DISCRETE OSTROWSKI TYPE MODULUS INEQUALITIES IN HERMITIAN UNITAL BANACH $*$-ALGEBRAS

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Abstract. Assume that A is a Hermitian unital Banach $*$-algebra. We can define the modulus of $a \in A$ by $|a| := (a^*a)^{1/2} \geq 0$. In this paper we show among others that, if $a_i \in A$, $i \in \{1, ..., n\}$, then

$$\left| a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \leq \left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2} \right)^2 \right] \sum_{j=1}^{n-1} |\Delta a_j|^2$$

for all $i \in \{1, ..., n\}$ where $\Delta a_j := a_{j+1} - a_j$ is the forward difference. Applications for Grüss type inequalities and discrete Fourier transform are also provided.

1. Introduction

In 2002, [3] we obtained the following discrete inequality of Ostrowski type:

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed linear space and x_i $(i = 1, \ldots, n)$ be vectors in X. Then we have the inequality*

$$\left\| x_i - \frac{1}{n} \sum_{k=1}^{n} x_k \right\| \leq \frac{1}{n} \left[\frac{n^2-1}{4} + \left(i - \frac{n+1}{2} \right)^2 \right] \max_{k=1,\ldots,n-1} \|\Delta x_k\| \tag{1.1}$$

for all $i \in \{1, \ldots, n\}$. The constant $c = \frac{1}{4}$ in the right hand side is best in the sense that it cannot be replaced by a smaller one.

As particular cases of interest, we note:

Corollary 1. *Under the above assumptions and if $n = 2m+1$, then we have the inequality*

$$\left\| x_{m+1} - \frac{1}{2m+1} \sum_{k=1}^{2m+1} x_k \right\| \leq \frac{m(m+1)}{2m+1} \max_{k=1,\ldots,2m} \|\Delta x_k\| . \tag{1.2}$$

The following corollary also holds.

Corollary 2. *Under the above assumptions, we have:*

a) *If $n = 2k$, then*

$$\left\| \frac{x_1 + x_{2k}}{2} - \frac{1}{2k} \sum_{j=1}^{2k} x_j \right\| \leq \frac{1}{2}(k-1) \max_{j=1,\ldots,2k-1} \|\Delta x_j\| . \tag{1.3}$$

1991 *Mathematics Subject Classification.* 47A63, 47A30, 15A60, 26D15, 26D10.

Key words and phrases. Weighted geometric mean, Weighted harmonic mean, Young's inequality, Operator modulus, Arithmetic mean-geometric mean-harmonic mean inequality.

RGMIA Res. Rep. Coll. 24 (2021), Art. 85, 15 pp. Received 28/06/21

b) *If* $n=2k+1$, *then*

$$\left\| \frac{x_1+x_{2k+1}}{2} - \frac{1}{2k+1}\sum_{j=1}^{2k+1} x_j \right\| \le \frac{2k^2+2k+1}{2(2k+1)} \max_{j=1,\dots,2k} \|\Delta x_j\| . \tag{1.4}$$

In order to extend these results for modulus in unital Hermitian Banach $*$-algebras, we need the following preparations.

Let A be a unital Banach $*$-algebra with unit 1. An element $a\in A$ is called *selfadjoint* if $a^*=a$. A is called *Hermitian* if every selfadjoint element a in A has real *spectrum* $\sigma(a)$, namely $\sigma(a)\subset\mathbb{R}$.

In what follows we assume that A is a Hermitian unital Banach $*$-algebra.

We say that an element a is *nonnegative* and write this as $a\ge 0$ if $a^*=a$ and $\sigma(a)\subset[0,\infty)$. We say that a is *positive* and write $a>0$ if $a\ge 0$ and $0\notin\sigma(a)$. Thus $a>0$ implies that its inverse a^{-1} exists. Denote the set of all invertible elements of A by $\mathrm{Inv}(A)$. If $a,b\in\mathrm{Inv}(A)$, then $ab\in\mathrm{Inv}(A)$ and $(ab)^{-1}=b^{-1}a^{-1}$. Also, saying that $a\ge b$ means that $a-b\ge 0$ and, similarly $a>b$ means that $a-b>0$.

The *Shirali-Ford theorem* asserts that [13] (see also [1, Theorem 41.5])

$$a^*a\ge 0 \text{ for every } a\in A. \tag{SF}$$

Based on this fact, Okayasu [12], Tanahashi and Uchiyama [14] proved the following fundamental properties (see also [8]):

(i) If $a,\,b\in A$, then $a\ge 0$, $b\ge 0$ imply $a+b\ge 0$ and $\alpha\ge 0$ implies $\alpha a\ge 0$;
(ii) If $a,\,b\in A$, then $a>0$, $b\ge 0$ imply $a+b>0$;
(iii) If $a,\,b\in A$, then either $a\ge b>0$ or $a>b\ge 0$ imply $a>0$;
(iv) If $a>0$, then $a^{-1}>0$;
(v) If $c>0$, then $0<b<a$ if and only if $cbc<cac$, also $0<b\le a$ if and only if $cbc\le cac$;
(vi) If $0<a<1$, then $1<a^{-1}$;
(vii) If $0<b<a$, then $0<a^{-1}<b^{-1}$, also if $0<b\le a$, then $0<a^{-1}\le b^{-1}$.

