ČEBYŠEV TYPE INEQUALITIES FOR SEQUENCES IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS WITH APPLICATIONS

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Abstract. In this paper we obtained some Čebyšev type inequalities for sequences in a Hermitian unital Banach $*$-algebra. Some applications for power series with examples for exponential, logarithm and other elementary functions are also provided.

1. Introduction

Consider the real sequences (n-tuples) $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ and the nonnegative sequence $\mathbf{p} = (p_1, \ldots, p_n)$ with $P_n := \sum_{i=1}^{n} p_i > 0$. Define the *weighted Čebyšev's functional*

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) := \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i. \tag{1.1}$$

In 1882 – 1883, Čebyšev [3] and [4] proved that if $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the same (opposite) sense, then

$$T_n(\mathbf{p}; \mathbf{a}, \mathbf{b}) \geq (\leq)\, 0. \tag{1.2}$$

In the special case $\mathbf{p} = \mathbf{a} \geq \mathbf{0}$, it appears that the inequality (1.2) has been obtained by Laplace long before Čebyšev (see for example [18, p. 240]).

The inequality (1.2) was mentioned by Hardy, Littlewood and Pólya in their book [16] in 1934 in the more general setting of synchronous sequences, i.e., if $\mathbf{a}$, $\mathbf{b}$ are synchronous (asynchronous), this means that

$$(a_i - a_j)(b_i - b_j) \geq (\leq)\, 0 \text{ for any } i, j \in \{1, \ldots, n\}, \tag{1.3}$$

then (1.2) holds true as well.

A relaxation of the synchronicity condition was provided by M. Biernacki in 1951, [1], which showed that, if $\mathbf{a}$, $\mathbf{b}$ are monotonic in mean in the same sense, i.e., for $P_k := \sum_{i=1}^{k} p_i$, $k = 1, \ldots, n-1$;

$$\frac{1}{P_k} \sum_{i=1}^{k} p_i a_i \leq (\geq) \frac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i a_i, \quad k \in \{1, \ldots, n-1\} \tag{1.4}$$

and

$$\frac{1}{P_k} \sum_{i=1}^{k} p_i b_i \leq (\geq) \frac{1}{P_{k+1}} \sum_{i=1}^{k+1} p_i b_i, \quad k \in \{1, \ldots, n-1\}, \tag{1.5}$$

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then (1.2) holds with " $\geq$ ". If if $\mathbf{a}$, $\mathbf{b}$ are monotonic in mean in the opposite sense then (1.2) holds with " $\leq$ ".

In 1989, Dragomir and Pečarić [13] proved the following refinement of Čebyšev's inequality for synchronous sequences. If $\mathbf{a}$, $\mathbf{b}$ are synchronous and by $|\mathbf{a}|$ we denote the $n-$tuple $(|a_1|, \dots, |a_n|)$, then

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b}) \geq \max\{|T_n(\mathbf{p};|\mathbf{a}|,\mathbf{b})|, |T_n(\mathbf{p};\mathbf{a},|\mathbf{b}|)|, |T_n(\mathbf{p};|\mathbf{a}|,|\mathbf{b}|)|\} \geq 0. \tag{1.6}$$

In 1990, Dragomir [6] considered the following class associated to a pair of synchronous sequences $\mathbf{a}$, $\mathbf{b}$;

$$\bar{S}(\mathbf{a},\mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n | (\mathbf{a}+\mathbf{x},\mathbf{b}) \text{ and } (\mathbf{a}-\mathbf{x},\mathbf{b}) \text{ are synchronous}\}.$$

It can be shown that $\bar{S}(\mathbf{a},\mathbf{b}) \neq \emptyset$ and one has the representation

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b}) = \sup_{\mathbf{x}\in\bar{S}(\mathbf{a},\mathbf{b})} |T_n(\mathbf{p};\mathbf{x},\mathbf{b})| \geq 0. \tag{1.7}$$

