LASOTA-OPIAL TYPE MODULUS INEQUALITIES FOR FORWARD DIFFERENCE IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a\in A$ by $|a|:=(a^*a)^{1/2}\geq 0$. Suppose that A has a continuous involution. In this paper we show among others that, if $\{\gamma_i\}_{i=0}^N\subset\mathbb{C}$ and $\{a_i\}_{i=0}^N\subset A$ are sequences with $a_0=0$ and $a_N=0$, then for $n\in\{2,...,N-1\}$,

$$\sum_{i=1}^{N-1}|\Delta\gamma_i|\,|a_i|\leq\left(\sum_{i=0}^{N-1}p_i(n)\,|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i(n)\,|\Delta a_i|^2\right)^{1/2}$$
$$\leq\frac{1}{2}\sum_{i=0}^{N-1}\left(p_i(n)\,|\Delta\gamma_i|^2+q_i(n)\,|\Delta a_i|^2\right),$$

where

$$p_i(n):=\begin{cases} i, \text{ if } 0\leq i\leq n-1,\\ \\ N-i, \text{ if } n\leq i\leq N-1\end{cases}$$

and

$$q_i(n):=\begin{cases} n-i-1, \text{ if } 0\leq i\leq n-1,\\ \\ i+1-n, \text{ if } n\leq i\leq N-1,\end{cases}$$

where $\Delta\gamma_j:=\gamma_{j+1}-\gamma_j$ is the forward difference.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b|u(t)\,u'(t)|\,dt\leq\frac{1}{4}(b-a)\int_a^b|u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) \ \textit{if } a\leq t\leq\frac{a+b}{2},\\ \\ c(b-t) \ \textit{if } \frac{a+b}{2}<t\leq b\end{cases}$$

where c is an arbitrary constant;

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(ii) *If* $u(a)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ *for some constant* c;

(iii) *If* $\int_a^b u(t)\,dt=0$, *then the inequality (1.1) holds with equality if and only if*

$$u(t)=c\left(t-\frac{a+b}{2}\right)$$

for any constant c.

The inequality (1.1) was obtained by Olech in [20] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [21].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

The inequality (1.1) in the case (iii), namely in the case that u satisfies the condition $\int_a^b u(t)\,dt=0$ was obtained by Brown and Plum in [5].

As mentioned in [5] the inequality (1.1) also holds if $u(a)+u(b)=0$.

For a sequence $\{x_i\}_{i=0}^n$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1}-x_i$, $i=0,...,n-1$. The summation by parts formula also holds

$$\sum_{k=m}^{n} a_k\Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1}\Delta a_k. \tag{1.3}$$

In [16], Lasota provided discrete versions of Opial inequality (1.1) about the forward difference operator as follows:

Theorem 2. *Let* $\{x_i\}_{i=0}^N$ *be a sequence of real numbers with* $x_0=x_N=0$. *Then, the following inequality holds*

$$\sum_{i=1}^{N-1} |x_i\Delta x_i| \le \frac{1}{2}\left\lfloor\frac{N+1}{2}\right\rfloor \sum_{i=0}^{N-1}|\Delta x_i|^2, \tag{1.4}$$

where $\lfloor\cdot\rfloor$ *is the integer part function. If* N *is even, then the inequality (1.4) is sharp.*

Also, we have the following results, see [1]:

Theorem 3. *Let* $\{x_i\}_{i=0}^N$ *be a sequence of real numbers. If* $x_0=0$, *then*

$$\sum_{i=1}^{\tau-1} |x_i\Delta x_i| \le \frac{1}{2}(\tau-1)\sum_{i=0}^{\tau-1}|\Delta x_i|^2,\ \tau\in\{2,...,N\}. \tag{1.5}$$

If $x_N=0$, *then*

$$\sum_{i=\tau}^{N-1} |x_i\Delta x_i| \le \frac{1}{2}(N-\tau+1)\sum_{i=0}^{N-1}|\Delta x_i|^2,\ \tau\in\{1,...,N-1\}. \tag{1.6}$$

For other discrete Opial type inequalities, see [14], [15] and [22]-[27].