In order to introduce the real power of a positive element, we need the following facts [1, Theorem 41.5]. Let G be an open subset of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a):=\frac{1}{2\pi i}\int_\gamma f(z)(z-a)^{-1}\,dz,$$

where γ is chosen to be close rectifiable curve in G such that $\sigma(a)\subset\mathrm{ins}(\gamma)$, the inside of γ. It is well known (see for instance [2, pp. 201-204]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem (SMT)

$$\sigma(f(a))=f(\sigma(a))$$

holds.

Let $a\in A$ and $a>0$, then $0\notin\sigma(a)$ and the fact that $\sigma(a)$ is a compact subset of $\mathbb{C}$ implies that $\inf\{z:z\in\sigma(a)\}>0$ and $\sup\{z:z\in\sigma(a)\}<\infty$. Choose γ to be close rectifiable curve in $\{\mathrm{Re}\,z>0\}$, the right half open plane of the complex plane, such that $\sigma(a)\subset\mathrm{ins}(\gamma)$, the inside of γ. For any $\alpha\in\mathbb{R}$ we define for $a\in A$ and $a>0$, the real power

$$a^\alpha:=\frac{1}{2\pi i}\int_\gamma z^\alpha(z-a)^{-1}\,dz,$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, then $a^{\alpha} \in A$. Moreover, since z^{α} is analytic in $\{\operatorname{Re} z > 0\}$, then by (SMT) we have

$$\sigma\left(a^{\alpha}\right) = \left(\sigma\left(a\right)\right)^{\alpha} = \left\{z^{\alpha} : z \in \sigma\left(a\right)\right\} \subset \left(0, \infty\right).$$

Following [8], we list below some important properties of real powers:

(viii) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$ and $\left(a^2\right)^{1/2} = a$, [14, Lemma 6];
(ix) If $0 < a \in A$ and α, $\beta \in \mathbb{R}$, then $a^{\alpha}a^{\beta} = a^{\alpha+\beta}$;
(x) If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $\left(a^{\alpha}\right)^{-1} = \left(a^{-1}\right)^{\alpha} = a^{-\alpha}$;
(xi) If $0 < a$, $b \in A$, α, $\beta \in \mathbb{R}$ and $ab = ba$, then $a^{\alpha}b^{\beta} = b^{\beta}a^{\alpha}$.

Okayasu [12] showed that the *Löwner-Heinz inequality* remains valid in a Hermitian unital Banach $*$-algebra with continuous involution, namely if a, $b \in A$ and $p \in [0, 1]$ then $a > b$ $(a \geq b)$ implies that $a^p > b^p$ $(a^p \geq b^p)$.

For several recent inequalities in Hermitian unital Banach $*$-algebra, see [4]-[7].

By *Shirali-Ford theorem* we have $a^*a \geq 0$ for every $a \in A$, so we can define the absolute value or modulus of a by $|a| := (a^*a)^{1/2} \geq 0$. It is well know that if $A = \mathcal{B}(H)$, the C^*-algebra of bounded linear operators on a complex Hilbert space H, then the triangle inequality for the modulus

$$|a + b| \leq |a| + |b|, \ a, \ b \in A$$

does not hold in general, so the inequalities based on this inequality cannot be extended to the modulus in general.

In this paper we show among others that, if $a_i \in A$, $i \in \{1, ..., n\}$, then

$$\left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right|^2 \leq \left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2}\right)^2\right]\sum_{j=1}^{n-1}\left|\Delta a_j\right|^2$$

for all $i \in \{1, ..., n\}$ where $\Delta a_j := a_{j+1} - a_j$ is the forward difference. Applications for Grüss type inequalities and discrete Fourier transform are also provided.

2. Some Identities

For a sequence $\{x_i\}_{i=0}^{n}$, we consider the forward operator Δ defined by $\Delta x_i = x_{i+1} - x_i$, $i = 0, ..., n-1$. The summation by parts formula also holds

$$\sum_{j=m}^{k} a_j \Delta b_j = a_k b_{k+1} - a_m b_m - \sum_{k=m}^{k-1} b_{j+1} \Delta a_j, \tag{2.1}$$

provided that the products exist.