Now, if $\mathbf{k} = (k, k, \dots, k)$ is a constant sequence and if we denote by $\mathbf{a} \vee \mathbf{k} := (\max\{a_1, k\}, \dots, \max\{a_n, k\})$ and by $\mathbf{a} \wedge \mathbf{k} := (\min\{a_1, k\}, \dots, \min\{a_n, k\})$, then we may state the following result obtained in the general setting of positive linear functionals by Dragomir in 1993, [7]

$$\begin{aligned} T_n(\mathbf{p};\mathbf{a},\mathbf{b}) &\geq \max\{|T_n(\mathbf{p};\mathbf{a}\vee\mathbf{k},\mathbf{b})| + |T_n(\mathbf{p};\mathbf{a}\wedge\mathbf{k},\mathbf{b})|, \\ &\qquad |T_n(\mathbf{p};\mathbf{a},\mathbf{b}\vee\mathbf{k})| + |T_n(\mathbf{p};\mathbf{a},\mathbf{b}\wedge\mathbf{k})|\} \\ &\geq 0, \end{aligned} \tag{1.8}$$

provided $\mathbf{a}$ and $\mathbf{b}$ are synchronous.

If $\mathbf{k} = \mathbf{0}$, and $\mathbf{a}_+ := \mathbf{a} \vee \mathbf{0}$, $\mathbf{a}_- := \mathbf{a} \wedge \mathbf{0}$, then for synchronous sequences $\mathbf{a}$, $\mathbf{b}$ one has

$$\begin{aligned} T_n(\mathbf{p};\mathbf{a},\mathbf{b}) &\geq \max\{|T_n(\mathbf{p};\mathbf{a}_+,\mathbf{b})| + |T_n(\mathbf{p};\mathbf{a}_-,\mathbf{b})|, \\ &\qquad |T_n(\mathbf{p};\mathbf{a},\mathbf{b}_+)| + |T_n(\mathbf{p};\mathbf{a},\mathbf{b}_-)|\} \\ &\geq 0. \end{aligned} \tag{1.9}$$

Note that, since, obviously

$$\begin{aligned} |T_n(\mathbf{p};\mathbf{a}_+,\mathbf{b})| + |T_n(\mathbf{p};\mathbf{a}_-,\mathbf{b})| &\geq |T_n(\mathbf{p};\mathbf{a}_+,\mathbf{b}) + T_n(\mathbf{p};\mathbf{a}_-,\mathbf{b})| \\ &= |T_n(\mathbf{p};|\mathbf{a}|,\mathbf{b})|, \end{aligned}$$

then by (1.6) and (1.9), we deduce the sequence of inequalities

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b}) \geq |T_n(\mathbf{p};\mathbf{a}_+,\mathbf{b})| + |T_n(\mathbf{p};\mathbf{a}_-,\mathbf{b})| \geq |T_n(\mathbf{p};|\mathbf{a}|,\mathbf{b})| \geq 0, \tag{1.10}$$

provided $\mathbf{a}$ and $\mathbf{b}$ are synchronous. This is a refinement of (1.6).

If one would like to drop the assumption of nonnegativity for the components of $\mathbf{p}$, then one may state the following inequality obtained by Mitrinović and Pečarić in 1991, [17]:

If $0 \leq P_i \leq P_n$ for each $i \in \{1, \dots, n-1\}$, then

$$T_n(\mathbf{p};\mathbf{a},\mathbf{b}) \geq 0 \tag{1.11}$$

provided $\mathbf{a}$ and $\mathbf{b}$ are sequences with the same monotonicity. If $\mathbf{a}$ and $\mathbf{b}$ are monotonic in the opposite sense, the sign of the inequality (1.11) reverses.

In order to extend some of the above results for sequences in Hermitian unital Banach $*$-algebra we need the following preparations.

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a, b \in \mathrm{Inv}(A)$, then $ab \in \mathrm{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [23] (see also [2, Theorem 41.5])

$$\text{(SF)} \qquad a^*a \geq 0 \text{ for every } a \in A.$$

Based on this fact, Okayasu [21], Tanahashi and Uchiyama [24] proved the following fundamental properties (see also [14]):

(i) If $a,\, b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If $a,\, b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If $a,\, b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;
(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \leq a$, then $0 < a^{-1} \leq b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [2, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z - a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \mathrm{ins}(\gamma)$, the inside of γ. It is well known (see for instance [5, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\mathrm{Re}\, z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \mathrm{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z - a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\mathrm{Re}\, z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [14], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$ and $\left(a^{2}\right)^{1/2} = a$, [24, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha,\ \beta \in \mathbb{R}$, then $a^{\alpha}a^{\beta} = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^{\alpha})^{-1} = \left(a^{-1}\right)^{\alpha} = a^{-\alpha}$;
(xi) If $0 < a,\ b \in A$, $\alpha,\ \beta \in \mathbb{R}$ and $ab = ba$, then $a^{\alpha}b^{\beta} = b^{\beta}a^{\alpha}$.

