REVERSE TRIANGLE INTEGRAL INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if U is an unitary operator and $B : \Omega \to \mathcal{B}(H)$ is strongly μ-measurable with $BU \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$ and such that

$$\left|B(s)U - \frac{\gamma+\Gamma}{2}U\right|^2 \le \frac{1}{4}|\Gamma-\gamma|^2 1_H \text{ for } \mu\text{-a.e. } s \in \Omega$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^* - B^*(s)\right)(B(s) - \gamma U)\right] \ge 0 \text{ for } \mu\text{-a.e. } s \in \Omega$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma}) > 0$, then

$$\begin{aligned}
&\int_\Omega w(s)|B(s)U|\,d\mu(s) \\
&\le \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right] \\
&\le \frac{|\gamma+\Gamma|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right|.
\end{aligned}$$

1. Introduction

Let $f : [a,b] \to \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality is the continuous version of the triangle inequality

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt, \tag{1.1}$$

and plays a fundamental role in Mathematical Analysis and its applications.

It appears, see [5, p. 492], that the first reverse inequality for (1.1) was obtained by J. Karamata in his book from 1949, [3]:

$$\cos\theta \int_a^b |f(t)|\,dt \le \left|\int_a^b f(t)\,dt\right| \tag{1.2}$$

provided

$$-\theta \le \arg[f(t)] \le \theta, \quad t \in [a,b]$$

for given $\theta \in \left(0, \frac{\pi}{2}\right)$.

In [2], the author has extended the above result for Bochner integrals of vector-valued functions in real or complex Hilbert spaces.

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If $(H;\langle\cdot,\cdot\rangle)$ is a Hilbert space over $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ and $f\in L([a,b];H)$, this means that $f:[a,b]\to H$ is strongly measurable on $[a,b]$ and the Lebesgue integral $\int_a^b\|f(t)\|\,dt$ exists and is finite, and there exists a constant $K\geq 1$ and a vector $e\in H$, $\|e\|=1$ such that

$$\|f(t)\|\leq K\operatorname{Re}\langle f(t),e\rangle \quad \text{for a.e.} \quad t\in[a,b], \tag{1.3}$$

then we have the inequality [2]

$$\int_a^b\|f(t)\|\,dt\leq K\left\|\int_a^b f(t)\,dt\right\|. \tag{1.4}$$

This provides a reverse inequality for the well know result for Bochner integrals and vector-valued functions:

$$\left\|\int_a^b f(t)\,dt\right\|\leq\int_a^b\|f(t)\|\,dt \tag{1.5}$$

for any $f\in L([a,b];H)$.

Note that the case of equality holds in (1.4) if and only if [2]

$$\int_a^b f(t)\,dt=\frac{1}{K}\left(\int_a^b\|f(t)\|\,dt\right)e. \tag{1.6}$$

For some particular cases of interest, see [2].

For reverses of the discrete generalised triangle inequality for complex numbers with applications for polynomials, see [4] and [7]. Generalizations in Hilbert and Banach spaces were obtained by Diaz & Metcalf in [1]. For other related results see Chapter XVII of [5].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

We have the following operator inequality:

Proposition 1. *Assume that* $B\in L_w(\Omega,\mu,\mathcal{B}(H))$, *then*

$$\int_\Omega w(t)\,|B(t)|\,d\mu(t)\geq\left(\int_\Omega w(t)\,|B(t)|^{1/2}\,d\mu(t)\right)^2. \tag{1.7}$$

Proof. We have

$$\left(|B(t)|^{1/2}-|B(s)|^{1/2}\right)^2\geq 0$$

for μ-a.e. $s,\,t\in\Omega$.

This gives that

$$|B(t)|+|B(s)|\geq|B(t)|^{1/2}|B(s)|^{1/2}+|B(s)|^{1/2}|B(t)|^{1/2}$$

for μ-a.e. $s,\,t\in\Omega$.

If we multiply by $w(s)w(t) \geq 0$ and integrate, then you get

$$\begin{aligned}
&\int_\Omega w(t)|B(t)|\,d\mu(t) + \int_\Omega w(s)|B(s)|\,d\mu(s) \\
&\geq \int_\Omega w(t)|B(t)|^{1/2}\,d\mu(t) \int_\Omega w(s)|B(s)|^{1/2}\,d\mu(s) \\
&+ \int_\Omega w(s)|B(s)|^{1/2}\,d\mu(s) \int_\Omega w(t)|B(t)|^{1/2}\,d\mu(t) \\
&= 2\left(\int_\Omega w(t)|B(t)|^{1/2}\,d\mu(t)\right)^2
\end{aligned}$$

which is equivalent to the desired inequality (1.7). □

Motivated by the above results, in this paper we provided some operator upper bunds for the integral

$$\int_\Omega w(s)|B(s)U|\,d\mu(s),$$

where U is an unitary operator and $B:\Omega \to \mathcal{B}(H)$ is strongly μ-measurable with $BU \in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$.

