REVERSES OF THE DISCRETE TRIANGLE INEQUALITY FOR MODULUS IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^* a)^{1/2} \geq 0$. In this paper we show among others that, if u is an unitary element in A, namely $u^* u = 1$ and $a_k \in A$ such that

$$\left| a_k u - \frac{\gamma + \Gamma}{2} u \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \text{ for } k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} u^* - a_k^* \right) (a_k - \gamma u) \right] \geq 0 \text{ for } k \in \{1, ..., n\},$$

then

$$\sum_{k=1}^{n} p_k |a_k u| \leq \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma \overline{\gamma})}} u^* \left(\sum_{k=1}^{n} p_k a_k \right) u \right] \leq \frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma \overline{\gamma})}} \left| u^* \left(\sum_{k=1}^{n} p_k a_k \right) u \right|.$$

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\operatorname{Inv}(A)$. If $a, b \in \operatorname{Inv}(A)$, then $ab \in \operatorname{Inv}(A)$ and $(ab)^{-1} = b^{-1} a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [13] (see also [1, Theorem 41.5])

$$a^* a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [12], Tanahashi and Uchiyama [14] proved the following fundamental properties (see also [8]):

- (i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
- (ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
- (iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
- (iv) If $a > 0$, then $a^{-1} > 0$;

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(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z) (z - a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [14, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a, b \in A$, $\alpha, \beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [12] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^* a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^* a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a + b| \leq |a| + |b|, \ a, \ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

In this paper we show among others that, if u is an unitary element in A, namely $u^*u = 1$ and $a_k \in A$ such that

$$\left|a_k u - \frac{\gamma+\Gamma}{2} u\right|^2 \leq \frac{1}{4}\left|\Gamma-\gamma\right|^2 \text{ for } k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} u^* - a_k^*\right)(a_k - \gamma u)\right] \geq 0 \text{ for } k \in \{1, ..., n\},$$

then

$$\begin{aligned}\sum_{k=1}^n p_k \left|a_k u\right| &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}} u^* \left(\sum_{k=1}^n p_k a_k\right) u\right] \\ &\leq \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left|u^*\left(\sum_{k=1}^n p_k a_k\right)u\right|.\end{aligned}$$

2. Main Results

For $a \in A$ we define the selfadjoint element

$$\operatorname{Re}(a) := \frac{1}{2}\left(a^* + a\right) = \operatorname{Re}\left(a^*\right)$$

We have the following identity of interest:

Lemma 1. *For any a, d, $c \in A$, we have*

$$\begin{aligned}\left|a - \frac{d+c}{2}\right|^2 - \frac{1}{4}\left|d-c\right|^2 &= \operatorname{Re}\left[\left(a^*-d^*\right)(a-c)\right] \\ &= \operatorname{Re}\left[\left(a^*-c^*\right)(a-d)\right].\end{aligned} \tag{2.1}$$

Proof. We have

$$\begin{aligned}&\left|a - \frac{d+c}{2}\right|^2 - \frac{1}{4}\left|d-c\right|^2 \\ &= \left|a\right|^2 - \frac{d^*+c^*}{2} a - a^* \frac{d+c}{2} + \frac{1}{4}\left(\left|d\right|^2 + d^*c + c^*d + \left|c\right|^2\right) \\ &- \frac{1}{4}\left(\left|d\right|^2 - d^*c - c^*d + \left|c\right|^2\right) \\ &= \left|a\right|^2 - \frac{d^*+c^*}{2} a - a^* \frac{d+c}{2} + \frac{1}{2}\left(d^*c + c^*d\right)\end{aligned}$$

and

$$\begin{aligned}&\operatorname{Re}\left[\left(a^*-d^*\right)(a-c)\right] \\ &= \operatorname{Re}\left[\left|a\right|^2 - d^*a - a^*c + d^*c\right] \\ &= \left|a\right|^2 - \operatorname{Re}\left(d^*a\right) - \operatorname{Re}\left(a^*c\right) + \operatorname{Re}\left(d^*c\right) \\ &= \left|a\right|^2 - \frac{1}{2}\left(d^*a + a^*d\right) - \frac{1}{2}\left(a^*c + c^*a\right) + \frac{1}{2}\left(d^*c + c^*d\right) \\ &= \left|a\right|^2 - \frac{1}{2}\left(d^* + c^*\right) a - \frac{1}{2} a^*\left(d+c\right) + \frac{1}{2}\left(d^*c + c^*d\right),\end{aligned}$$

which proves the desired identity (2.1). □

Corollary 1. *Let a, d, $c \in A$. The following statements are equivalent*

$$\left| a - \frac{d+c}{2} \right|^2 \leq \frac{1}{4} \left| d - c \right|^2 \tag{2.2}$$

and

$$\operatorname{Re}\left[\left(d^* - a^* \right) \left(a - c \right) \right] = \operatorname{Re}\left[\left(a^* - c^* \right) \left(d - a \right) \right] \geq 0. \tag{2.3}$$