Okayasu [21] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a,\ b \in A$ and $p \in [0,1]$ then $a > b$ $(a \geq b)$ implies that $a^{p} > b^{p}$ $(a^{p} \geq b^{p})$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [8]-[11].

By *Shirali-Ford theorem* we have $a^{*}a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^{*}a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^{*}-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a+b| \leq |a| + |b|\,,\ a,\ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

In this paper we establish some Čebyšev type inequalities for sequences in a Hermitian unital Banach $*$-algebra. Some applications for power series with examples for exponential, logarithm and other elementary functions are also provided.

2. Čebyšev Type Inequalities

We say that the sequence $\mathbf{b} = (b_1, \ldots, b_n) \in A^n$ is *monotonic nondecreasing (nonincreasing)* if $b_1\,(\geq) \leq (\geq) \ldots \leq (\geq)\, b_n$ in the order of A. Let $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \ldots, b_n) \in A^n$. The pair $(\boldsymbol{\gamma}, \mathbf{b})$ is called *synchronous (asynchronous)* if

$$\left(\gamma_i - \gamma_j\right)(b_i - b_j) \geq (\leq)\, 0 \text{ for all } i,\ j \in \{1, \ldots, n\}\,.$$

Using the Korkine type identity

$$\begin{aligned} T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}) &:= \frac{1}{P_n}\sum_{i=1}^{n} p_i\gamma_i b_i - \frac{1}{P_n}\sum_{i=1}^{n} p_i\gamma_i \frac{1}{P_n}\sum_{i=1}^{n} p_i b_i \\ &= \frac{1}{P_n^2}\sum_{i,j=1}^{n} p_i p_j \left(\gamma_i - \gamma_j\right)(b_i - b_j) \\ &= \frac{1}{P_n^2}\sum_{1\leq i<j\leq n} p_i p_j \left(\gamma_i - \gamma_j\right)(b_i - b_j)\,, \end{aligned}$$

where $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ with $P_n := \sum_{i=1}^{n} p_i \neq 0$, we can state the Čebyšev type inequality:

Theorem 1. *Assume that the pair* $(\boldsymbol{\gamma}, \mathbf{b})$ *is synchronous (asynchronous) and* $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ *with* $P_n > 0$, *then* $T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}) \geq 0$.

Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$ and $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n$ be sequences of real numbers. Define

$$P_i := \sum_{k=1}^{i} p_k, \quad \bar{P}_i = P_n - P_i,$$
$$\Gamma_i(\mathbf{p}) = \sum_{k=1}^{i} p_k \gamma_k, \quad \bar{\Gamma}_i(\mathbf{p}) = \Gamma_n(\mathbf{p}) - \Gamma_i(\mathbf{p}).$$

The following result holds.

Theorem 2. *Let $\gamma = (\gamma_1, \dots, \gamma_n)$, $\mathbf{p} = (p_1, \dots, p_n)$ be sequences of real numbers and $\mathbf{b} = (b_1, \dots, b_n)$ a sequence in A. Assume that $p_i \geq 0$ $(i \in \{1, \dots, n\})$ such that $P_i \neq 0$ $(i \in \{1, \dots, n\})$. If either*

(i) $\mathbf{b}$ *is increasing and γ is a last-max in mean sequence, i.e., γ satisfies the condition*

$$\frac{\Gamma_n(\mathbf{p})}{P_n} \geq \frac{\Gamma_i(\mathbf{p})}{P_i}$$

for each $i \in \{1, \dots, n-1\}$;