In order to extend this result for modulus of elements in Banach $*$-algebra we need the following preparations.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [24] (see also [3, Theorem 41.5])

$$\text{(SF)} \qquad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [19], Tanahashi and Uchiyama [25] proved the following fundamental properties (see also [12]):

(i) If $a, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [3, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z - a)^{-1} \, dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \mathrm{ins}(\gamma)$, the inside of γ. It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\mathrm{Re}\, z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \mathrm{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} \, dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\mathrm{Re}\, z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [12], we list below some important properties of real powers:

(viii) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $a^{\alpha}\in A$ with $a^{\alpha}>0$ and $\left(a^{2}\right)^{1/2}=a$, [25, Lemma 6];
(ix) If $0<a\in A$ and α, $\beta\in\mathbb{R}$, then $a^{\alpha}a^{\beta}=a^{\alpha+\beta}$;
(x) If $0<a\in A$ and $\alpha\in\mathbb{R}$, then $\left(a^{\alpha}\right)^{-1}=\left(a^{-1}\right)^{\alpha}=a^{-\alpha}$;
(xi) If $0<a$, $b\in A$, α, $\beta\in\mathbb{R}$ and $ab=ba$, then $a^{\alpha}b^{\beta}=b^{\beta}a^{\alpha}$.

Okayasu [19] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b\in A$ and $p\in[0,1]$ then $a>b$ $(a\geq b)$ implies that $a^{p}>b^{p}$ $(a^{p}\geq b^{p})$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [7]-[10].

By *Shirali-Ford theorem* we have $a^{*}a\geq0$ for every $a\in A$, so we can define the absolute value or modulus of a by $|a|:=(a^{*}a)^{1/2}\geq0$. It is well know that if $A=\mathcal{B}(H)$, the C^{*}-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a+b|\leq|a|+|b|\,,\ a,\ b\in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

In this paper we show among others that, if $\{\gamma_{i}\}_{i=0}^{N}\subset\mathbb{C}$ and $\{a_{i}\}_{i=0}^{N}\subset A$ are sequences with $a_{0}=0$ and $a_{N}=0$, then for $n\in\{2,...,N-1\}$,

$$\sum_{i=1}^{N-1}|\Delta\gamma_{i}|\,|a_{i}|\leq\left(\sum_{i=0}^{N-1}p_{i}(n)\,|\Delta\gamma_{i}|^{2}\right)^{1/2}\left(\sum_{i=0}^{N-1}q_{i}(n)\,|\Delta a_{i}|^{2}\right)^{1/2}$$
$$\leq\frac{1}{2}\sum_{i=0}^{N-1}\left(p_{i}(n)\,|\Delta\gamma_{i}|^{2}+q_{i}(n)\,|\Delta a_{i}|^{2}\right),$$

where

$$p_{i}(n):=\begin{cases}i,\ \text{if } 0\leq i\leq n-1,\\ \\ N-i,\ \text{if } n\leq i\leq N-1\end{cases}$$

and

$$q_{i}(n):=\begin{cases}n-i-1,\ \text{if } 0\leq i\leq n-1,\\ \\ i+1-n,\ \text{if } n\leq i\leq N-1,\end{cases}$$

where $\Delta\gamma_{j}:=\gamma_{j+1}-\gamma_{j}$ is the forward difference.