We have the following identity that is of interest in itself:

Lemma 1. *Let $x_i \in X$, $i \in \{1, ..., n\}$ vectors in the linear spaces X. Then*

$$x_i - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=0}^{i-1} j\Delta x_j + \frac{1}{n}\sum_{j=i}^{n}\left(j - n\right)\Delta x_j \tag{2.2}$$

for all $i \in \{1, ..., n\}$, where in the right side x_0 and x_{n+1} can be any vector in X.

In particular,

$$x_1 - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=1}^{n-1}\left(j - n\right)\Delta x_j, \tag{2.3}$$

$$x_n - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=1}^{n-1} j\Delta x_j \tag{2.4}$$

and

$$x_{\lfloor \frac{n+1}{2} \rfloor} - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor - 1} j\Delta x_j + \frac{1}{n}\sum_{j=\lfloor \frac{n+1}{2} \rfloor}^{n} (j-n)\,\Delta x_j. \tag{2.5}$$

Also

$$\frac{x_1 + x_n}{2} - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=1}^{n-1}\left(j - \frac{n}{2}\right)\Delta x_j. \tag{2.6}$$

Proof. If we apply (2.1) for $m = 1$, $k = i - 1$, $a_j = j$ and $b_j = x_j$, we have

$$\begin{aligned}\sum_{j=1}^{i-1} j\Delta x_j &= (i-1)\,x_i - x_1 - \sum_{j=1}^{i-2} x_{j+1} = ix_i - x_i - x_1 - \sum_{j=1}^{i-2} x_{j+1} \\ &= ix_i - \sum_{j=1}^{i} x_j.\end{aligned} \tag{2.7}$$

If we apply (2.1) for $m = i$, $k = n - 1$, $a_j = j - n$, $b_j = x_j$

$$\begin{aligned}\sum_{j=i}^{n-1} (j-n)\,\Delta x_j &= (n-1-n)\,x_n - (i-n)\,x_i - \sum_{k=i}^{n-2} x_{j+1} \\ &= -x_n + (n-i)\,x_i - \sum_{k=i}^{n-2} x_{j+1} = (n-i)\,x_i - \sum_{j=i+1}^{n} x_j.\end{aligned} \tag{2.8}$$

So, by adding (2.7) with (2.8) we get

$$\begin{aligned}\sum_{j=1}^{i-1} j\Delta x_j + \sum_{j=i}^{n-1} (j-n)\,\Delta x_j &= ix_i - \sum_{j=1}^{i} x_j + (n-i)\,x_i - \sum_{j=i+1}^{n} x_j \\ &= nx_i - \sum_{j=1}^{n} x_j.\end{aligned}$$

Since

$$\sum_{j=1}^{i-1} j\Delta x_j = \sum_{j=0}^{i-1} j\Delta x_j \text{ with } x_0 \text{ any vector}$$

and

$$\sum_{j=i}^{n-1} (j-n)\,\Delta x_j = \sum_{j=i}^{n} (j-n)\,\Delta x_j \text{ with } x_{n+1} \text{ any vector,}$$

hence we get

$$x_i - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=0}^{i-1} j\Delta x_j + \frac{1}{n}\sum_{j=i}^{n} (j-n)\,\Delta x_j$$

for all $i \in \{1, ..., n\}$, and the identity (2.2) is proved.

The rest follows by (2.2). □

Remark 1. *Consider the kernel*

$$p(i,j) := \begin{cases} j, \ 0 \le j \le i-1, \\ \\ j-n, \ i \le j \le n, \end{cases} \tag{2.9}$$

where $i \in \{1, ..., n\}$, *then we have from (2.2) that*

$$x_i - \frac{1}{n}\sum_{j=1}^{n} x_j = \frac{1}{n}\sum_{j=0}^{n} p(i,j)\,\Delta x_j \tag{2.10}$$

for $i \in \{1, ..., n\}$.

We start to the following identities of interest:

Lemma 2. *Let* $a_k \in A$, $\alpha_k \in \mathbb{C}$ *and* $p_k \ge 0$ *for* $k \in \{1, ..., n\}$. *Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|^2 = \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k \left|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right|^2. \tag{2.11}$$

In particular,

$$\sum_{k=1}^{n} |\alpha_k|^2 \sum_{k=1}^{n} |a_k|^2 - \left|\sum_{j=1}^{n} \alpha_j a_j\right|^2 = \frac{1}{2}\sum_{j,k=1}^{n} \left|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right|^2 \tag{2.12}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{j=1}^{n} p_j a_j\right|^2 = \frac{1}{2}\sum_{j,k=1}^{n} p_j p_k |a_k - a_j|^2. \tag{2.13}$$

Proof. Observe that

$$\begin{aligned}
&\left|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right|^2 \\
&= \left(\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right)^* \left(\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right) = \left(\alpha_j a_k^* - \alpha_k a_j^*\right)\left(\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right) \\
&= \alpha_j a_k^* \overline{\alpha_j} a_k - \alpha_j a_k^* \overline{\alpha}_k a_j - \alpha_k a_j^* \overline{\alpha_j} a_k + \alpha_k a_j^* \overline{\alpha}_k a_j \\
&= |\alpha_j|^2 |a_k|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + |\alpha_k|^2 |a_j|^2
\end{aligned}$$

for all $j, k \in \{1, ..., n\}$.