We have the following reverse of the triangle inequality:

Theorem 1. *Let u be an unitary element in A, namely $u^* u = 1$ and $a_k \in A$ such that*

$$\left| a_k - \frac{\gamma + \Gamma}{2} u \right|^2 \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2 \text{ for } k \in \{1, ..., n\} \tag{2.4}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} u^* - a_k^* \right) \left(a_k - \gamma u \right) \right] \geq 0 \text{ for } k \in \{1, ..., n\} \tag{2.5}$$

for some complex constants γ, Γ with $\operatorname{Re}\left(\Gamma \overline{\gamma} \right) > 0$. Then for $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$,

$$\begin{aligned} \sum_{k=1}^n p_k \left| a_k u \right| &\leq \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma \overline{\gamma} \right)}} u^* \left(\sum_{k=1}^n p_k a_k \right) u \right] \\ &\leq \frac{\left| \gamma + \Gamma \right|}{2\sqrt{\operatorname{Re}\left(\Gamma \overline{\gamma} \right)}} \left| u^* \left(\sum_{k=1}^n p_k a_k \right) u \right|. \end{aligned} \tag{2.6}$$

Proof. The equivalence of the statements (2.4) and (2.5) follows by Corollary 1 for $a = a_k$, $d = \Gamma u$ and $c = \gamma u$ and taking into account that $\left| u \right|^2 = 1$.

By the properties of operator modulus, we have

$$\left| a_k u \right|^2 - 2 \operatorname{Re}\left[\left(\frac{\gamma + \Gamma}{2} u \right)^* a_k u \right] + \left| \frac{\gamma + \Gamma}{2} u \right|^2 \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2,$$

namely

$$\left| a_k u \right|^2 - 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right] + \left| \frac{\gamma + \Gamma}{2} \right|^2 \leq \frac{1}{4} \left| \Gamma - \gamma \right|^2,$$

or

$$\left| a_k u \right|^2 + \left| \frac{\gamma + \Gamma}{2} \right|^2 - \frac{1}{4} \left| \Gamma - \gamma \right|^2 \leq 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right], \tag{2.7}$$

for $k \in \{1, ..., n\}$.

Observe that

$$\begin{aligned} \frac{1}{4} \left| \Gamma + \gamma \right|^2 - \frac{1}{4} \left| \Gamma - \gamma \right|^2 &= \frac{1}{4} \left(\left| \Gamma \right|^2 + 2 \operatorname{Re}\left(\Gamma \overline{\gamma} \right) + \left| \gamma \right|^2 \right) \\ &\quad - \frac{1}{4} \left(\left| \Gamma \right|^2 - 2 \operatorname{Re}\left(\Gamma \overline{\gamma} \right) + \left| \gamma \right|^2 \right) = \operatorname{Re}\left(\Gamma \overline{\gamma} \right) > 0, \end{aligned}$$

then by (2.7) we get

$$\left| a_k u \right|^2 + \operatorname{Re}\left(\Gamma \overline{\gamma} \right) \leq 2 \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right], \tag{2.8}$$

for $k \in \{1, ..., n\}$.

If we multiply (2.8) by $p_k \geq 0$ and sum, then we get

$$\sum_{k=1}^{n} p_k |a_k u|^2 + \operatorname{Re}(\Gamma\overline{\gamma}) \leq 2\sum_{k=1}^{n} p_k \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2} u^* a_k u\right] = 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right]. \tag{2.9}$$

Using the elementary operator inequality

$$2\alpha b \leq b^2 + \alpha^2,$$

where $b \geq 0$ in the operator order and real number $\alpha \geq 0$, then we also have

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\,|a_k u| \leq |a_k u|^2 + \operatorname{Re}(\Gamma\overline{\gamma}), \tag{2.10}$$

for $k \in \{1, ..., n\}$.

By multiplying this inequality with $p_k \geq 0$ and sum, then we get

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\sum_{k=1}^{n} p_k |a_k u| \leq \sum_{k=1}^{n} p_k |a_k u|^2 + \operatorname{Re}(\Gamma\overline{\gamma}), \tag{2.11}$$

and by (2.9) and (2.11) we derive

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\sum_{k=1}^{n} p_k |a_k u| \leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right],$$

which is equivalent to the first inequality in (2.6).

For an element a we consider the selfadjoint elements

$$\operatorname{Re}(a) := \frac{a^*+a}{2}, \quad \operatorname{Im}(a) := \frac{a-a^*}{2i}.$$

Then $a = \operatorname{Re}(a) + i\operatorname{Im}(a)$, $|a|^2 = (\operatorname{Re}(a))^2 + (\operatorname{Im}(a))^2$. We have $|a|^2 \geq (\operatorname{Re}(a))^2$ which implies, by taking the square root, that $|a| \geq |\operatorname{Re}(a)|$.