2. Main Results

We start to the following identities of interest:

Lemma 1. *Let $a_{k}\in A$, $\gamma_{k}\in\mathbb{C}$ and $p_{k}\geq0$ for $k\in\{1,...,n\}$. Then*

$$\sum_{k=1}^{n}p_{k}|\gamma_{k}|^{2}\sum_{k=1}^{n}p_{k}|a_{k}|^{2}-\left|\sum_{j=1}^{n}p_{j}\gamma_{j}a_{j}\right|^{2}=\frac{1}{2}\sum_{j,k=1}^{n}p_{j}p_{k}\left|\overline{\gamma_{j}}a_{k}-\overline{\gamma}_{k}a_{j}\right|^{2}. \tag{2.1}$$

In particular,

$$\sum_{k=1}^{n}|\gamma_{k}|^{2}\sum_{k=1}^{n}|a_{k}|^{2}-\left|\sum_{j=1}^{n}\gamma_{j}a_{j}\right|^{2}=\frac{1}{2}\sum_{j,k=1}^{n}\left|\overline{\gamma_{j}}a_{k}-\overline{\gamma}_{k}a_{j}\right|^{2} \tag{2.2}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{j=1}^{n} p_j a_j \right|^2 = \frac{1}{2} \sum_{j,k=1}^{n} p_j p_k |a_k - a_j|^2 . \tag{2.3}$$

Proof. Observe that

$$\begin{aligned}
&\left|\overline{\gamma_j} a_k - \overline{\gamma}_k a_j\right|^2 \\
&= \left(\overline{\gamma_j} a_k - \overline{\gamma}_k a_j\right)^* \left(\overline{\gamma_j} a_k - \overline{\gamma}_k a_j\right) = \left(\gamma_j a_k^* - \gamma_k a_j^*\right) \left(\overline{\gamma_j} a_k - \overline{\gamma}_k a_j\right) \\
&= \gamma_j a_k^* \overline{\gamma_j} a_k - \gamma_j a_k^* \overline{\gamma}_k a_j - \gamma_k a_j^* \overline{\gamma_j} a_k + \gamma_k a_j^* \overline{\gamma}_k a_j \\
&= \left|\gamma_j\right|^2 |a_k|^2 - \overline{\gamma}_k a_k^* \gamma_j a_j - \overline{\gamma_j} a_j^* \gamma_k a_k + |\gamma_k|^2 |a_j|^2
\end{aligned}$$

for all $j,\, k \in \{1, ..., n\}$.

This implies that

$$\begin{aligned}
&\sum_{j,k=1}^{n} p_j p_k \left|\overline{\gamma_j} a_k - \overline{\gamma}_k a_j\right|^2 \\
&= \sum_{j,k=1}^{n} p_j p_k \left[\left|\gamma_j\right|^2 |a_k|^2 - \overline{\gamma}_k a_k^* \gamma_j a_j - \overline{\gamma_j} a_j^* \gamma_k a_k + |\gamma_k|^2 |a_j|^2\right] \\
&= \sum_{j,k=1}^{n} p_j p_k \left|\gamma_j\right|^2 |a_k|^2 - \sum_{j,k=1}^{n} p_j p_k \overline{\gamma}_k a_k^* \gamma_j a_j \\
&- \sum_{j,k=1}^{n} p_j p_k \overline{\gamma_j} a_j^* \gamma_k a_k + \sum_{j,k=1}^{n} p_j p_k |\gamma_k|^2 |a_j|^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} p_j \left|\gamma_j\right|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \sum_{k=1}^{n} p_k \overline{\gamma}_k a_k^* \sum_{j=1}^{n} p_j \gamma_j a_j \\
&- \sum_{j=1}^{n} p_j \overline{\gamma_j} a_j^* \sum_{k=1}^{n} p_k \gamma_k a_k + \sum_{k=1}^{n} p_k |\gamma_k|^2 \sum_{j=1}^{n} p_j |a_j|^2 \\
&= \sum_{j=1}^{n} p_j \left|\gamma_j\right|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left(\sum_{k=1}^{n} p_k \gamma_k a_k\right)^* \sum_{j=1}^{n} p_j \gamma_j a_j \\
&- \left(\sum_{j=1}^{n} p_j \gamma_j a_j\right)^* \sum_{k=1}^{n} p_k \gamma_k a_k + \sum_{k=1}^{n} p_k |\gamma_k|^2 \sum_{j=1}^{n} p_j |a_j|^2 \\
&= 2 \left[\sum_{k=1}^{n} p_k |\gamma_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k \gamma_k a_k\right|^2\right],
\end{aligned}$$

which is equivalent to the desired identity (2.1). □

We have the following Cauchy-Bunyakowsky-Schwarz (CBS) type inequalities:

