CBS TYPE MODULUS INEQUALITIES FOR FORWARD DIFFERENCE IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS WITH APPLICATIONS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$. Suppose that A has a continuous involution. In this paper we show among others thst, if $\{\gamma_i\}_{i=0}^N \subset \mathbb{C}$ and $\{a_i\}_{i=0}^N \subset A$ are sequences with $\gamma_0 = 0$ and $a_N = 0$, then

$$\sum_{i=1}^{N-1} |\gamma_i a_i| \leq \frac{1}{2} N \left[\sum_{i=0}^{N-1} (N-i) |\Delta\gamma_i|^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) |\Delta a_i|^2 \right]^{1/2}$$
$$\leq \frac{1}{2} N^2 \left[\sum_{i=0}^{N-1} |\Delta\gamma_i|^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} |\Delta a_i|^2 \right]^{1/2},$$

where $\Delta\gamma_j := \gamma_{j+1} - \gamma_j$ is the forward difference.

1. Introduction

Let A be a unital Banach $*$-algebra with unit 1. An element $a \in A$ is called *selfadjoint* if $a^* = a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a) \subset \mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a \geq 0$ if $a^* = a$ and $\sigma(a) \subset [0, \infty)$. We say that a is *positive* and write $a > 0$ if $a \geq 0$ and $0 \notin \sigma(a)$. Thus $a > 0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\text{Inv}(A)$. If $a, b \in \text{Inv}(A)$, then $ab \in \text{Inv}(A)$ and $(ab)^{-1} = b^{-1}a^{-1}$. Also, saying that $a \geq b$ means that $a - b \geq 0$ and, similarly $a > b$ means that $a - b > 0$.

The *Shirali-Ford theorem* asserts that [13] (see also [1, Theorem 41.5])

$$a^*a \geq 0 \text{ for every } a \in A. \tag{SF}$$

Based on this fact, Okayasu [12], Tanahashi and Uchiyama [14] proved the following fundamental properties (see also [8]):

(i) If a, $b \in A$, then $a \geq 0$, $b \geq 0$ imply $a + b \geq 0$ and $\alpha \geq 0$ implies $\alpha a \geq 0$;
(ii) If a, $b \in A$, then $a > 0$, $b \geq 0$ imply $a + b > 0$;
(iii) If a, $b \in A$, then either $a \geq b > 0$ or $a > b \geq 0$ imply $a > 0$;
(iv) If $a > 0$, then $a^{-1} > 0$;
(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$, also $0 < b \leq a$ if and only if $cbc \leq cac$;

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Weighted geometric mean, Weighted harmonic mean, Young's inequality, Operator modulus, Arithmetic mean-geometric mean-harmonic mean inequality.

RGMIA Res. Rep. Coll. 24 (2021), Art. 84, 15 pp. Received 28/06/21

(vi) If $0 < a < 1$, then $1 < a^{-1}$;
(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$, also if $0 < b \le a$, then $0 < a^{-1} \le b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) := \frac{1}{2\pi i} \int_\gamma f(z)(z-a)^{-1} dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z : z \in \sigma(a)\} > 0$ and $\sup\{z : z \in \sigma(a)\} < \infty$. Choose γ to be close rectifiable curve in $\{\operatorname{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \operatorname{ins}(\gamma)$, the inside of γ. For any $\alpha \in \mathbb{R}$ we define for $a \in A$ and $a > 0$, the real power

$$a^\alpha := \frac{1}{2\pi i} \int_\gamma z^\alpha (z-a)^{-1} dz,$$

where z^α is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^\alpha \in A$. Moreover, since z^α is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma(a^\alpha) = (\sigma(a))^\alpha = \{z^\alpha : z \in \sigma(a)\} \subset (0, \infty).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^\alpha \in A$ with $a^\alpha > 0$ and $(a^2)^{1/2} = a$, [14, Lemma 6];
(ix) If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^\alpha a^\beta = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $(a^\alpha)^{-1} = (a^{-1})^\alpha = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^\alpha b^\beta = b^\beta a^\alpha$.

Okayasu [12] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if $a, b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \ge b)$ implies that $a^p > b^p$ $(a^p \ge b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [3]-[6].

By *Shirali-Ford theorem* we have $a^* a \ge 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^* a)^{1/2} \ge 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a + b| \le |a| + |b|, \ a, \ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

Suppose that A has a continuous involution. In this paper we show among others that, if $\{\gamma_i\}_{i=0}^N \subset C$ and $\{a_i\}_{i=0}^N \subset A$ are sequences with $\gamma_0 = 0$ and $a_N = 0$, then

$$\sum_{i=1}^{N-1} |\gamma_i a_i| \le \frac{1}{2} N \left[\sum_{i=0}^{N-1} (N-i) |\Delta\gamma_i|^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) |\Delta a_i|^2 \right]^{1/2} \le \frac{1}{2} N^2 \left[\sum_{i=0}^{N-1} |\Delta\gamma_i|^2 \right]^{1/2} \left[\sum_{i=0}^{N-1} |\Delta a_i|^2 \right]^{1/2},$$

where $\Delta\gamma_j := \gamma_{j+1} - \gamma_j$ is the forward difference.