This implies that

$$\begin{aligned}
&\sum_{j,k=1}^{n} p_j p_k \left|\overline{\alpha_j} a_k - \overline{\alpha}_k a_j\right|^2 \\
&= \sum_{j,k=1}^{n} p_j p_k \left[|\alpha_j|^2 |a_k|^2 - \overline{\alpha}_k a_k^* \alpha_j a_j - \overline{\alpha_j} a_j^* \alpha_k a_k + |\alpha_k|^2 |a_j|^2\right] \\
&= \sum_{j,k=1}^{n} p_j p_k |\alpha_j|^2 |a_k|^2 - \sum_{j,k=1}^{n} p_j p_k \overline{\alpha}_k a_k^* \alpha_j a_j \\
&- \sum_{j,k=1}^{n} p_j p_k \overline{\alpha_j} a_j^* \alpha_k a_k + \sum_{j,k=1}^{n} p_j p_k |\alpha_k|^2 |a_j|^2
\end{aligned}$$

$$= \sum_{j=1}^{n} p_j |\alpha_j|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \sum_{k=1}^{n} p_k \overline{\alpha}_k a_k^* \sum_{j=1}^{n} p_j \alpha_j a_j$$
$$- \sum_{j=1}^{n} p_j \overline{\alpha_j} a_j^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{j=1}^{n} p_j |a_j|^2$$
$$= \sum_{j=1}^{n} p_j |\alpha_j|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left(\sum_{k=1}^{n} p_k \alpha_k a_k\right)^* \sum_{j=1}^{n} p_j \alpha_j a_j$$
$$- \left(\sum_{j=1}^{n} p_j \alpha_j a_j\right)^* \sum_{k=1}^{n} p_k \alpha_k a_k + \sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{j=1}^{n} p_j |a_j|^2$$
$$= 2\left[\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 - \left|\sum_{k=1}^{n} p_k \alpha_k a_k\right|^2\right],$$

which is equivalent to the desired identity (2.11). □

We have the following Cauchy-Bunyakowsky-Schwarz (CBS) type inequalities:

Corollary 3. *Let $a_k \in A$, $\alpha_k \in \mathbb{C}$ and $p_k > 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \sum_{k=1}^{n} p_k |a_k|^2 \geq \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|^2. \tag{2.14}$$

In particular,

$$\sum_{k=1}^{n} |\alpha_k|^2 \sum_{k=1}^{n} |a_k|^2 \geq \left|\sum_{j=1}^{n} \alpha_j a_j\right|^2 \tag{2.15}$$

and

$$\sum_{k=1}^{n} p_k \sum_{k=1}^{n} p_k |a_k|^2 \geq \left|\sum_{j=1}^{n} p_j a_j\right|^2. \tag{2.16}$$

The equality holds in (2.16) if and only if $a_k = a$ for some $a \in A$ and all $k \in \{1, ..., n\}$.

Remark 2. *If A has a continuous involution, then we can take the square root in (2.14) to get*

$$\left(\sum_{k=1}^{n} p_k |\alpha_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} p_k |a_k|^2\right)^{1/2} \geq \left|\sum_{j=1}^{n} p_j \alpha_j a_j\right|. \tag{2.17}$$

Recall that a C^**-algebra* A is a Banach $*$-algebra such that the norm satisfies the condition

$$\|a^* a\| = \|a\|^2 \text{ for any } a \in A.$$

If a C^*-algebra A has a unit 1, then automatically $\|1\| = 1$.

It is well know that, if A is a C^*-algebra, then (see for instance [10, 2.2.5 Theorem])

$$b \geq a \geq 0 \text{ implies that } \|b\| \geq \|a\|.$$

By utilising this property, we get from (2.14) the norm inequality

$$\sum_{k=1}^{n} p_k |\alpha_k|^2 \left\| \sum_{k=1}^{n} p_k |a_k|^2 \right\| \geq \left\| \sum_{j=1}^{n} p_j \alpha_j a_j \right\|^2 . \tag{2.18}$$

Theorem 2. *Let a_i, $i \in \{1, ..., n\}$ be elements in A. Then*

$$\left| a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \leq \left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2} \right)^2 \right] \sum_{j=1}^{n-1} |\Delta a_j|^2 \tag{2.19}$$

for all $i \in \{2, ..., n-1\}$ and

$$\left| a_{\lfloor \frac{n+1}{2} \rfloor} - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \leq \left[\frac{n^2-1}{12} + \left(\left\lfloor \frac{n+1}{2} \right\rfloor - \frac{n+1}{2} \right)^2 \right] \sum_{j=1}^{n-1} |\Delta a_j|^2 . \tag{2.20}$$