or

(ii) $\mathbf{b}$ *is decreasing and γ is a first-max in mean sequence, i.e.,*

$$\frac{\Gamma_n(\mathbf{p})}{P_n} \leq \frac{\Gamma_i(\mathbf{p})}{P_i}$$

for each $i \in \{1, \dots, n-1\}$; then one has the inequality

$$T_n(\mathbf{p}; \boldsymbol{\gamma}, \mathbf{b}) \geq 0. \tag{2.1}$$

Proof. If we assume that $P_n \neq 0$, then we have the identities

$$\begin{aligned} T_n(\mathbf{p}; \boldsymbol{\gamma}, \mathbf{b}) &= \frac{1}{P_n^2} \sum_{i=1}^{n-1} \det \begin{bmatrix} P_i & P_n \\ \Gamma_i(\mathbf{p}) & \Gamma_n(\mathbf{p}) \end{bmatrix} \Delta b_i \\ &= \frac{1}{P_n} \sum_{i=1}^{n-1} P_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \frac{\Gamma_i(\mathbf{p})}{P_i} \right) \Delta b_i \\ &= \frac{1}{P_n^2} \sum_{i=1}^{n-1} P_i \bar{P}_i \left(\frac{\bar{\Gamma}_i(\mathbf{p})}{\bar{P}_i} - \frac{\Gamma_i(\mathbf{p})}{P_i} \right) \Delta b_i, \end{aligned} \tag{2.2}$$

where $\Delta b_i := b_{i+1} - b_i$ $(i = 0, \dots, n-1)$ is the forward difference.

We use the following well known summation by parts formula

$$\sum_{\ell=p}^{q-1} d_\ell \Delta v_\ell = d_\ell v_\ell \Big|_p^q - \sum_{\ell=p}^{q-1} (\Delta d_\ell) v_{\ell+1}, \tag{2.3}$$

where $d_\ell, v_\ell \in \mathbb{R}$, $\ell = p, \dots, q$ ($q > p$, p, q are natural numbers).

If we choose in (2.3), $p=1$, $q=n$, $d_i = P_i\Gamma_n(\bar{p}) - P_n\Gamma_i(\bar{p})$ and $v_i = b_i$ $(i=1,\dots,n)$, then we get

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left[P_i\Gamma_n(\mathbf{p}) - P_n\Gamma_i(\mathbf{p})\right]\Delta b_i \\
&= \left[P_i\Gamma_n(\mathbf{p}) - P_n\Gamma_i(\mathbf{p})\right]b_i\Big|_1^n - \sum_{i=1}^{n-1}\Delta\left(P_i\Gamma_n(\mathbf{p}) - P_n\Gamma_i(\mathbf{p})\right)b_{i+1} \\
&= \left[P_n\Gamma_n(\mathbf{p}) - P_n\Gamma_n(\mathbf{p})\right]b_n - \left[P_1\Gamma_n(\mathbf{p}) - P_n\Gamma_1(\mathbf{p})\right]b_1 \\
&- \sum_{i=1}^{n-1}\left[P_{i+1}\Gamma_n(\mathbf{p}) - P_n\Gamma_{i+1}(\mathbf{p}) - P_i\Gamma_n(\mathbf{p}) + P_n\Gamma_i(\mathbf{p})\right]b_{i+1} \\
&= P_n p_1\gamma_1 b_1 - p_1 b_1\Gamma_n(\mathbf{p}) - \sum_{i=1}^{n-1}\left(p_{i+1}\Gamma_n(\mathbf{p}) - P_n p_{i+1}\gamma_{i+1}\right)b_{i+1} \\
&= P_n p_1\gamma_1 b_1 - p_1 b_1\Gamma_n(\mathbf{p}) - \Gamma_n(\mathbf{p})\sum_{i=1}^{n-1}p_{i+1}b_{i+1} + P_n\sum_{i=1}^{n-1}p_{i+1}\gamma_{i+1}b_{i+1} \\
&= P_n\sum_{i=1}^{n}p_i\gamma_i b_i - \sum_{i=1}^{n}p_i\gamma_i\sum_{i=1}^{n}p_i b_i = P_n^2 T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}),
\end{aligned}$$

which produces the first identity in (2.2). The second and third are obvious and we omit the details.

The inequality (2.1) follows by the equalities in (2.2). □

The following result holds.