2. Main Results

For a sequence $\{x_i\}_{i=0}^N$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., N-1$. Recall the summation by parts formula stated as

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k, \tag{2.1}$$

where a_k and b_k are some sequences for which the products above exist.

We start to the following identities of interest:

Lemma 1. *Let $a_k \in A$, $\gamma_k \in \mathbb{C}$ and $p_k \ge 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k |\gamma_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{j=1}^{n} p_j \gamma_j a_j \right|^2 = \frac{1}{2} \sum_{j,k=1}^{n} p_j p_k \left| \overline{\gamma_j} a_k - \overline{\gamma}_k a_j \right|^2. \tag{2.2}$$

In particular,

$$\sum_{k=1}^{n} |\gamma_k|^2 \sum_{k=1}^{n} |a_k|^2 - \left| \sum_{j=1}^{n} \gamma_j a_j \right|^2 = \frac{1}{2} \sum_{j,k=1}^{n} \left| \overline{\gamma_j} a_k - \overline{\gamma}_k a_j \right|^2 \tag{2.3}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 - \left| \sum_{j=1}^{n} p_j a_j \right|^2 = \frac{1}{2} \sum_{j,k=1}^{n} p_j p_k |a_k - a_j|^2. \tag{2.4}$$

Proof. Observe that

$$\begin{aligned}
&\left| \overline{\gamma_j} a_k - \overline{\gamma}_k a_j \right|^2 \\
&= \left(\overline{\gamma_j} a_k - \overline{\gamma}_k a_j \right)^* \left(\overline{\gamma_j} a_k - \overline{\gamma}_k a_j \right) = \left(\gamma_j a_k^* - \gamma_k a_j^* \right) \left(\overline{\gamma_j} a_k - \overline{\gamma}_k a_j \right) \\
&= \gamma_j a_k^* \overline{\gamma_j} a_k - \gamma_j a_k^* \overline{\gamma}_k a_j - \gamma_k a_j^* \overline{\gamma_j} a_k + \gamma_k a_j^* \overline{\gamma}_k a_j \\
&= \left| \gamma_j \right|^2 |a_k|^2 - \overline{\gamma}_k a_k^* \gamma_j a_j - \overline{\gamma_j} a_j^* \gamma_k a_k + |\gamma_k|^2 |a_j|^2
\end{aligned}$$

for all j, $k \in \{1, ..., n\}$.

This implies that

$$\begin{aligned}
&\sum_{j,k=1}^{n} p_j p_k \left|\overline{\gamma_j} a_k - \overline{\gamma}_k a_j\right|^2 \\
&= \sum_{j,k=1}^{n} p_j p_k \left[\left|\gamma_j\right|^2 \left|a_k\right|^2 - \overline{\gamma}_k a_k^* \gamma_j a_j - \overline{\gamma_j} a_j^* \gamma_k a_k + \left|\gamma_k\right|^2 \left|a_j\right|^2\right] \\
&= \sum_{j,k=1}^{n} p_j p_k \left|\gamma_j\right|^2 \left|a_k\right|^2 - \sum_{j,k=1}^{n} p_j p_k \overline{\gamma}_k a_k^* \gamma_j a_j \\
&- \sum_{j,k=1}^{n} p_j p_k \overline{\gamma_j} a_j^* \gamma_k a_k + \sum_{j,k=1}^{n} p_j p_k \left|\gamma_k\right|^2 \left|a_j\right|^2
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n} p_j \left|\gamma_j\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \sum_{k=1}^{n} p_k \overline{\gamma}_k a_k^* \sum_{j=1}^{n} p_j \gamma_j a_j \\
&- \sum_{j=1}^{n} p_j \overline{\gamma_j} a_j^* \sum_{k=1}^{n} p_k \gamma_k a_k + \sum_{k=1}^{n} p_k \left|\gamma_k\right|^2 \sum_{j=1}^{n} p_j \left|a_j\right|^2 \\
&= \sum_{j=1}^{n} p_j \left|\gamma_j\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left(\sum_{k=1}^{n} p_k \gamma_k a_k\right)^* \sum_{j=1}^{n} p_j \gamma_j a_j \\
&- \left(\sum_{j=1}^{n} p_j \gamma_j a_j\right)^* \sum_{k=1}^{n} p_k \gamma_k a_k + \sum_{k=1}^{n} p_k \left|\gamma_k\right|^2 \sum_{j=1}^{n} p_j \left|a_j\right|^2 \\
&= 2\left[\sum_{k=1}^{n} p_k \left|\gamma_k\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 - \left|\sum_{k=1}^{n} p_k \gamma_k a_k\right|^2\right],
\end{aligned}$$

which is equivalent to the desired identity (2.2). □

We have the following Cauchy-Bunyakowsky-Schwarz (CBS) type inequalities:

Corollary 1. *Let $a_k \in A$, $\gamma_k \in \mathbb{C}$ and $p_k > 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k \left|\gamma_k\right|^2 \sum_{k=1}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} p_j \gamma_j a_j\right|^2. \tag{2.5}$$

In particular,

$$\sum_{k=1}^{n} \left|\gamma_k\right|^2 \sum_{k=1}^{n} \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} \gamma_j a_j\right|^2 \tag{2.6}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k \left|a_k\right|^2 \geq \left|\sum_{j=1}^{n} p_j a_j\right|^2. \tag{2.7}$$

The equality holds in (2.7) if and only if $a_k = a$ for some $a \in A$ and all $k \in \{1, ..., n\}$.

Remark 1. *If A has a continuous involution, then we can take the square root to obtain the inequalities*

$$\left(\sum_{k=1}^{n} p_k |\gamma_k|^2\right) \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} p_j \gamma_j a_j\right|. \tag{2.8}$$

In particular,

$$\left(\sum_{k=1}^{n} |\gamma_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} \gamma_j a_j\right| \tag{2.9}$$

and

$$\left(\sum_{k=1}^{n} p_k\right)^{1/2} \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} p_j a_j\right|. \tag{2.10}$$

We have the following result for two sequences:

Theorem 1. *Suppose that A has a continuous involution. Assume that $\{\gamma_i\}_{i=0}^{N} \subset \mathbb{C}$ and $\{a_i\}_{i=0}^{N} \subset A$ are sequences with $\gamma_0 = 0$ and $a_0 = 0$, then*

$$\begin{aligned} \sum_{i=1}^{N} |\gamma_i a_i| &\leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta\gamma_i|^2\right)^{1/2} \\ &\quad \times \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta a_i|^2\right)^{1/2} \\ &\leq \frac{1}{4} \sum_{i=0}^{N-1} (N-i)(N+i+1) \left(|\Delta\gamma_i|^2 + |\Delta a_i|^2\right). \end{aligned} \tag{2.11}$$

Proof. Since $\gamma_0 = 0$ and $a_0 = 0$, hence $\gamma_i = \sum_{j=0}^{i-1} \Delta\gamma_j$ and $a_i = \sum_{j=0}^{i-1} \Delta a_j$ for $i = 1, ..., N$. Then

$$\begin{aligned} \sum_{i=1}^{N} |\gamma_i| |a_i| &= \sum_{i=1}^{N} \left|\sum_{j=0}^{i-1} \Delta\gamma_j\right| \left|\sum_{j=0}^{i-1} \Delta a_j\right| \\ &= \sum_{i=1}^{N} i \frac{1}{\sqrt{i}} \left|\sum_{j=0}^{i-1} \Delta\gamma_j\right| \frac{1}{\sqrt{i}} \left|\sum_{j=0}^{i-1} \Delta a_j\right| =: K. \end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality for scalars and elements in A, (2.10), we have

$$\frac{1}{\sqrt{i}} \left|\sum_{j=0}^{i-1} \Delta\gamma_j\right| \leq \left(\sum_{j=0}^{i-1} |\Delta\gamma_j|^2\right)^{1/2}$$

and

$$\frac{1}{\sqrt{i}} \left|\sum_{j=0}^{i-1} \Delta a_j\right| \leq \left(\sum_{j=0}^{i-1} |\Delta a_j|^2\right)^{1/2},$$

which gives

$$K\leq\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}\right)^{1/2}\left(\sum_{j=0}^{i-1}\left|\Delta a_{j}\right|^{2}\right)^{1/2}.$$

By the weighted (CBS) inequality in A we also have

$$\begin{aligned}&\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}\right)^{1/2}\left(\sum_{j=0}^{i-1}\left|\Delta a_{j}\right|^{2}\right)^{1/2}\\&\leq\left(\sum_{i=1}^{N}i\left[\left(\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}\right)^{1/2}\right]^{2}\right)^{1/2}\left(\sum_{i=1}^{N}i\left[\left(\sum_{j=0}^{i-1}\left|\Delta a_{j}\right|^{2}\right)^{1/2}\right]^{2}\right)^{1/2}\\&=\left(\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}\right)\right)^{1/2}\left(\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta a_{j}\right|^{2}\right)\right)^{1/2},\end{aligned}$$

which implies that

$$A\leq\left(\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}\right)\right)^{1/2}\left(\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta a_{j}\right|^{2}\right)\right)^{1/2}=:B.\tag{2.12}$$