Therefore

$$\begin{aligned}
0 &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right] \\
&\leq \left|\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right| \\
&= \left|\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\right| \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right| \\
&= \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \left|u^* \left(\sum_{k=1}^{n} p_k a_k\right) u\right|,
\end{aligned}$$

and the last part of (2.6) is thus proved. □

Remark 1. *Observe that for $z = \alpha + i\beta$ and $a \in A$, we have*

$$\begin{aligned}
\operatorname{Re}(\overline{z}a) &= \operatorname{Re}[(\alpha - i\beta)(\operatorname{Re} a + i\operatorname{Im} a)] \\
&= \operatorname{Re}[\alpha\operatorname{Re} a + \beta\operatorname{Im} a - i(\beta\operatorname{Re} a - \alpha\operatorname{Im} a)] \\
&= \alpha\operatorname{Re} a + \beta\operatorname{Im} a = \operatorname{Re} z\operatorname{Re} a + \operatorname{Im} z\operatorname{Im} a
\end{aligned}$$

and then

$$\begin{aligned}
&\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}u^{*}\left(\sum_{k=1}^{n}p_{k}a_{k}\right)u\right] \\
&=\frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\left[u^{*}\left(\sum_{k=1}^{n}p_{k}a_{k}\right)u\right] \\
&+\frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\left[u^{*}\left(\sum_{k=1}^{n}p_{k}a_{k}\right)u\right] \\
&=\frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Re}\left(\gamma+\Gamma\right)u^{*}\left(\sum_{k=1}^{n}p_{k}\operatorname{Re}\left(a_{k}\right)\right)u \\
&+\frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Im}\left(\gamma+\Gamma\right)u^{*}\left(\sum_{k=1}^{n}p_{k}\operatorname{Im}\left(a_{k}\right)\right)u.
\end{aligned}$$

Therefore by (2.6) we have the unpacked inequality

$$\begin{aligned}
\sum_{k=1}^{n}p_{k}\left|a_{k}u\right| &\leq \frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left[\operatorname{Re}\left(\gamma+\Gamma\right)u^{*}\left(\sum_{k=1}^{n}p_{k}\operatorname{Re}\left(a_{k}\right)\right)u\right. \\
&\quad\left.+\operatorname{Im}\left(\gamma+\Gamma\right)u^{*}\left(\sum_{k=1}^{n}p_{k}\operatorname{Im}\left(a_{k}\right)\right)u\right] \\
&\leq \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left|u^{*}\left(\sum_{k=1}^{n}p_{k}a_{k}\right)u\right|.
\end{aligned} \tag{2.12}$$

Corollary 2. *Let u be an unitary element in A and $a_k \in A$ and such that*

$$\left|a_{k}u-\frac{m+M}{2}u\right|^{2}\leq\frac{1}{4}\left(M-m\right)^{2}\text{ for }k\in\{1,...,n\} \tag{2.13}$$

or, equivalently

$$\operatorname{Re}\left[\left(Mu^{*}-a_{k}^{*}\right)\left(a_{k}-mu\right)\right]\geq 0\text{ for }k\in\{1,...,n\} \tag{2.14}$$

for some real numbers $M > m > 0$. Then

$$\begin{aligned}
\sum_{k=1}^{n}p_{k}\left|a_{k}u\right| &\leq \frac{m+M}{2\sqrt{mM}}\operatorname{Re}\left[u^{*}\left(\sum_{k=1}^{n}p_{k}a_{k}\right)u\right] \\
&\leq \frac{m+M}{2\sqrt{Mm}}\left|u^{*}\left(\sum_{k=1}^{n}p_{k}a_{k}\right)u\right|.
\end{aligned} \tag{2.15}$$

Recall that a C^*-*algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\left\|a^{*}a\right\|=\left\|a\right\|^{2}\text{ for any }a\in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [10, 2.2.5 Theorem])

$$b\geq a\geq 0\text{ implies that }\left\|b\right\|\geq\left\|a\right\|.$$

With the assumptions of Theorem 1 we have the norm inequality

$$\left\| \sum_{k=1}^{n} p_k |a_k u| \right\| \leq \frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \left\| u^* \left(\sum_{k=1}^{n} p_k a_k \right) u \right\|. \tag{2.16}$$