2. Main Results

We have the following identity of interest:

Lemma 1. *For any A, X, $Y \in \mathcal{B}(H)$, we have*

$$\left|A - \frac{X+Y}{2}\right|^2 - \frac{1}{4}|X-Y|^2 = \operatorname{Re}\left[(A^* - X^*)(A-Y)\right]. \tag{2.1}$$

Proof. We have

$$\begin{aligned}
&\left|A - \frac{X+Y}{2}\right|^2 - \frac{1}{4}|X-Y|^2 \\
&= |A|^2 - \frac{X^*+Y^*}{2}A - A^*\frac{X+Y}{2} + \frac{1}{4}\left(|X|^2 + X^*Y + Y^*X + |Y|^2\right) \\
&- \frac{1}{4}\left(|X|^2 - X^*Y - Y^*X + |Y|^2\right) \\
&= |A|^2 - \frac{X^*+Y^*}{2}A - A^*\frac{X+Y}{2} + \frac{1}{2}(X^*Y + Y^*X)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\left[(A^* - X^*)(A-Y)\right] \\
&= \operatorname{Re}\left[|A|^2 - X^*A - A^*Y + X^*Y\right] \\
&= |A|^2 - \operatorname{Re}(X^*A) - \operatorname{Re}(A^*Y) + \operatorname{Re}(X^*Y) \\
&= |A|^2 - \frac{1}{2}(X^*A + A^*X) - \frac{1}{2}(A^*Y + Y^*A) + \frac{1}{2}(X^*Y + Y^*X) \\
&= |A|^2 - \frac{1}{2}(X^* + Y^*)A - \frac{1}{2}A^*(X+Y) + \frac{1}{2}(X^*Y + Y^*X),
\end{aligned}$$

which proves the desired identity (2.1). □

Corollary 1. *Let A, X, $Y \in \mathcal{B}(H)$. The following statements are equivalent*

$$\left|A-\frac{X+Y}{2}\right|^{2}\leq\frac{1}{4}\left|X-Y\right|^{2}$$

and

$$\operatorname{Re}\left[\left(X^{*}-A^{*}\right)\left(A-Y\right)\right]\geq 0.$$

We have the following reverse inequality of Cauchy-Bunyakowsky-Schwarz integral inequality for operator modulus:

Theorem 1. *Let U be an unitary operator and $B:\Omega\rightarrow\mathcal{B}(H)$ strongly μ-measurable with $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that*

$$\left|B(s)U-\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}\text{ for }\mu\text{-a.e. }s\in\Omega \tag{2.2}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma}U^{*}-B^{*}(s)\right)\left(B(s)-\gamma U\right)\right]\geq 0\text{ for }\mu\text{-a.e. }s\in\Omega \tag{2.3}$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma\overline{\gamma})>0$. Then

$$\begin{aligned}&\int_{\Omega}w(s)\left|B(s)U\right|d\mu(s)\\&\leq\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^{*}\left(\int_{\Omega}w(s)B(s)d\mu(s)\right)U\right]\\&\leq\frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}\left|U^{*}\left(\int_{\Omega}w(s)B(s)d\mu(s)\right)U\right|.\end{aligned} \tag{2.4}$$

Proof. The equivalence of the statements (2.2) and (2.3) follows by Corollary 1 for $A=B(s)$, $X=\Gamma U$ and $Y=\gamma U$ and taking into account that $\left|U\right|^{2}=1_{H}$.

By the properties of operator modulus, we have

$$\left|B(s)U\right|^{2}-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}U\right)^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H},$$

namely

$$\left|B(s)U\right|^{2}-2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\leq\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H},$$

or

$$\left|B(s)U\right|^{2}+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}-\frac{1}{4}\left|\Gamma-\gamma\right|^{2}1_{H}\leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right], \tag{2.5}$$

for μ-a.e. $s\in\Omega$.

Observe that

$$\begin{aligned}\frac{1}{4}\left|\Gamma+\gamma\right|^{2}-\frac{1}{4}\left|\Gamma-\gamma\right|^{2}&=\frac{1}{4}\left(\left|\Gamma\right|^{2}+2\operatorname{Re}(\Gamma\overline{\gamma})+\left|\gamma\right|^{2}\right)\\&\quad-\frac{1}{4}\left(\left|\Gamma\right|^{2}-2\operatorname{Re}(\Gamma\overline{\gamma})+\left|\gamma\right|^{2}\right)\\&=\operatorname{Re}(\Gamma\overline{\gamma})>0,\end{aligned}$$

then by (2.5) we get

$$|B(s)U|^2 + \operatorname{Re}(\Gamma\overline{\gamma})1_H \le 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^*B(s)U\right], \tag{2.6}$$

for μ-a.e. $s \in \Omega$.

Using the elementary operator inequality

$$2aA \le A^2 + a^2 1_H, \tag{2.7}$$

we also have

$$2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\,|B(s)U| \le |B(s)U|^2 + \operatorname{Re}(\Gamma\overline{\gamma})1_H \tag{2.8}$$

for μ-a.e. $s \in \Omega$.