Corollary 1. *Let $a_k \in A$, $\gamma_k \in \mathbb{C}$ and $p_k > 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k |\gamma_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 \geq \left| \sum_{j=1}^{n} p_j \gamma_j a_j \right|^2 . \tag{2.4}$$

In particular,

$$\sum_{k=1}^{n} |\gamma_k|^2 \sum_{k=1}^{n} |a_k|^2 \geq \left| \sum_{j=1}^{n} \gamma_j a_j \right|^2 \tag{2.5}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 \geq \left| \sum_{j=1}^{n} p_j a_j \right|^2 . \tag{2.6}$$

The equality holds in (2.6) if and only if $a_k = a$ for some $a \in A$ and all $k \in \{1, ..., n\}$.

Remark 1. *If A has a continuous involution, then we can take the square root to obtain the inequalities*

$$\left(\sum_{k=1}^{n} p_k |\gamma_k|^2 \right) \left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} p_j \gamma_j a_j \right| . \tag{2.7}$$

In particular,

$$\left(\sum_{k=1}^{n} |\gamma_k|^2 \right)^{1/2} \left(\sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} \gamma_j a_j \right| \tag{2.8}$$

and

$$\left(\sum_{k=1}^{n} p_k \right)^{1/2} \left(\sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} p_j a_j \right| . \tag{2.9}$$

We have the following result for two sequences:

Theorem 4. *Suppose that A has a continuous involution. Assume that $\{\gamma_i\}_{i=0}^{N} \subset \mathbb{C}$ and $\{a_i\}_{i=0}^{N} \subset A$ are sequences with $a_0 = 0$, then for $n \in \{2, ..., N\}$,*

$$\begin{aligned} \sum_{i=1}^{n-1} |\Delta \gamma_i| |a_i| &\leq \left(\sum_{i=0}^{n-1} i |\Delta \gamma_i|^2 \right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) |\Delta a_i|^2 \right)^{1/2} \\ &\leq \frac{1}{2} \sum_{i=0}^{n-1} \left[i |\Delta \gamma_i|^2 + (n-i-1) |\Delta a_i|^2 \right] \\ &\leq \frac{1}{2} (n-1) \sum_{i=0}^{n-1} \left(|\Delta \gamma_i|^2 + |\Delta a_i|^2 \right) . \end{aligned} \tag{2.10}$$

Proof. Let $n\in\{2,...,N\}$. Since $a_0=0$, hence $a_i=\sum_{j=0}^{i-1}\Delta a_j$ for $i=1,...,n-1$. Then

$$\sum_{i=1}^{n-1}|\Delta\gamma_i|\,|a_i|=\sum_{i=1}^{n-1}|\Delta\gamma_i|\left|\sum_{j=0}^{i-1}\Delta a_j\right|=\sum_{i=1}^{n-1}\sqrt{i}\,|\Delta\gamma_i|\,\frac{1}{\sqrt{i}}\left|\sum_{j=0}^{i-1}\Delta a_j\right|=:K.$$

By the discrete Cauchy-Bunyakowsky-Schwarz (CBS) inequality (2.8) for elements in A, we have

$$K\le\left(\sum_{i=1}^{n-1}i\,|\Delta\gamma_i|^2\right)^{1/2}\left[\sum_{i=1}^{n-1}\frac{1}{i}\left|\sum_{j=0}^{i-1}\Delta a_j\right|^2\right]^{1/2}=:B.$$

