) \end{aligned} \tag{4.4}$$

for all $m \in \{1, \dots, n\}$.

As a simple calculation reveals that

$$\begin{aligned}
&\sum_{k=1}^{n} \exp(2wimk) \\
&= \exp(2wim) \times \left[\frac{\exp(2wimn) - 1}{\exp(2wim) - 1}\right] \\
&= \exp(2wim) \times \left[\frac{\cos(2wmn) + i\sin(2wmn) - 1}{\cos(2wm) + i\sin(2wm) - 1}\right] \\
&= \exp(2wim) \times \frac{\sin(wmn)}{\sin(wm)} \left[\frac{\cos(wmn) + i\sin(wmn)}{\cos(wm) + i\sin(wm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp(2wim) \left[\frac{\exp(iwmn)}{\exp(iwm)}\right] \\
&= \frac{\sin(wmn)}{\sin(wm)} \times \exp[w(n+1)im].
\end{aligned}$$

Also, we observe that

$$\begin{aligned}
&\exp(2wim(k+1)) - \exp(2wimk) \\
&= \cos(2wm(k+1)) + i\sin(2wm(k+1)) - \cos(2wmk) - i\sin(2wmk) \\
&= \cos(2wm(k+1)) - \cos(2wmk) + i[\sin(2wm(k+1)) - \sin(2wmk)] \\
&= -2\sin\left[\frac{2wm(k+1) + 2wmk}{2}\right] \sin\left[\frac{2wm(k+1) - 2wmk}{2}\right] \\
&+ i2\cos\left[\frac{2wm(k+1) + 2wmk}{2}\right] \sin\left[\frac{2wm(k+1) - 2wmk}{2}\right] \\
&= -2\sin((2k+1)wm)\sin(wm) + 2i\cos((2k+1)wm)\sin(wm) \\
&= 2i\sin(wm)[\cos[(2k+1)mw] + i\sin[(2k+1)mw]] \\
&= 2i\sin(wm)\exp[(2k+1)mwi],
\end{aligned}$$

and then

$$|\exp(2wim(k+1)) - \exp(2wimk)| = 2|\sin(wm)|$$

for all $k = 1, ..., n-1$. Then by (4.4) we obtain (4.2). □

Also we have:

Theorem 6. *Let* $a = (a_1, \ldots, a_n)$ *be a sequence of elements in* A *and such that* $|\Delta a_i|^2 \le b$ *for* $i \in \{1, ..., n-1\}$ *and some* $b \in A$, $b > 0$. *Then*

$$\left|\mathcal{F}_w(\mathbf{a})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \frac{1}{n}\sum_{k=1}^{n} a_k\right|^2 \le \frac{(n-1)n^3(n+1)}{12}\left[1 - \frac{\sin^2(wmn)}{n^2\sin^2(wm)}\right] b \tag{4.5}$$

for all $m \in \{1, \ldots, n\}$ *and* $w \in \mathbb{R}$, $w \ne \frac{l}{m}\pi$, $l \in \mathbb{Z}$.

Proof. From (3.22) we have

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k a_k-\frac{1}{n}\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2 \le \frac{(n-1)\,n^2\,(n+1)}{12}\left(\frac{1}{n}\sum_{i=1}^{n}|\alpha_i|^2-\left|\frac{1}{n}\sum_{i=1}^{n}\alpha_i\right|^2\right)b \tag{4.6}$$

for all $\alpha_k\in\mathbb{C},\ a_k\in A\ \ (k=1,\ldots,n)$ with $|\Delta a_i|^2\le b$ for $i\in\{1,...,n-1\}$. By (4.6) we get

$$\left|\mathcal{F}_w(\mathbf{a})(m)-\sum_{k=1}^{n}\exp(2wimk)\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2 \le \frac{(n-1)\,n^3\,(n+1)}{12} \times\left(\frac{1}{n}\sum_{k=1}^{n}|\exp(2wimk)|^2-\left|\frac{1}{n}\sum_{k=1}^{n}\exp(2wimk)\right|^2\right)b. \tag{4.7}$$

Since $\sum_{k=1}^{n}|\exp(2wimk)|^2=n$ and

$$\left|\sum_{k=1}^{n}\exp(2wimk)\right|^2=\frac{\sin^2(wmn)}{\sin^2(wm)},\ \text{ for } w\neq\frac{l}{m}\pi,\ l\in\mathbb{Z},$$

hence by (4.7) we derive (4.5). □

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