Also

$$\left| a_1 - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \leq \frac{1}{6} (n-1)(2n-1) \sum_{j=1}^{n-1} |\Delta a_j|^2 , \tag{2.21}$$

$$\left| a_n - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \leq \frac{1}{6} (n-1)(2n-1) \sum_{j=1}^{n-1} |\Delta a_j|^2 \tag{2.22}$$

and

$$\left| \frac{a_1 + a_n}{2} - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \leq \frac{1}{12} (n-2)(n-1) \sum_{j=1}^{n-1} |\Delta a_j|^2 \tag{2.23}$$

for $n \geq 2$.

Proof. Let $i \in \{2, ..., n-1\}$. By taking the modulus in (2.10) and using Cauchy-Bunyakowsky-Schwarz inequality (2.15) we get

$$\left| a_i - \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 = \frac{1}{n} \left| \sum_{j=1}^{n-1} p(i,j) \Delta a_j \right|^2 \leq \frac{1}{n} \sum_{j=1}^{n-1} |p(i,j)|^2 \sum_{j=1}^{n-1} |\Delta a_j|^2 . \tag{2.24}$$

Observe that for

$$\begin{aligned}&\sum_{j=1}^{n-1}|p(i,j)|^2\\&=\sum_{j=1}^{i-1}|p(i,j)|^2+\sum_{j=i}^{n-1}|p(i,j)|^2=\sum_{j=1}^{i-1}j^2+\sum_{j=i}^{n-1}(n-j)^2\\&=\frac{1}{6}(i-1)i(2(i-1)+1)+\frac{1}{6}(n-i)(n-i+1)(2(n-i)+1)\\&=\frac{1}{6}[(i-1)i(2i-1)+(n-i)(n-i+1)(2n-2i+1)].\end{aligned}$$

Notice that, by doing the required calculations, we have

$$\begin{aligned}&\frac{1}{6}[(i-1)i(2i-1)+(n-i)(n-i+1)(2n-2i+1)]\\&=n\left[\frac{n^2-1}{12}+\left(i-\frac{n+1}{2}\right)^2\right],\end{aligned}$$

which proves the required bound.

Now, by (2.3) we have

$$\begin{aligned}\left|a_1-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^2&=\frac{1}{n}\left|\sum_{j=1}^{n-1}(j-n)\Delta a_j\right|^2\le\frac{1}{n}\sum_{j=1}^{n-1}(j-n)^2\sum_{j=1}^{n-1}|\Delta a_j|^2\\&=\frac{1}{6}(n-1)(2n-1)\sum_{j=1}^{n-1}|\Delta a_j|^2,\end{aligned}$$

which proves (2.21).

By (2.4),

$$\begin{aligned}\left|a_n-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^2&=\frac{1}{n}\left|\sum_{j=1}^{n-1}j\Delta a_j\right|^2\le\frac{1}{n}\sum_{j=1}^{n-1}j^2\sum_{j=1}^{n-1}|\Delta a_j|^2\\&=\frac{1}{6}(n-1)(2n-1)\sum_{j=1}^{n-1}|\Delta a_j|^2,\end{aligned}$$

which proves (2.22).

By (2.6) we have

$$\begin{aligned}\left|\frac{a_1+a_n}{2}-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^2&=\frac{1}{n}\left|\sum_{j=1}^{n-1}\left(j-\frac{n}{2}\right)\Delta a_j\right|^2\\&\le\frac{1}{n}\sum_{j=1}^{n-1}\left(j-\frac{n}{2}\right)^2\sum_{j=1}^{n-1}|\Delta a_j|^2\\&=\frac{1}{12}(n-2)(n-1)\sum_{j=1}^{n-1}|\Delta a_j|^2,\end{aligned}$$

which proves (2.23). □

Remark 3. *Observe that for $i = 1$ we have*

$$\frac{n^2-1}{12} + \left(1 - \frac{n+1}{2}\right)^2 = \frac{1}{6}(n-1)(2n-1)$$

also for $i = n$, we have

$$\frac{n^2-1}{12} + \left(n - \frac{n+1}{2}\right)^2 = \frac{1}{6}(n-1)(2n-1),$$

so, in fact we obtain from Theorem 2 that

$$\left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right|^2 \leq \left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2}\right)^2\right] \sum_{j=1}^{n-1} |\Delta a_j|^2 \tag{2.25}$$

for all $i \in \{1, ..., n\}$.