By (CBS) inequality we also have

$$\frac{1}{i}\left|\sum_{j=0}^{i-1}\Delta a_j\right|^2\le\sum_{j=0}^{i-1}|\Delta a_j|^2,$$

which, by taking the square root, gives

$$B\le\left(\sum_{i=1}^{n-1}i\,|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}|\Delta a_j|^2\right)\right)^{1/2}. \tag{2.11}$$

From (1.3), we have for $m=1$ and n is replaced by $n-1$ that

$$\sum_{i=1}^{n-1}(\Delta\alpha_i)\,b_i=\alpha_{n-1}b_n-\alpha_1b_1-\sum_{i=1}^{n-2}\alpha_{i+1}\Delta b_i,$$

which by taking $b_i=\sum_{j=0}^{i-1}|\Delta a_j|^2$, $\alpha_i=i$, produces that

$$\begin{aligned}&\sum_{i=1}^{n-1}\left(\sum_{j=0}^{i-1}|\Delta a_j|^2\right)\\&=n\sum_{j=0}^{n-2}|\Delta a_j|^2-|\Delta a_0|^2-\sum_{i=1}^{n-2}(i+1)\,|\Delta a_i|^2\\&=n\,|\Delta a_0|^2-|\Delta a_0|^2+n\sum_{j=1}^{n-2}|\Delta a_j|^2-\sum_{i=1}^{n-2}(i+1)\,|\Delta a_i|^2\\&=(n-1)\,|\Delta a_0|^2+\sum_{i=1}^{n-2}(n-i-1)\,|\Delta a_i|^2=\sum_{i=0}^{n-2}(n-i-1)\,|\Delta a_i|^2.\end{aligned}$$

Now, it is obvious that

$$\sum_{i=1}^{n-1}i\,|\Delta\gamma_i|^2=\sum_{i=0}^{n-1}i\,|\Delta\gamma_i|^2$$

and

$$\sum_{i=0}^{n-2}(n-i-1)\,|\Delta a_i|^2=\sum_{i=0}^{n-1}(n-i-1)\,|\Delta a_i|^2.$$

By utilising (2.11) we derive the first part of (2.10). The second part follows by the A-G-means inequality,

$$\alpha b \leq \frac{\alpha^2 + b^2}{2},\ \alpha \text{ positive real number},\ b \geq 0 \text{ in } A.$$

Now, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left[i\left|\Delta\gamma_i\right|^2 + (n-i-1)\left|\Delta a_i\right|^2\right] \\
&\leq \max_{i\in\{0,\dots,n\}}\{i, n-i-1\}\sum_{i=0}^{n-1}\left[\left|\Delta\gamma_i\right|^2 + \left|\Delta a_i\right|^2\right] \\
&= (n-1)\sum_{i=0}^{n-1}\left[\left|\Delta\gamma_i\right|^2 + \left|\Delta a_i\right|^2\right],
\end{aligned}$$

which proves the last part. □

The case of one sequence, is as follows:

Corollary 2. *With the assumptions of Theorem 4, we also have*

$$\begin{aligned}
\sum_{i=1}^{n-1}\left|\Delta\gamma_i\right|\left|a_i\right| &\leq \left(\sum_{i=0}^{n-1} i\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left|\Delta a_i\right|^2\right)^{1/2} \\
&\leq (n-1)\left(\sum_{i=0}^{n-1}\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}\left|\Delta a_i\right|^2\right)^{1/2}.
\end{aligned} \tag{2.12}$$

The last inequality follows by the fact that $i,\ n-i-1 \leq n-1$ for $i \in \{0,\dots,n\}$.