Remark 1. *We observe that if* $\gamma = (\gamma_1,\dots,\gamma_n)$ *is monotonic increasing in mean for a given* $\mathbf{p}$ *positive, i.e.,*

$$\frac{1}{P_i}\Gamma_i(\mathbf{p}) \le \frac{1}{P_{i+1}}\Gamma_{i+1}(\mathbf{p}),$$

then obviously

$$\frac{1}{P_i}\Gamma_i(\mathbf{p}) \le \frac{1}{P_n}\Gamma_n(\mathbf{p}) \tag{2.4}$$

for each $i \in \{1,\dots,n-1\}$*, i.e.,* γ *is a last-max in mean sequence for that specific weight vector* $\mathbf{p}$*. The converse is not true, generally.*

We also note that if γ *is monotonic nondecreasing, then for any positive* $\mathbf{p}$*, it is increasing in mean and, a fortiori, a last-max in mean sequence.*

Remark 2. *We observe, for* $\Gamma_i(\mathbf{p}) := \Gamma_n(\mathbf{p}) - \Gamma_i(\mathbf{p})$, $i \in \{1,\dots,n-1\}$*, that*

$$\frac{\Gamma_n(\mathbf{p})}{P_n} - \frac{\Gamma_i(\mathbf{p})}{P_i} = \frac{\bar{P}_i}{P_n}\left(\frac{\bar{\Gamma}_i(\mathbf{p})}{\bar{P}_i} - \frac{\Gamma_i(\mathbf{p})}{P_i}\right)$$

for each $i \in \{1,\dots,n-1\}$*, and thus, if we assume that* $\mathbf{p}$ *is positive, then*

$$\frac{\Gamma_n(\mathbf{p})}{P_n} \ge \frac{\Gamma_i(\mathbf{p})}{P_i} \quad \text{for every} \quad i \in \{1,\dots,n-1\}$$

if and only if

$$\frac{\bar{\Gamma}_i(\mathbf{p})}{\bar{P}_i} \ge \frac{\Gamma_i(\mathbf{p})}{P_i} \quad \text{for every} \quad i \in \{1,\dots,n-1\}.$$

If we would like to omit the assumption of positivity for the sequence $\mathbf{p}$, then the following result providing sufficient conditions for the functional $T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b})$ to be positive (negative) holds.

Theorem 3. *Let* $\gamma = (\gamma_1,\dots,\gamma_n)$ *and* $\mathbf{p} = (p_1,\dots,p_n)$ *be sequences of real numbers. If* $\mathbf{b} = (b_1,\dots,b_n)$ *is monotonic nondecreasing and either*

(*i*)

$$\det\begin{bmatrix} P_i & P_n \\ \Gamma_i(\mathbf{p}) & \Gamma_n(\mathbf{p}) \end{bmatrix} \geq 0 \text{ for each } i \in \{1,\dots,n-1\};$$

or

(*ii*) $P_i > 0$ *for any* $i \in \{1,\dots,n\}$ *and* γ *is a last-max in mean sequence*

or

(*iii*) $0 < P_i < P_n$ *for every* $i \in \{1,\dots,n-1\}$ *and*

$$\frac{\bar{\Gamma}_i(\mathbf{p})}{P_i} \geq \frac{\Gamma_i(\mathbf{p})}{P_i} \quad \text{for each } \ i \in \{1,\dots,n-1\};$$

then

$$T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}) \geq 0. \tag{2.5}$$

If $\mathbf{b}$ *is monotonic nonincreasing and either* (*i*) *or* (*ii*) *or* (*iii*) *from above holds, then the reverse inequality in (2.5) holds true.*

The proof of the theorem follows from the identities incorporated in (2.2) and we omit the details.

3. Some Inequalities for Convex (Concave) Sequences

The following result holds.

Theorem 4. *Let* $\gamma = (\gamma_1,\dots,\gamma_n)$ *be a sequence of real numbers and* $\mathbf{p} = (p_1,\dots,p_n)$ *a sequence of positive real numbers.*

If $\mathbf{b} = (b_1,\dots,b_n) \in A^n$ *is convex (concave), i.e.,*

$$\frac{b_{i+2}+b_i}{2} \geq (\leq)\, b_{i+1} \quad \text{for each } \ i \in \{1,\dots,n-2\} \tag{3.1}$$

and γ *satisfies the property*

$$\gamma_{i+1} \leq (\geq) \frac{\Gamma_n(\mathbf{p})}{P_n}, \quad \text{for each } \ i \in \{1,\dots,n-1\}; \tag{3.2}$$

then we have the inequality

$$T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}) \geq \frac{1}{(n-1)P_n}\sum_{i=1}^{n-1}(n-i)\,p_i\left(\frac{\Gamma_n(\mathbf{p})}{P_n}-\gamma_i\right)(b_n-b_1). \tag{3.3}$$