From the formula (2.1), we get

$$\sum_{i=1}^{N}\beta_{i}\Delta b_{i}=\beta_{N}b_{N+1}-\beta_{1}b_{1}-\sum_{i=1}^{N-1}\left(\Delta\beta_{i}\right)b_{i+1}.\tag{2.13}$$

Now, if we take $\beta_{i}=\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2},\ i=1,...,N,\ b_{i}=\frac{1}{2}i\left(i-1\right),$ then $\beta_{N}=\sum_{j=0}^{N-1}\left|\Delta\gamma_{j}\right|^{2},$

$$\Delta b_{i}=b_{i+1}-b_{i}=\frac{1}{2}i\left(i+1\right)-\frac{1}{2}i\left(i-1\right)=i,$$

and

$$\Delta\beta_{i}=\beta_{i+1}-\beta_{i}=\sum_{j=0}^{i}\left|\Delta\gamma_{j}\right|^{2}-\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}=\left|\Delta\gamma_{i}\right|^{2}.$$

By (2.13) we derive

$$\begin{aligned}\sum_{i=1}^{N}i\left(\sum_{j=0}^{i-1}\left|\Delta\gamma_{j}\right|^{2}\right)&=\frac{1}{2}N\left(N+1\right)\sum_{j=0}^{N-1}\left|\Delta\gamma_{j}\right|^{2}-\sum_{k=1}^{N-1}\frac{1}{2}i\left(i+1\right)\left|\Delta\gamma_{i}\right|^{2}\\&=\sum_{i=0}^{N-1}\left[\frac{1}{2}N\left(N+1\right)-\frac{1}{2}i\left(i+1\right)\right]\left|\Delta\gamma_{i}\right|^{2}\\&=\frac{1}{2}\sum_{i=0}^{N-1}\left(N-i\right)\left(N+i+1\right)\left|\Delta\gamma_{i}\right|^{2}\end{aligned}\tag{2.14}$$

and, similarly

$$\sum_{i=1}^{N} i\left(\sum_{j=0}^{i-1}\left|\Delta a_j\right|^2\right)=\frac{1}{2}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta a_i\right|^2. \tag{2.15}$$

Therefore

$$B=\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta a_i\right|^2\right)^{1/2}$$

and by (2.12) we derive the first inequality in (2.11).

The last inequality in (2.11) follows by the elementary inequality

$$\beta b\leq\frac{1}{2}\left(b^2+\beta^2\right) \tag{2.16}$$

that holds for real β and an element $b\in A$, b selfadjoint. □

Remark 2. *Now observe that*

$$\begin{aligned}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta\gamma_i\right|^2&\leq\max_{i\in\{0,\dots,N-1\}}\left|\Delta\gamma_i\right|^2\sum_{i=0}^{N-1}(N-i)(N+i+1)\\&=\frac{1}{3}N(N+1)(2N+1)\max_{i\in\{0,\dots,N-1\}}\left|\Delta\gamma_i\right|^2,\end{aligned}$$

which proves the inequality

$$\begin{aligned}\sum_{i=1}^{N}\left|\gamma_i a_i\right|&\leq\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta\gamma_i\right|^2\right)^{1/2}\\&\quad\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta a_i\right|^2\right)^{1/2}\\&\leq\frac{1}{2}\left(\frac{1}{3}N(N+1)(2N+1)\right)^{1/2}\max_{i\in\{0,\dots,N-1\}}\left|\Delta\gamma_i\right|\\&\quad\times\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta a_i\right|^2\right)^{1/2}.\end{aligned} \tag{2.17}$$

Observe also that

$$\begin{aligned}\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta\gamma_i\right|^2&\leq\max_{i\in\{0,\dots,N-1\}}\left[(N-i)(N+i+1)\right]\sum_{i=0}^{N-1}\left|\Delta\gamma_i\right|^2\\&=\max_{i\in\{0,\dots,N-1\}}\left[N(N+1)-i(i+1)\right]\sum_{i=0}^{N-1}\left|\Delta\gamma_i\right|^2\\&=N(N+1)\sum_{i=0}^{N-1}\left|\Delta\gamma_i\right|^2,\end{aligned}$$

which proves that

$$\sum_{i=1}^{N}|\gamma_i a_i| \le \frac{1}{2}\sqrt{N(N+1)}\left(\sum_{i=0}^{N-1}|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta a_i|^2\right)^{1/2}. \tag{2.18}$$