Corollary 3. *With the assumptions of Theorem 4 and if* $\left|\Delta a_i\right|^2 \leq M^2\left|\Delta\gamma_i\right|^2$ *for* $i \in \{0,\dots,n\}$ *and* $M > 0$, *then we also have*

$$\begin{aligned}
\sum_{i=1}^{n-1}\left|\Delta\gamma_i\right|\left|a_i\right| &\leq M\left(\sum_{i=0}^{n-1} i\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)\left|\Delta\gamma_i\right|^2\right)^{1/2} \\
&\leq \frac{1}{2}M(n-1)\sum_{i=0}^{n-1}\left|\Delta\gamma_i\right|^2.
\end{aligned} \tag{2.13}$$

We also have:

Theorem 5. *With the assumptions of Theorem 4 and if* $a_N = 0$, *then for* $n \in \{1,\dots,N-1\}$,

$$\begin{aligned}
\sum_{i=n}^{N-1}\left|\Delta\gamma_i\right|\left|a_i\right| &\leq \left(\sum_{i=n}^{N-1}(N-i)\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)\left|\Delta a_i\right|^2\right)^{1/2} \\
&\leq \frac{1}{2}\sum_{i=n}^{N-1}\left[(N-i)\left|\Delta\gamma_i\right|^2 + (i+1-n)\left|\Delta a_i\right|^2\right] \\
&\leq \frac{1}{2}(N-n)\left[\sum_{i=n}^{N-1}\left(\left|\Delta\gamma_i\right|^2 + \left|\Delta a_i\right|^2\right)\right].
\end{aligned} \tag{2.14}$$

Proof. If $a_N = 0$, then $a_i = -\sum_{j=i}^{N-1} \Delta a_j$ for $i = n+1, ..., N-1$. Then

$$\begin{aligned}\sum_{i=n}^{N-1} |\Delta\gamma_i| \, |a_i| &= \sum_{i=n}^{N-1} |\Delta\gamma_i| \left| \sum_{j=i}^{N-1} \Delta a_j \right| \\ &= \sum_{i=n}^{N-1} \sqrt{N-i} \, |\Delta\gamma_i| \, \frac{1}{\sqrt{N-i}} \left| \sum_{j=i}^{N-1} \Delta a_j \right| =: C.\end{aligned}$$

By the discrete Cauchy-Bunyakowsky-Schwarz (CBS) inequality, we have

$$\begin{aligned}C &\le \left(\sum_{i=n}^{N-1} (N-i) \, |\Delta\gamma_i|^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} \frac{1}{N-i} \left| \sum_{j=i}^{N-1} \Delta a_j \right|^2 \right)^{1/2} \qquad (2.15) \\ &\le \left(\sum_{i=n}^{N-1} (N-i) \, |\Delta\gamma_i|^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} \left(\sum_{j=i}^{N-1} |\Delta a_j|^2 \right) \right)^{1/2} =: D\end{aligned}$$

From (1.3) we have

$$\sum_{i=n}^{N-1} (\Delta\alpha_i) \, b_i = \alpha_N b_{N-1} - \alpha_n b_n - \sum_{i=n}^{N-2} \alpha_{i+1} \Delta b_i,$$

and by $b_i = \sum_{j=i}^{N-1} |\Delta a_i|^2$ and $\alpha_i = i$, we have

$$\begin{aligned}&\sum_{i=n}^{N-1} \left(\sum_{j=i}^{N-1} |\Delta a_i|^2 \right) \\ &= N \, |\Delta a_{N-1}|^2 - n \sum_{j=n}^{N-1} |\Delta a_i|^2 - \sum_{i=n}^{N-2} (i+1) \left(\sum_{j=i+1}^{N-1} |\Delta a_i|^2 - \sum_{j=i}^{N-1} |\Delta a_j|^2 \right) \\ &= N \, |\Delta a_{N-1}|^2 + \sum_{i=n}^{N-2} (i+1) \, |\Delta a_i|^2 - n \sum_{j=n}^{N-1} |\Delta a_i|^2 . \\ &= N \, |\Delta a_{N-1}|^2 + \sum_{i=n}^{N-2} (i+1) \, |\Delta a_i|^2 - n \sum_{j=n}^{N-2} |\Delta a_i|^2 - n \, |\Delta a_{N-1}|^2 \\ &= \sum_{i=n}^{N-2} (i+1) \, |\Delta a_i|^2 - n \sum_{j=n}^{N-2} |\Delta a_i|^2 \\ &= \sum_{i=n}^{N-2} (i+1-n) \, |\Delta a_i|^2 + (N-n) \, |\Delta a_{N-1}|^2 = \sum_{i=n}^{N-1} (i+1-n) \, |\Delta a_i|^2 .\end{aligned}$$