Proof. We know, by Čebyšev's inequality that if $\bar{\mathbf{z}} \in A^n$ and $\bar{\mathbf{u}} \in \mathbb{R}^n$ are synchronous, then

$$(n-1)\sum_{i=1}^{n-1}u_i z_i \geq \sum_{i=1}^{n-1}u_i\sum_{i=1}^{n-1}z_i. \tag{3.4}$$

Define $z_i := b_{i+1}-b_i$ and $u_i := P_i\Gamma_n(\mathbf{p}) - \Gamma_i(\mathbf{p})P_n$ for $i \in \{1,\dots,n-1\}$. Then

$$z_{i+1}-z_i = 2\left(\frac{b_{i+2}+b_i}{2}-b_{i+1}\right) \geq (\leq)\, 0 \quad \text{for each } i \in \{1,\dots,n-1\}$$

and

$$\begin{aligned}u_{i+1}-u_i &= P_{i+1}\Gamma_n(\mathbf{p})-\Gamma_{i+1}(\mathbf{p})P_n-P_i\Gamma_n(\mathbf{p})+\Gamma_i(\mathbf{p})P_n\\ &= p_{i+1}\Gamma_n(\mathbf{p})-\gamma_{i+1}p_{i+1}P_n\\ &= p_{i+1}P_n\left(\frac{\Gamma_n(\mathbf{p})}{P_n}-\gamma_{i+1}\right)\geq(\leq)\,0\end{aligned}$$

for each $i\in\{1,\dots,n-1\}$, showing that $\bar{\mathbf{z}}$ and $\bar{\mathbf{u}}$ are synchronous. Applying (3.4) and the first identity in (2.2), we have

$$\begin{aligned}T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}) &= \frac{1}{P_n^2}\sum_{i=1}^{n-1}\left(P_i\Gamma_n(\mathbf{p})-\Gamma_i(\mathbf{p})P_n\right)(b_{i+1}-b_i)\\ &\geq \frac{1}{(n-1)P_n^2}\sum_{i=1}^{n-1}\left(P_i\Gamma_n(\mathbf{p})-\Gamma_i(\mathbf{p})P_n\right)\sum_{i=1}^{n-1}(b_{i+1}-b_i)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{(n-1)P_n^2}\left[\Gamma_n(\mathbf{p})\sum_{i=1}^{n-1}P_i-P_n\sum_{i=1}^{n-1}\Gamma_i(\mathbf{p})\right](b_n-b_1)\\ &= \frac{1}{(n-1)}\left[\frac{\Gamma_n(\mathbf{p})}{P_n}\cdot\frac{1}{P_n}\sum_{i=1}^{n-1}(n-i)p_i-\frac{1}{P_n}\sum_{i=1}^{n-1}(n-i)p_i\gamma_i\right](b_n-b_1)\\ &= \frac{1}{(n-1)}\frac{1}{P_n}\sum_{i=1}^{n-1}(n-i)p_i\left(\frac{\Gamma_n(\mathbf{p})}{P_n}-\gamma_i\right)(b_n-b_1)\end{aligned}$$

and the inequality (3.3) is proved. □

The second result which does not require positivity for the weights $\mathbf{p}=(p_1,\dots,p_n)$, is enclosed in the following theorem.

Theorem 5. *Let γ and $\mathbf{p}$ be sequences of real numbers and $\mathbf{b}\in A^n$. Assume $P_i:=\sum_{k=1}^{i}p_k>0$ for $i=1,\dots,n$, $\mathbf{b}$ is convex (concave) and γ satisfies the following monotonicity in mean condition*

$$\frac{\Gamma_i(\mathbf{p})}{P_i}\geq(\leq)\,\frac{\Gamma_{i+1}(\mathbf{p})}{P_{i+1}},\quad \text{for } i\in\{1,\dots,n-1\}. \tag{3.5}$$

Then one has the inequality

$$T_n(\mathbf{p};\boldsymbol{\gamma},\mathbf{b}) \geq \frac{1}{\sum_{i=1}^{n-1}(n-i)p_i}\sum_{i=1}^{n-1}(n-i)p_i\left(\frac{\Gamma_n(\mathbf{p})}{P_n}-\gamma_i\right)\left(b_n-\frac{B_n(\mathbf{p})}{P_n}\right), \tag{3.6}$$

where $B_n(\mathbf{p}):=\sum_{i=1}^{n}p_ib_i$.