Moreover, since

$$\begin{aligned}\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta a_i|^2 &\le \max_{i\in\{0,\dots,N-1\}}[(N-i)(N+i+1)]\sum_{i=0}^{N-1}|\Delta a_i|^2\\ &= N(N+1)\sum_{i=0}^{N-1}|\Delta a_i|^2\end{aligned}$$

hence by taking the square root, we get

$$\left(\sum_{i=0}^{N-1}(N-i)(N+i+1)|\Delta a_i|^2\right)^{1/2} \le \sqrt{N(N+1)}\left(\sum_{i=0}^{N-1}|\Delta a_i|^2\right)^{1/2}$$

and we derive the simpler inequality

$$\sum_{i=1}^{N}|\gamma_i a_i| \le \frac{1}{2}N(N+1)\left(\sum_{i=0}^{N-1}|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}|\Delta a_i|^2\right)^{1/2}. \tag{2.19}$$

Also, we have:

Theorem 2. *With the assumptions of Theorem 1 and if* $\gamma_N = 0$, $a_N = 0$, *then*

$$\begin{aligned}\sum_{i=0}^{N-1}|\gamma_i a_i| &\le \frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)|\Delta a_i|^2\right)^{1/2}\\ &\le \frac{1}{4}\sum_{i=0}^{N-1}(i+1)(2N-i)\left(|\Delta\gamma_i|^2+|\Delta a_i|^2\right).\end{aligned} \tag{2.20}$$

Proof. If $\gamma_N = 0$ and $a_N = 0$, then $\gamma_i = -\sum_{j=i}^{N-1}\Delta\gamma_j$ and $a_i = -\sum_{j=i}^{N-1}\Delta a_j$, $i \in \{0,...,N-1\}$. Then

$$\begin{aligned}\sum_{i=0}^{N-1}|\gamma_i||a_i| &= \sum_{i=0}^{N-1}\left|\sum_{j=i}^{N-1}\Delta\gamma_j\right|\left|\sum_{j=i}^{N-1}\Delta a_j\right|\\ &= \sum_{i=0}^{N-1}(N-i)\frac{1}{\sqrt{N-i}}\left|\sum_{j=i}^{N-1}\Delta\gamma_j\right|\frac{1}{\sqrt{N-i}}\left|\sum_{j=i}^{N-1}\Delta a_j\right| =: C.\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz (CBS) inequality we have

$$\frac{1}{\sqrt{N-i}}\left|\sum_{j=i}^{N-1}\Delta\gamma_j\right| \le \left(\sum_{j=i}^{N-1}|\Delta\gamma_j|^2\right)^{1/2}$$

and

$$\frac{1}{\sqrt{N-i}}\left|\sum_{j=i}^{N-1}\Delta a_j\right| \leq \left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)^{1/2},$$

which gives

$$C \leq \sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2\right)^{1/2}\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)^{1/2}.$$

By the weighted (CBS) inequality we have

$$\begin{aligned}&\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2\right)^{1/2}\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)^{1/2}\\&\leq\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2\right)\right)^{1/2}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)\right)^{1/2},\end{aligned}$$

which gives

$$\begin{aligned}C &\leq \left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2\right)\right)^{1/2}\left(\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)\right)^{1/2}\\&=: D.\end{aligned} \tag{2.21}$$

From (2.1) we get for $m=0$ and $n=N-1$ that

$$\sum_{i=0}^{N-1}\beta_i\Delta b_i = \beta_{N-1}b_N - \beta_0 b_0 - \sum_{i=0}^{N-2}(\Delta\beta_i)\, b_{i+1}. \tag{2.22}$$

Take $\beta_i = \sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2$ and $b_i = -\frac{1}{2}(N-i)(N-i+1)$, then we get

$$\begin{aligned}\Delta b_i = b_{i+1} - b_i &= -\frac{1}{2}(N-i-1)(N-i) + \frac{1}{2}(N-i)(N-i+1)\\&= \frac{1}{2}(N-i)(-N+i+1+N-i+1) = N-i\end{aligned}$$

and

$$\Delta\beta_i = \beta_{i+1} - \beta_i = \sum_{j=i+1}^{N-1}\left|\Delta\gamma_j\right|^2 - \sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2 = -\left|\Delta\gamma_i\right|^2.$$

Then

$$\begin{aligned}
&\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta\gamma_j\right|^2\right)\\
&=\frac{1}{2}N(N+1)\sum_{j=0}^{N-1}\left|\Delta\gamma_j\right|^2-\frac{1}{2}\sum_{i=0}^{N-2}(N-i-1)(N-i)\left|\Delta\gamma_i\right|^2\\
&=\frac{1}{2}N(N+1)\sum_{i=0}^{N-1}\left|\Delta\gamma_j\right|^2-\frac{1}{2}\sum_{i=0}^{N-1}(N-i-1)(N-i)\left|\Delta\gamma_i\right|^2\\
&=\frac{1}{2}\sum_{i=0}^{N-1}\left[N(N+1)-(N-i-1)(N-i)\right]\left|\Delta\gamma_i\right|^2\\
&=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta\gamma_i\right|^2
\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(N-i)\left(\sum_{j=i}^{N-1}\left|\Delta a_j\right|^2\right)=\frac{1}{2}\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta a_i\right|^2.$$

Therefore

$$D=\frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta a_i\right|^2\right)^{1/2},$$

and by (2.21) we derive the first inequality in (2.20).