Then

$$D = \left(\sum_{i=n}^{N-1} (N-i) \, |\Delta\gamma_i|^2 \right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \, |\Delta a_i|^2 \right)^{1/2},$$

which, by (2.15), proves the first inequality in (2.14).

The second part follows by A-G-means inequality. The last part is obvious. □

Corollary 4. *With the assumptions of Theorem 5, we also have*

$$\sum_{i=n}^{N-1}|\Delta\gamma_i||a_i| \le \left(\sum_{i=n}^{N-1}(N-i)|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)|\Delta a_i|^2\right)^{1/2} \le (N-n)\left(\sum_{i=n}^{N-1}|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}|\Delta a_i|^2\right)^{1/2} \tag{2.16}$$

The last inequality follows by the fact that $N-i,\ i+1-n \le N-n$ for $i\in\{n,...,N-1\}$.

Corollary 5. *With the assumptions of Theorem 5 and if* $|\Delta a_i|^2 \le M^2|\Delta\gamma_i|^2$ *for* $i\in\{n,...,N-1\}$ *and* $M>0$, *then*

$$\sum_{i=n}^{N-1}|\Delta\gamma_i||a_i| \le M\left(\sum_{i=n}^{N-1}(N-i)|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)|\Delta\gamma_i|^2\right)^{1/2} \le \frac{1}{2}M(N-n+1)\sum_{i=n}^{N-1}|\Delta\gamma_i|^2. \tag{2.17}$$

We also have the following result that incorporates both cases:

Theorem 6. *With the assumptions of Theorem 4 and if* $a_0=a_N=0$, *then for* $n\in\{2,...,N-1\}$,

$$\sum_{i=1}^{N-1}|\Delta\gamma_i||a_i| \le \left(\sum_{i=0}^{N-1}p_i(n)|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}q_i(n)|\Delta a_i|^2\right)^{1/2} \le \frac{1}{2}\sum_{i=0}^{N-1}\left(p_i(n)|\Delta\gamma_i|^2+q_i(n)|\Delta a_i|^2\right), \tag{2.18}$$

where

$$p_i(n) := \begin{cases} i, & \text{if } 0\le i\le n-1,\\ N-i, & \text{if } n\le i\le N-1\end{cases}$$

and

$$q_i(n) := \begin{cases} n-i-1, & \text{if } 0\le i\le n-1,\\ i+1-n, & \text{if } n\le i\le N-1.\end{cases}$$

Proof. We have for $n\in\{2,...,N-1\}$ that

$$\sum_{i=1}^{n-1}|\Delta\gamma_i||a_i| \le \left(\sum_{i=0}^{n-1}i|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{n-1}(n-i-1)|\Delta a_i|^2\right)^{1/2}$$

and

$$\sum_{i=n}^{N-1}|\Delta\gamma_i||a_i| \le \left(\sum_{i=n}^{N-1}(N-i)|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=n}^{N-1}(i+1-n)|\Delta a_i|^2\right)^{1/2}.$$

Observe that

$$(\alpha b+\beta d)^2 \le \left(\alpha^2+\beta^2\right)\left(b^2+d^2\right), \tag{2.19}$$

where α, β are positive real numbers and c, $d \geq 0$ in A.