Proof. We use the following Čebyšev weighted inequality

$$\sum_{i=1}^{n-1}q_i\sum_{i=1}^{n-1}q_iu_iz_i\geq\sum_{i=1}^{n-1}q_iu_i\sum_{i=1}^{n-1}q_iz_i, \tag{3.7}$$

provided $q_i\geq 0$ and $\bar{\mathbf{z}}$, $\bar{\mathbf{u}}$ are monotonic in the same sense.

Now, if we define $q_i := P_i$, $z_i := b_{i+1} - b_i$ and $u_i := \frac{\Gamma_n(\mathbf{p})}{P_n} - \frac{\Gamma_{i+1}(\mathbf{p})}{P_{i+1}}$ for $i \in \{1, \dots, n-1\}$, then by Čebyšev's inequality, (3.7) and the second identity in (2.2), we have

$$T_n(\mathbf{p}; \boldsymbol{\gamma}, \mathbf{b}) = \frac{1}{P_n} \sum_{i=1}^{n-1} P_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \frac{\Gamma_i(\mathbf{p})}{P_i} \right) \Delta b_i \tag{3.8}$$
$$\geq \frac{1}{P_n \sum_{i=1}^{n-1} P_i} \sum_{i=1}^{n-1} P_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \frac{\Gamma_i(\mathbf{p})}{P_i} \right) \sum_{i=1}^{n-1} P_i \Delta b_i.$$

Since

$$\sum_{i=1}^{n-1} P_i = \sum_{i=1}^{n-1} (n-i) p_i,$$

$$\sum_{i=1}^{n-1} P_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \frac{\Gamma_i(\mathbf{p})}{P_i} \right) = \sum_{i=1}^{n-1} (n-i) p_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \gamma_i \right)$$

and

$$\sum_{i=1}^{n-1} P_i \Delta b_i = P_i b_i \Big|_1^n - \sum_{i=1}^{n-1} \Delta P_i b_{i+1} = P_n b_n - \sum_{i=1}^{n} p_i b_i$$

thus, by (3.8), we get

$$T_n(\mathbf{p}; \boldsymbol{\gamma}, \mathbf{b})$$
$$\geq \frac{1}{P_n \sum_{i=1}^{n-1} (n-i) p_i} \sum_{i=1}^{n-1} (n-i) p_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \gamma_i \right) \left(P_n b_n - \sum_{i=1}^{n} b_i p_i \right)$$
$$= \frac{1}{\sum_{i=1}^{n-1} (n-i) p_i} \sum_{i=1}^{n-1} (n-i) p_i \left(\frac{\Gamma_n(\mathbf{p})}{P_n} - \gamma_i \right) \left(b_n - \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right)$$

and the theorem is completely proved. □

4. Applications for Power Series

First we need the following facts:

Lemma 1 ([20, Lemma 2.4])**.** *Let A be a unital Hermitian Banach $*$-algebra with continuous involution, and let $\{a_n\}$ be a sequence of positive elements such that $a_n \to a$ in the norm topology. Then a is positive.*

Corollary 1. *Let A be a unital Hermitian Banach $*$-algebra with continuous involution, and let $\{a_n\}$, $\{b_n\}$ sequences of selfadjoint elements with $a_n \geq b_n$ for all $n \geq 0$. If $a_n \to a$, $b_n \to b$ in the norm topology, then $a \geq b$.*

We denote by $\mathbb{C}$ the set of all complex numbers. Let ρ_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\rho_n|^{\frac{1}{n}}}.$$

Clearly $0 \leq R \leq \infty$, but we consider only the case $0 < R \leq \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{z \in \mathbb{C} : |z| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the function:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C},\ f(\lambda) := \sum_{n=0}^{\infty} \rho_n \lambda^n.$$

We have the following examples of power series with positive coefficients:

$$\sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda},\ \lambda \in D(0,1); \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda,\ \lambda \in \mathbb{C};$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda,\ \lambda \in \mathbb{C};$$

$$\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{4.2}$$

$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function.*

We also have:

Theorem 6. *Let A be a unital Hermitian Banach $*$-algebra with continuous involution. Consider the function $\lambda \mapsto f(\lambda) : D(0,1) \to \mathbb{C},\ f(\lambda) := \sum_{n=0}^{\infty} \rho_n \lambda^n$ with $\rho_n \geq 0$ for all $n \geq 0$. If $a \in A$ is selfadjoint in A, $\lambda,\ \alpha \in \mathbb{R}$ with $0 \leq \lambda < 1,\ 0 \leq \alpha < 1,\ 0 \leq a < 1$, then*

$$f(\lambda) f(\lambda \alpha a) \geq f(\lambda \alpha) f(\lambda a) \tag{4.3}$$

in the order of A.