If A has a continuous involution, then we can take the square root in (2.19) to get

$$\left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right| \leq \left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2}\right)^2\right]^{1/2} \left(\sum_{j=1}^{n-1} |\Delta a_j|^2\right)^{1/2} \tag{2.26}$$

for all $i \in \{1, ..., n\}$.

If A is a C^*-algebra, then we have the norm inequality

$$\left\|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right\|^2 \leq \left[\frac{n^2-1}{12} + \left(i - \frac{n+1}{2}\right)^2\right] \left\|\sum_{j=1}^{n-1} |\Delta a_j|^2\right\| \tag{2.27}$$

for all $i \in \{1, ..., n\}$.

3. An Application for Grüss Inequality

We start with the following reverse of unweighted CBS inequality:

Lemma 3. *Let a_i, $i \in \{1, ..., n\}$ be elements in A. Then*

$$0 \leq \sum_{i=1}^{n} \left|a_i - \frac{1}{n}\sum_{j=1}^{n} a_j\right|^2 \leq \frac{1}{6}(n-1)n(n+1)\sum_{j=1}^{n-1} |\Delta a_j|^2, \tag{3.1}$$

or, equivalently,

$$0 \leq \frac{1}{n}\sum_{i=1}^{n} |a_i|^2 - \left|\frac{1}{n}\sum_{j=1}^{n} a_j\right|^2 \leq \frac{1}{6}(n-1)(n+1)\sum_{j=1}^{n-1} |\Delta a_j|^2. \tag{3.2}$$

Proof. Indeed, if we sum in inequality (2.25) from 1 to n, we get

$$\begin{aligned}
\sum_{i=1}^{n}\left|a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^2 &\leq \sum_{i=1}^{n}\left[\frac{n^2-1}{12}+\left(i-\frac{n+1}{2}\right)^2\right]\sum_{j=1}^{n-1}|\Delta a_j|^2 \\
&= \left[\left(\frac{n^2-1}{12}\right)n+\sum_{i=1}^{n}\left(i-\frac{n+1}{2}\right)^2\right]\sum_{j=1}^{n-1}|\Delta a_j|^2 \\
&= \left[\left(\frac{n^2-1}{12}\right)n+\left(\frac{n^2-1}{12}\right)n\right]\sum_{j=1}^{n-1}|\Delta a_j|^2 \\
&= \frac{1}{6}(n-1)\,n\,(n+1)\sum_{j=1}^{n-1}|\Delta a_j|^2,
\end{aligned} \tag{3.3}$$

which proves (3.1).

Now, observe that

$$\begin{aligned}
\sum_{i=1}^{n}\left|a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right|^2 &= \sum_{i=1}^{n}\left(a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right)^*\left(a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right) \\
&= \sum_{i=1}^{n}\left(a_i^*-\frac{1}{n}\left(\sum_{j=1}^{n}a_j\right)^*\right)\left(a_i-\frac{1}{n}\sum_{j=1}^{n}a_j\right) \\
&= \sum_{i=1}^{n}\left[|a_i|^2-\frac{1}{n}\left(\sum_{j=1}^{n}a_j\right)^*a_i-\frac{1}{n}a_i^*\sum_{j=1}^{n}a_j+\frac{1}{n^2}\left|\sum_{j=1}^{n}a_j\right|^2\right] \\
&= \sum_{i=1}^{n}|a_i|^2-\frac{1}{n}\left(\sum_{j=1}^{n}a_j\right)^*\sum_{i=1}^{n}a_i \\
&\quad -\frac{1}{n}\sum_{i=1}^{n}a_i^*\sum_{j=1}^{n}a_j+\frac{1}{n}\left|\sum_{j=1}^{n}a_j\right|^2 \\
&= \sum_{i=1}^{n}|a_i|^2-\frac{2}{n}\left|\sum_{j=1}^{n}a_j\right|^2+\frac{1}{n}\left|\sum_{j=1}^{n}a_j\right|^2 \\
&= \sum_{i=1}^{n}|a_i|^2-\frac{1}{n}\left|\sum_{j=1}^{n}a_j\right|^2,
\end{aligned}$$

which proves (3.2). □

Remark 4. *If A has a continuous involution, then we can take the square root in (3.2) to get*

$$0 \leq \left(\frac{1}{n} \sum_{i=1}^{n} |a_i|^2 - \left| \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \right)^{1/2} \leq \frac{\sqrt{6}}{6} \left[(n-1)(n+1) \right]^{1/2} \left(\sum_{j=1}^{n-1} |\Delta a_j|^2 \right)^{1/2}. \tag{3.4}$$

If A is a C^*-algebra, then we have the norm inequality

$$0 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} |a_i|^2 - \left| \frac{1}{n} \sum_{j=1}^{n} a_j \right|^2 \right\| \leq \frac{1}{6} (n-1)(n+1) \left\| \sum_{j=1}^{n-1} |\Delta a_j|^2 \right\|. \tag{3.5}$$