If we add these inequalities, then we get, by the elementary inequality

$$\alpha b + \beta d \leq \left(\alpha^2 + \beta^2\right)^{1/2} \left(b^2 + d^2\right)^{1/2}$$

that we derive by taking the square root in (2.19),

$$\begin{aligned}
\sum_{i=1}^{N-1} |\Delta\gamma_i| \, |a_i| &\leq \left(\sum_{i=0}^{n-1} i \, |\Delta\gamma_i|^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (n-i-1) \, |\Delta a_i|^2\right)^{1/2} \\
&\quad + \left(\sum_{i=n}^{N-1} (N-i) \, |\Delta\gamma_i|^2\right)^{1/2} \left(\sum_{i=n}^{N-1} (i+1-n) \, |\Delta a_i|^2\right)^{1/2} \\
&\leq \left(\sum_{i=0}^{n-1} i \, |\Delta\gamma_i|^2 + \sum_{i=n}^{N-1} (N-i) \, |\Delta\gamma_i|^2\right)^{1/2} \\
&\quad \times \left(\sum_{i=0}^{n-1} (n-i-1) \, |\Delta a_i|^2 + \sum_{i=n}^{N-1} (i+1-n) \, |\Delta a_i|^2\right)^{1/2} \\
&= \left(\sum_{i=0}^{N-1} p_i \, |\Delta\gamma_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q_i \, |\Delta a_i|^2\right)^{1/2},
\end{aligned}$$

which proves the first inequality in (2.18).

The second inequality follows by A-G-means inequality. □

Corollary 6. *With the assumptions of Theorem 6 and if* $|\Delta a_i|^2 \leq M^2 |\Delta\gamma_i|^2$ *for* $i \in \{0, ..., N-1\}$ *and* $M > 0$*, then for* $n \in \{1, ..., N-1\}$

$$\begin{aligned}
\sum_{i=1}^{N-1} |\Delta\gamma_i| \, |a_i| &\leq M \left(\sum_{i=0}^{N-1} p_i(n) \, |\Delta\gamma_i|^2\right)^{1/2} \left(\sum_{i=0}^{N-1} q(n)_i \, |\Delta\gamma_i|^2\right)^{1/2} \\
&\leq \frac{1}{2} M \sum_{i=0}^{N-1} s_i(n) \, |\Delta\gamma_i|^2,
\end{aligned} \tag{2.20}$$

where

$$s_i(n) := \begin{cases} n-1, & \text{if } 0 \leq i \leq n-1, \\ N-n+1, & \text{if } n \leq i \leq N-1. \end{cases}$$

Remark 2. *If we take in (2.20) $n = \left\lfloor \frac{N+1}{2} \right\rfloor + 1$, then by (2.20) we get*

$$\sum_{i=1}^{N-1} |\Delta\gamma_i| |a_i| \le M \left(\sum_{i=0}^{N-1} p_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta\gamma_i|^2 \right)^{1/2} \times \left(\sum_{i=0}^{N-1} q \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta\gamma_i|^2 \right)^{1/2} \le \frac{1}{2} M \sum_{i=0}^{N-1} s_i \left(\left\lfloor \frac{N+1}{2} \right\rfloor + 1 \right) |\Delta\gamma_i|^2 \le \frac{1}{2} M \left\lfloor \frac{N+1}{2} \right\rfloor \sum_{i=0}^{N-1} |\Delta\gamma_i|^2 , \tag{2.21}$$

since

$$s_i(n) := \begin{cases} \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } 0 \le i \le \left\lfloor \frac{N+1}{2} \right\rfloor, \\ N - \left\lfloor \frac{N+1}{2} \right\rfloor, & \text{if } \left\lfloor \frac{N+1}{2} \right\rfloor + 1 \le i \le N-1 \end{cases} \le \left\lfloor \frac{N+1}{2} \right\rfloor .$$

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