Proof. From $0 \leq a < 1$ we get by multiplying both sides by $a^{1/2}$ that $0 \leq a^2 \leq a$ and in general

$$0 \leq ... \leq a^n \leq ... \leq a^2 \leq a < 1$$

for all $n \geq 1$, which shows that the sequence $b_n := a^n$ is monotonic nonincreasing.

Since $\gamma_n := \alpha^n$ is also nonincreasing and $p_n := \rho_n \lambda^n$ is nonnegative, then by Čebyšev's inequality for synchronous sequences, we have

$$\sum_{n=0}^{m} \rho_n \lambda^n \sum_{n=0}^{m} \rho_n \lambda^n \alpha^n a^n \geq \sum_{n=0}^{m} \rho_n \lambda^n \alpha^n \sum_{n=0}^{m} \rho_n \lambda^n a^n,$$

namely

$$\sum_{n=0}^{m} \rho_n \lambda^n \sum_{n=0}^{m} \rho_n (\lambda \alpha a)^n \geq \sum_{n=0}^{m} \rho_n (\lambda \alpha)^n \sum_{n=0}^{m} \rho_n (\lambda a)^n . \tag{4.4}$$

Since the series $\sum_{n=0}^{\infty} \rho_n \lambda^n$, $\sum_{n=0}^{\infty} \rho_n (\lambda \alpha a)^n$, $\sum_{n=0}^{\infty} \rho_n (\lambda \alpha)^n$ and $\sum_{n=0}^{\infty} \rho_n (\lambda a)^n$ are convergent and

$$\sum_{n=0}^{\infty} \rho_n \lambda^n = f(\lambda), \quad \sum_{n=0}^{\infty} \rho_n (\lambda \alpha a)^n = f(\lambda \alpha a), \quad \sum_{n=0}^{\infty} \rho_n (\lambda \alpha)^n = f(\lambda \alpha)$$

and $\sum_{n=0}^{\infty} \rho_n (\lambda a)^n = f(\lambda a)$, then by taking the limit over $m \to \infty$ and using Corollary 1, we deduce the required inequality (4.3) □

Remark 3. *If $a \in A$ is selfadjoint in A, λ, $\alpha \in \mathbb{R}$ with $0 \leq \lambda < 1$, $0 \leq \alpha < 1$, $0 \leq a < 1$, then*

$$(1-\lambda)^{-1}(1-\lambda\alpha a)^{-1} \geq (1-\lambda\alpha)^{-1}(1-\lambda a)^{-1}, \tag{4.5}$$

$$\ln(1-\lambda)^{-1} \ln(1-\lambda\alpha a)^{-1} \geq \ln(1-\lambda\alpha)^{-1} \ln(1-\lambda a)^{-1} \tag{4.6}$$

and

$$\sin^{-1}(\lambda) \sin^{-1}(\lambda \alpha a) \geq \sin^{-1}(\lambda \alpha) \sin^{-1}(\lambda a) . \tag{4.7}$$

We also have

$$\exp[\lambda(1+\alpha a)] \geq \exp[\lambda(\alpha + a)] . \tag{4.8}$$

We remark that if the radius of convergence $R = \infty$, then we can prove in a similar way that

$$f(\lambda) f(\lambda \alpha a) \geq f(\lambda \alpha) f(\lambda a) \tag{4.9}$$

provided that $\lambda \geq 0$ and $\alpha \geq 1$, $a \geq 1$.

We observe also that the sign of inequality reverses in (4.9) if either $0 \leq \alpha < 1$ and $1 \leq a$ or $0 \leq a < 1$ and $1 \leq \alpha$.

The inequality (4.8) also holds for $\lambda \geq 0$ and $\alpha \geq 1$, $a \geq 1$. The sign of inequality reverses in (4.8) if either $0 \leq \alpha < 1$ and $1 \leq a$ or $0 \leq a < 1$ and $1 \leq \alpha$.

Similar inequalities may be stated for the other functions listed above, however the details are omitted.

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