Theorem 3. *Assume that $\alpha_i \in \mathbb{C}$ and $a_i \in A$, $i \in \{1, ..., n\}$, then*

$$\left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k a_k - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \frac{1}{n} \sum_{j=1}^{n} a_k \right|^2 \leq \frac{1}{6} (n-1)(n+1) \left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^2 - \left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k \right|^2 \right) \sum_{j=1}^{n-1} |\Delta a_j|^2. \tag{3.6}$$

Proof. We have the following Korkine type identity in Hermitian unital Banach $*$-algebra A,

$$\begin{aligned}
&\frac{1}{n^2} \sum_{k,j=1}^{n} (\alpha_j - \alpha_k)(a_j - a_k) \\
&= \frac{1}{n^2} \sum_{k,j=1}^{n} (\alpha_j a_j - \alpha_j a_k - \alpha_k a_j + \alpha_k a_k) \\
&= \frac{1}{n^2} \left(\sum_{k,j=1}^{n} \alpha_j a_j - \sum_{k,j=1}^{n} \alpha_j a_k - \sum_{k,j=1}^{n} \alpha_k a_j + \sum_{k,j=1k}^{n} \alpha_k a_k \right) \\
&= \frac{1}{n} \sum_{j=1}^{n} \alpha_j a_j - \frac{1}{n} \sum_{j=1}^{n} \alpha_j \frac{1}{n} \sum_{k=1}^{n} a_k - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \frac{1}{n} \sum_{j=1}^{n} a_j + \frac{1}{n} \sum_{k=1}^{n} \alpha_k a_k \\
&= 2 \left(\frac{1}{n} \sum_{k=1}^{n} \alpha_k a_k - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \frac{1}{n} \sum_{j=1}^{n} a_k \right),
\end{aligned}$$

namely

$$\frac{1}{n} \sum_{k=1}^{n} \alpha_k a_k - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \frac{1}{n} \sum_{j=1}^{n} a_k = \frac{1}{2} \frac{1}{n^2} \sum_{k,j=1}^{n} (\alpha_j - \alpha_k)(a_j - a_k).$$

Using CBS weighted inequality for double sums, we have

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k a_k-\frac{1}{n}\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{j=1}^{n}a_k\right|^2=\left|\frac{1}{2}\frac{1}{n^2}\sum_{k,j=1}^{n}\left(\alpha_j-\alpha_k\right)\left(a_j-a_k\right)\right|^2 \tag{3.7}$$
$$\leq\frac{1}{2}\frac{1}{n^2}\sum_{k,j=1}^{n}\left|\alpha_j-\alpha_k\right|^2\times\frac{1}{2}\frac{1}{n^2}\sum_{k,j=1}^{n}\left|a_j-a_k\right|^2.$$

Since

$$\frac{1}{2}\frac{1}{n^2}\sum_{k,j=1}^{n}\left|\alpha_j-\alpha_k\right|^2=\frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|^2$$

and,

$$\frac{1}{2}\frac{1}{n^2}\sum_{k,j=1}^{n}\left|a_j-a_k\right|^2=\frac{1}{n}\sum_{k=1}^{n}\left|a_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2,$$

hence by (3.7), we derive

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k a_k-\frac{1}{n}\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{j=1}^{n}a_k\right|^2 \tag{3.8}$$
$$\leq\left(\frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|^2\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left|a_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2\right).$$

By (3.2) we also have

$$\left(\frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|^2\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left|a_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}a_k\right|^2\right) \tag{3.9}$$
$$\leq\frac{1}{6}\left(n-1\right)\left(n+1\right)\left(\frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|^2\right)\sum_{j=1}^{n-1}\left|\Delta a_j\right|^2.$$

By utilising (3.8) and (3.9) we derive (3.6). □

Remark 5. *If A has a continuous involution, then we can take the square root in (3.6) to get*

$$\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k a_k-\frac{1}{n}\sum_{k=1}^{n}\alpha_k\frac{1}{n}\sum_{j=1}^{n}a_k\right| \tag{3.10}$$
$$\leq\frac{\sqrt{6}}{6}\left[\left(n-1\right)\left(n+1\right)\right]^{1/2}$$
$$\times\left(\frac{1}{n}\sum_{k=1}^{n}\left|\alpha_k\right|^2-\left|\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right|^2\right)^{1/2}\left(\sum_{j=1}^{n-1}\left|\Delta a_j\right|^2\right)^{1/2}.$$

4. Application to the Discrete Fourier Transform

Let A be a Hermitian unital Banach $*$-algebra and $a= (a_1, \dots, a_n)$ be a sequence of vectors in A.