The last part follows by (2.16). □

Remark 3. *In a similar way we can prove that*

$$\begin{aligned}
&\sum_{i=0}^{N-1}\left|\gamma_i a_i\right| \\
&\leq\frac{1}{2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta a_i\right|^2\right)^{1/2}\\
&\leq\frac{1}{8}(2N+1)^2\left(\sum_{i=0}^{N-1}\left|\Delta\gamma_i\right|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}\left|\Delta a_i\right|^2\right)^{1/2},
\end{aligned} \tag{2.23}$$

since

$$(i+1)(2N-i)\leq\frac{1}{4}(i+1+2N-i)^2=\frac{1}{4}(2N+1)^2$$

for $i\in\{0,...,N-1\}$.

We also have:

Theorem 3. *With the assumptions of Theorem 1 and if* $\gamma_0 = 0$ *and* $a_N = 0$, *then*

$$\sum_{i=1}^{N-1} |\gamma_i a_i| \leq \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta\gamma_i|^2 \right)^{1/2} \times \left(\sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta a_i|^2 \right)^{1/2} \leq \frac{1}{8} (2N+1)^2 \left(\sum_{i=0}^{N-1} |\Delta\gamma_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} |\Delta a_i|^2 \right)^{1/2}. \tag{2.24}$$

Proof. Since $\gamma_0 = 0$ and $a_N = 0$, hence $\gamma_i = \sum_{j=0}^{i-1} \Delta\gamma_j$ and $a_i = -\sum_{j=i}^{N-1} \Delta a_j$. Then by (CBS) inequality

$$\begin{aligned} \sum_{i=1}^{N-1} |\gamma_i a_i| &= \sum_{i=1}^{N-1} |\gamma_i| |a_i| = \sum_{i=1}^{N-1} \left| \sum_{j=0}^{i-1} \Delta\gamma_j \right| \left| \sum_{j=i}^{N-1} \Delta a_j \right| \\ &\leq \sum_{i=1}^{N-1} \sqrt{i} \left(\sum_{j=0}^{i-1} |\Delta\gamma_j|^2 \right)^{1/2} \sqrt{N-i} \left(\sum_{j=i}^{N-1} |\Delta a_j|^2 \right)^{1/2} \\ &\leq \left(\sum_{i=1}^{N-1} \left[\sqrt{i} \left(\sum_{j=0}^{i-1} |\Delta\gamma_j|^2 \right)^{1/2} \right]^2 \right)^{1/2} \\ &\quad \times \left(\sum_{i=1}^{N-1} \left[\sqrt{N-i} \left(\sum_{j=i}^{N-1} |\Delta a_j|^2 \right)^{1/2} \right]^2 \right)^{1/2} \\ &= \left(\sum_{i=1}^{N-1} i \left(\sum_{j=0}^{i-1} |\Delta\gamma_j|^2 \right) \right)^{1/2} \left(\sum_{i=1}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} |\Delta a_j|^2 \right) \right)^{1/2} =: E. \end{aligned} \tag{2.25}$$

Since

$$\sum_{i=1}^{N} i \left(\sum_{j=0}^{i-1} |\Delta\gamma_j|^2 \right) = \frac{1}{2} \sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta\gamma_i|^2$$

and

$$\sum_{i=0}^{N-1} (N-i) \left(\sum_{j=i}^{N-1} |\Delta a_j|^2 \right) = \frac{1}{2} \sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta a_i|^2,$$

hence

$$E = \frac{1}{2} \left(\sum_{i=0}^{N-1} (N-i)(N+i+1) |\Delta\gamma_i|^2 \right)^{1/2} \left(\sum_{i=0}^{N-1} (i+1)(2N-i) |\Delta a_i|^2 \right)^{1/2}.$$

By employing (2.25) we get the first part of (2.24).