If $a_k \in A$ such that

$$\left| a_k - \frac{\gamma + \Gamma}{2} \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \text{ for } k \in \{1, ..., n\}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} u - a_k^* \right) (a_k - \gamma) \right] \geq 0 \text{ for } k \in \{1, ..., n\}$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$. Then for $p_k \geq 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^{n} p_k = 1$,

$$\left\| \sum_{k=1}^{n} p_k |a_k| \right\| \leq \frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}} \left\| \sum_{k=1}^{n} p_k a_k \right\|. \tag{2.17}$$

We have the following result as well:

Theorem 2. *Let u be an unitary element in A and $a_k \in A$ such that either (2.4) or (2.5) is satisfied for some complex constants γ, Γ with $\gamma + \Gamma \neq 0$. Then for $p_k \geq 0$ with $\sum_{k=1}^{n} p_k = 1$,*

$$\sum_{k=1}^{n} p_k |a_k u| \leq \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{|\gamma + \Gamma|} u^* \left(\sum_{k=1}^{n} p_k a_k \right) u \right] + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\gamma + \Gamma|}. \tag{2.18}$$

Proof. By the properties of operator modulus, we have

$$|a_k u|^2 - 2\operatorname{Re}\left[\left(\frac{\gamma + \Gamma}{2} u \right)^* a_k u \right] + \left| \frac{\gamma + \Gamma}{2} u \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2,$$

namely

$$|a_k u|^2 - 2\operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right] + \left| \frac{\gamma + \Gamma}{2} \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2,$$

or

$$|a_k u|^2 + \left| \frac{\gamma + \Gamma}{2} \right|^2 \leq 2\operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right] + \frac{1}{4} |\Gamma - \gamma|^2, \tag{2.19}$$

for $k \in \{1, ..., n\}$.

If we multiply (2.19) by $p_k \geq 0$ and sum, then we get

$$\begin{aligned} \sum_{k=1}^{n} p_k |a_k u|^2 + \left| \frac{\gamma + \Gamma}{2} \right|^2 & \\ \leq 2 \sum_{k=1}^{n} p_k \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* a_k u \right] + \frac{1}{4} |\Gamma - \gamma|^2 & \\ = 2\operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2} u^* \left(\sum_{k=1}^{n} p_k a_k \right) u \right] + \frac{1}{4} |\Gamma - \gamma|^2. & \end{aligned} \tag{2.20}$$

Using the elementary operator inequality

$$2\alpha a \leq a^2 + \alpha^2, \tag{2.21}$$

where $a \geq 0$ in the operator order and $\alpha \geq 0$, then we also have

$$2\left|\frac{\gamma+\Gamma}{2}\right|\left|a_k u\right| \leq \left|a_k u\right|^2+\left|\frac{\gamma+\Gamma}{2}\right|^2, \tag{2.22}$$

which by summation produces

$$2\left|\frac{\gamma+\Gamma}{2}\right|\sum_{k=1}^{n} p_k\left|a_k u\right| \leq \sum_{k=1}^{n} p_k\left|a_k u\right|^2+\left|\frac{\gamma+\Gamma}{2}\right|^2. \tag{2.23}$$

By utilising (2.20) and (2.23) we deduce

$$2\left|\frac{\gamma+\Gamma}{2}\right|\sum_{k=1}^{n} p_k\left|a_k u\right| \leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right]+\frac{1}{4}\left|\Gamma-\gamma\right|^2.$$

Since $\gamma+\Gamma \neq 0$, hence by dividing with $|\gamma+\Gamma| \neq 0$, we get

$$\sum_{k=1}^{n} p_k\left|a_k u\right| \leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right]+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{|\gamma+\Gamma|},$$

which proves the first inequality in (2.18). □

Corollary 3. *Let u be an unitary element in A and $a_k \in A$ such that either (2.13) or (2.14) is satisfied for some real numbers $M > m > 0$. Then*

$$\sum_{k=1}^{n} p_k\left|a_k u\right| \leq \operatorname{Re}\left[u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right]+\frac{1}{4}\frac{(M-m)^2}{m+M}. \tag{2.24}$$

Remark 2. *If a is selfadjoint, then $|a|-a$ is selfadjoint and $\sigma(|a|-a) \subseteq [0,\infty)$ which gives that $|a| \geq a$. Then we have*

$$\begin{aligned}\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right] &\leq \left|\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right]\right| \\ &\leq \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right|.\end{aligned}$$

Therefore, under the assumptions of Theorem 1 we can also state the inequality

$$\sum_{k=1}^{n} p_k\left|a_k u\right| \leq \left|u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right|+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{|\gamma+\Gamma|}. \tag{2.25}$$

Assume that A is a C^*-algebra. With the assumption of Theorem 2 we have the norm inequality

$$\left\|\sum_{k=1}^{n} p_k\left|a_k u\right|\right\| \leq \left\|u^*\left(\sum_{k=1}^{n} p_k a_k\right)u\right\|+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{|\gamma+\Gamma|}. \tag{2.26}$$

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