For a given $w \in \mathbb{R}$, define the *discrete Fourier transform* as

$$\mathcal{F}_w(\mathbf{a})(m) := \sum_{k=1}^{n} \exp(2wimk)\, a_k,\ m = 1, \dots, n. \tag{4.1}$$

The following approximation result holds:

Theorem 4. *Let $a= (a_1, \dots, a_n)$ be a sequence of elements in A. Then*

$$\begin{aligned} &\left| \mathcal{F}_w(\mathbf{a})(m) - \frac{\sin(wmn)}{\sin(wm)} \exp[w(n+1)im] \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{1}{6}(n-1)(n+1)\left[1 - \frac{\sin^2(wmn)}{n^2 \sin^2(wm)}\right] \sum_{j=1}^{n-1} |\Delta a_j|^2 \end{aligned} \tag{4.2}$$

for all $m \in \{1, \dots, n\}$ and $w \in \mathbb{R},\ w \neq \frac{l}{m}\pi,\ l \in \mathbb{Z}$.

Proof. From the inequality (3.6), we can state that

$$\begin{aligned} &\left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k a_k - \frac{1}{n} \sum_{k=1}^{n} \alpha_k \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{1}{6}(n-1)(n+1)\left(\frac{1}{n} \sum_{k=1}^{n} |\alpha_k|^2 - \left| \frac{1}{n} \sum_{k=1}^{n} \alpha_k \right|^2 \right) \sum_{j=1}^{n-1} |\Delta a_j|^2 \end{aligned} \tag{4.3}$$

for all $\alpha_k \in \mathbb{C},\ a_k \in A\ (k = 1, \dots, n)$.

We now choose in (4.3), $\alpha_k = \exp(2wimk)$ to obtain

$$\begin{aligned} &\left| \mathcal{F}_w(\mathbf{a})(m) - \sum_{k=1}^{n} \exp(2wimk) \frac{1}{n} \sum_{k=1}^{n} a_k \right|^2 \\ &\leq \frac{1}{6}(n-1)(n+1)\left(\frac{1}{n} \sum_{k=1}^{n} |\exp(2wimk)|^2 - \left| \frac{1}{n} \sum_{k=1}^{n} \exp(2wimk) \right|^2 \right) \\ &\times \sum_{j=1}^{n-1} |\Delta a_j|^2 \end{aligned} \tag{4.4}$$

for all $m \in \{1, \dots, n\}$.

A simple calculation reveals that

$$\begin{aligned}
&\sum_{k=1}^{n}\exp\left(2wimk\right)\\
&=\exp\left(2wim\right)\times\left[\frac{\exp\left(2wimn\right)-1}{\exp\left(2wim\right)-1}\right]\\
&=\exp\left(2wim\right)\times\left[\frac{\cos\left(2wmn\right)+i\sin\left(2wmn\right)-1}{\cos\left(2wm\right)+i\sin\left(2wm\right)-1}\right]\\
&=\exp\left(2wim\right)\times\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\left[\frac{\cos\left(wmn\right)+i\sin\left(wmn\right)}{\cos\left(wm\right)+i\sin\left(wm\right)}\right]\\
&=\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\times\exp\left(2wim\right)\left[\frac{\exp\left(iwmn\right)}{\exp\left(iwm\right)}\right]\\
&=\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\times\exp\left[w\left(n+1\right)im\right].
\end{aligned}$$

Since $\sum_{k=1}^{n}\left|\exp\left(2wimk\right)\right|^{2}=n$ and

$$\left|\sum_{k=1}^{n}\exp\left(2wimk\right)\right|^{2}=\frac{\sin^{2}\left(wmn\right)}{\sin^{2}\left(wm\right)},\ \text{for } w\neq\frac{l}{m}\pi,\ l\in\mathbb{Z},$$

hence by (4.4) we derive (4.2). □

Remark 6. *If A has a continuous involution, then we can take the square root in (4.2) to get*

$$\begin{aligned}
&\left|\mathcal{F}_{w}\left(\mathbf{a}\right)\left(m\right)-\frac{\sin\left(wmn\right)}{\sin\left(wm\right)}\exp\left[w\left(n+1\right)im\right]\frac{1}{n}\sum_{k=1}^{n}a_{k}\right|\\
&\leq\frac{\sqrt{6}}{6}\left[\left(n-1\right)\left(n+1\right)\right]^{1/2}\left[1-\frac{\sin^{2}\left(wmn\right)}{n^{2}\sin^{2}\left(wm\right)}\right]^{1/2}\left(\sum_{j=1}^{n-1}\left|\Delta a_{j}\right|^{2}\right)^{1/2}
\end{aligned}\tag{4.5}$$

for all $m\in\{1,\ldots,n\}$ and $w\in\mathbb{R},\ w\neq\frac{l}{m}\pi,\ l\in\mathbb{Z}$.

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