The rest follows by the fact that

$$(N-i)(N+i+1) \leq \frac{1}{4} (N+i+1+N-i)^2 = \frac{1}{4} (2N+1)^2$$

and

$$(i+1)(2N-i) \le \frac{1}{4}(2N+1)^2$$

for $i \in \{0, ..., N-1\}$. □

3. Related Results

From a different perspective, we also have:

Theorem 4. *With the assumptions of Theorem 1, if* $\gamma_0 = 0$ *and* $a_N = 0$, *then*

$$\sum_{i=1}^{N-1} |\gamma_i a_i| \tag{3.1}$$
$$\le \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i-1)(N-i)|\Delta\gamma_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta a_i|^2\right]^{1/2}$$
$$\le \frac{1}{2}N(N-1)\left(\sum_{i=0}^{N-1}|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}|\Delta a_i|^2\right)^{1/2}.$$

Proof. From (2.25) we also have

$$\sum_{i=1}^{N-1} |\gamma_i a_i| \tag{3.2}$$
$$= \sum_{i=1}^{N-1}|\gamma_i||a_i| = \sum_{i=1}^{N-1}\left|\sum_{j=0}^{i-1}\Delta\gamma_j\right|\left|\sum_{j=i}^{N-1}\Delta a_j\right|$$
$$\le \sum_{i=1}^{N-1}\sqrt{i}\left(\sum_{j=0}^{i-1}|\Delta\gamma_j|^2\right)^{1/2}\sqrt{N-i}\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)^{1/2}$$
$$= \sum_{i=1}^{N-1}\sqrt{N-i}\left(\sum_{j=0}^{i-1}|\Delta\gamma_j|^2\right)^{1/2}\sqrt{i}\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)^{1/2}$$
$$\le \left[\sum_{i=1}^{N-1}(N-i)\left(\sum_{j=0}^{i-1}|\Delta\gamma_j|^2\right)\right]^{1/2}\left[\sum_{i=1}^{N-1} i\left(\sum_{j=i}^{N-1}|\Delta a_j|^2\right)\right]^{1/2} =: F.$$

We use the identity

$$\sum_{i=1}^{N}\beta_i\Delta b_i = \beta_N b_{N+1} - \beta_1 b_1 - \sum_{i=1}^{N-1}(\Delta\beta_i)\, b_{i+1}. \tag{3.3}$$

Take $\beta_i = \sum_{j=0}^{i-1}|\Delta\gamma_j|^2$ and $b_i = -\frac{1}{2}(N-i)(N-i+1)$, then

$$\Delta b_i = b_{i+1} - b_i = -\frac{1}{2}(N-i-1)(N-i) + \frac{1}{2}(N-i)(N-i+1)$$
$$= \frac{1}{2}(N-i)[N-i+1-(N-i-1)] = (N-i),$$

$$\Delta\beta_i = \beta_{i+1} - \beta_i = \sum_{j=0}^{i} \left|\Delta\gamma_j\right|^2 - \sum_{j=0}^{i-1} \left|\Delta\gamma_j\right|^2 = \left|\Delta\gamma_i\right|^2$$

and by (3.3)

$$\begin{aligned}(3.4)\quad &\sum_{i=1}^{N-1} (N-i) \left(\sum_{j=0}^{i-1} \left|\Delta\gamma_j\right|^2\right) \\ &= 0 \cdot \sum_{j=0}^{N-1} \left|\Delta\gamma_j\right|^2 + \frac{1}{2} N (N-1) \left|\Delta\gamma_0\right|^2 + \frac{1}{2} \sum_{i=1}^{N-1} (N-i-1)(N-i) \left|\Delta\gamma_i\right|^2 \\ &= \frac{1}{2} \sum_{i=0}^{N-1} (N-i-1)(N-i) \left|\Delta\gamma_i\right|^2 .\end{aligned}$$

Take $\beta_i = \sum_{j=i}^{N-1} \left|\Delta a_j\right|^2$ and $b_i = \frac{1}{2} i (i-1)$, then

$$\Delta b_i = b_{i+1} - b_i = \frac{1}{2}(i+1)i - \frac{1}{2} i (i-1) = \frac{1}{2} i (i+1-i+1) = i,$$

$$\Delta\beta_i = \beta_{i+1} - \beta_i = \sum_{j=i+1}^{N-1} \left|\Delta a_j\right|^2 - \sum_{j=i}^{N-1} \left|\Delta a_j\right|^2 = -\left|\Delta a_i\right|^2$$

and by the identity

$$\sum_{i=1}^{N-1} \beta_i \Delta b_i = \beta_{N-1} b_N - \beta_1 b_1 - \sum_{i=1}^{N-2} (\Delta\beta_i) b_{i+1}.$$

we get

$$\begin{aligned}(3.5)\quad \sum_{i=1}^{N-1} i \left(\sum_{j=i}^{N-1} \left|\Delta a_j\right|^2\right) &= \frac{1}{2} N (N-1) \left|\Delta a_{N-1}\right|^2 + \frac{1}{2} \sum_{i=1}^{N-2} (i+1) i \left|\Delta a_i\right|^2 \\ &= \frac{1}{2} \sum_{i=1}^{N-1} (i+1) i \left|\Delta a_i\right|^2 = \frac{1}{2} \sum_{i=0}^{N-1} (i+1) i \left|\Delta a_i\right|^2 .\end{aligned}$$

Therefore

$$\begin{aligned}F &= \left[\frac{1}{2} \sum_{i=0}^{N-1} (N-i-1)(N-i) \left|\Delta\gamma_i\right|^2\right]^{1/2} \left[\frac{1}{2} \sum_{i=0}^{N-1} (i+1) i \left|\Delta a_i\right|^2\right]^{1/2} \\ &= \frac{1}{2} \left[\sum_{i=0}^{N-1} (N-i-1)(N-i) \left|\Delta\gamma_i\right|^2\right]^{1/2} \left[\sum_{i=0}^{N-1} (i+1) i \left|\Delta a_i\right|^2\right]^{1/2},\end{aligned}$$

which proves the first inequality in (3.1).

Observe also that

$$\begin{aligned}\sum_{i=0}^{N-1} (N-i-1)(N-i) \left|\Delta\gamma_i\right|^2 &\le \max_{i\in\{0,\dots,N-1\}} \left[(N-i-1)(N-i)\right] \sum_{i=0}^{N-1} \left|\Delta\gamma_i\right|^2 \\ &= N (N-1) \sum_{i=0}^{N-1} \left|\Delta\gamma_i\right|^2\end{aligned}$$

and

$$\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta a_i|^2 \leq \max_{i\in\{0,\dots,N-1\}}[i\,(i+1)]\sum_{i=0}^{N-1}|\Delta a_i|^2 \leq N\,(N-1)\sum_{i=0}^{N-1}|\Delta a_i|^2\,,$$

which proves the second branch in (3.1). □

We also have the following simpler result:

Theorem 5. *With the assumptions of Theorem 1, if* $\gamma_0 = a_N = 0$, *then*

$$\sum_{i=1}^{N-1}|\gamma_i a_i| \leq \frac{1}{2}N\left[\sum_{i=0}^{N-1}(N-i)\,|\Delta\gamma_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\,|\Delta a_i|^2\right]^{1/2} \tag{3.6}$$
$$\leq \frac{1}{2}N^2\left[\sum_{i=0}^{N-1}|\Delta\gamma_i|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}|\Delta a_i|^2\right]^{1/2}.$$

Proof. From (2.24) we have

$$\sum_{i=1}^{N-1}|\gamma_i a_i| \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)\,(N+i+1)\,|\Delta\gamma_i|^2\right)^{1/2} \times\left(\sum_{i=0}^{N-1}(i+1)\,(2N-i)\,|\Delta a_i|^2\right)^{1/2}$$

and from (3.1)

$$\sum_{i=1}^{N-1}|\gamma_i a_i| \leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)\,(N-i)\,|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta a_i|^2\right)^{1/2}.$$

If we add these two inequalities, then we get by the elementary inequality

$$(\gamma b+\beta d)^2 \leq \left(\gamma^2+\beta^2\right)\left(b^2+d^2\right)$$

where γ, $\beta \geq 0$, a, $b \geq 0$, $a, b \in A$, which gives that

$$0 \leq \gamma b+\beta d \leq \left(\gamma^2+\beta^2\right)^{1/2}\left(b^2+d^2\right)^{1/2},$$

the following bounds

$$2\sum_{i=1}^{N-1}|\gamma_i a_i|$$
$$\leq \frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i)\,(N+i+1)\,|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)\,(2N-i)\,|\Delta a_i|^2\right)^{1/2}$$
$$+\frac{1}{2}\left(\sum_{i=0}^{N-1}(N-i-1)\,(N-i)\,|\Delta\gamma_i|^2\right)^{1/2}\left(\sum_{i=0}^{N-1}(i+1)\,i\,|\Delta a_i|^2\right)^{1/2}$$

$$\leq \frac{1}{2}\left[\sum_{i=0}^{N-1}(N-i)(N+i+1)\left|\Delta\gamma_i\right|^2+\sum_{i=0}^{N-1}(N-i-1)(N-i)\left|\Delta\gamma_i\right|^2\right]^{1/2}$$
$$\times\left[\sum_{i=0}^{N-1}(i+1)(2N-i)\left|\Delta a_i\right|^2+\sum_{i=0}^{N-1}(i+1)\,i\left|\Delta a_i\right|^2\right]^{1/2}$$
$$= N\left[\sum_{i=0}^{N-1}(N-i)\left|\Delta\gamma_i\right|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}(i+1)\left|\Delta a_i\right|^2\right]^{1/2}$$
$$\leq N^2\left[\sum_{i=0}^{N-1}\left|\Delta\gamma_i\right|^2\right]^{1/2}\left[\sum_{i=0}^{N-1}\left|\Delta a_i\right|^2\right]^{1/2},$$

which proves (3.6). □

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