Then by (2.6) and (2.8) we derive

$$\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\,|B(s)U| \le \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^*B(s)U\right]$$

for μ-a.e. $s \in \Omega$.

If we multiply with $w(s) \ge 0$ and integrate, then we get

$$\begin{aligned}
&\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}\int_\Omega w(s)\,|B(s)U|\,d\mu(s) \\
&\le \int_\Omega w(s)\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^*B(s)U\right]d\mu(s) \\
&= \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right],
\end{aligned}$$

which implies that

$$\int_\Omega w(s)\,|B(s)U|\,d\mu(s) \le \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma\overline{\gamma})}}U^*\left(\int_\Omega w(s)B(s)\,d\mu(s)\right)U\right]$$

that proves the first inequality in (2.4).

For an operator T we consider the selfadjoint operators

$$\operatorname{Re}(T) := \frac{T^*+T}{2},\quad \operatorname{Im}(T) := \frac{T-T^*}{2i}.$$

Then

$$T = \operatorname{Re}(T) + i\operatorname{Im}(T),\ |T|^2 = (\operatorname{Re}(T))^2 + (\operatorname{Im}(T))^2.$$

We have $|T|^2 \ge (\operatorname{Re}(T))^2$ which implies, by taking the square root, that $|T| \ge |\operatorname{Re}(T)|$.

Therefore

$$\begin{aligned}0 &\leq \operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right]\\ &\leq \left|\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right|\\ &= \left|\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\right|\left|U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right|\\ &= \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left|U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right|,\end{aligned}$$

and the last part of (2.4) is thus proved. □

Remark 1. *Observe that for $z=\alpha+i\beta$ and $A\in\mathcal{B}(H)$, we have*

$$\begin{aligned}\operatorname{Re}\left(\overline{z}A\right) &= \operatorname{Re}\left[\left(\alpha-i\beta\right)\left(\operatorname{Re}A+i\operatorname{Im}A\right)\right]\\ &= \operatorname{Re}\left[\alpha\operatorname{Re}A+\beta\operatorname{Im}A-i\left(\beta\operatorname{Re}A-\alpha\operatorname{Im}A\right)\right]\\ &= \alpha\operatorname{Re}A+\beta\operatorname{Im}A=\operatorname{Re}z\operatorname{Re}A+\operatorname{Im}z\operatorname{Im}A\end{aligned}$$

and then

$$\begin{aligned}&\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right]\\ &= \frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Re}\left(\gamma+\Gamma\right)\operatorname{Re}\left[U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right]\\ &+\frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Im}\left(\gamma+\Gamma\right)\operatorname{Im}\left[U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right]\\ &= \frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Re}\left(\gamma+\Gamma\right)U^{*}\left(\int_{\Omega}w\left(s\right)\operatorname{Re}\left(B\left(s\right)\right)d\mu\left(s\right)\right)U\\ &+\frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\operatorname{Im}\left(\gamma+\Gamma\right)U^{*}\left(\int_{\Omega}w\left(s\right)\operatorname{Im}\left(B\left(s\right)\right)d\mu\left(s\right)\right)U.\end{aligned}$$

Therefore by (2.4) we have the unpacked inequality

$$\begin{aligned}&\int_{\Omega}w\left(s\right)\left|B\left(s\right)U\right|d\mu\left(s\right)\\ &\leq \frac{1}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left[\operatorname{Re}\left(\gamma+\Gamma\right)U^{*}\left(\int_{\Omega}w\left(s\right)\operatorname{Re}\left(B\left(s\right)\right)d\mu\left(s\right)\right)U\right.\\ &\left.+\operatorname{Im}\left(\gamma+\Gamma\right)U^{*}\left(\int_{\Omega}w\left(s\right)\operatorname{Im}\left(B\left(s\right)\right)d\mu\left(s\right)\right)U\right]\\ &\leq \frac{\left|\gamma+\Gamma\right|}{2\sqrt{\operatorname{Re}\left(\Gamma\overline{\gamma}\right)}}\left|U^{*}\left(\int_{\Omega}w\left(s\right)B\left(s\right)d\mu\left(s\right)\right)U\right|.\end{aligned}\tag{2.9}$$

Corollary 2. *Let U be an unitary operator and $B:\Omega\to\mathcal{B}(H)$ strongly μ-measurable with $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that*

$$\left|B\left(s\right)U-\frac{m+M}{2}U\right|^{2}\leq\frac{1}{4}\left(M-m\right)^{2}1_{H}\text{ for }\mu\text{-a.e. }s\in\Omega\tag{2.10}$$

or, equivalently

$$\operatorname{Re}\left[\left(MU^{*}-B^{*}(s)\right)(B(s)-mU)\right]\geq 0 \text{ for } \mu\text{-a.e. } s\in\Omega \tag{2.11}$$

for some real numbers $M>m>0$. Then

$$\begin{aligned}&\int_{\Omega}w(s)\left|B(s)U\right|d\mu(s)\\&\leq\frac{m+M}{2\sqrt{mM}}\operatorname{Re}\left[U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right]\\&\leq\frac{m+M}{2\sqrt{Mm}}\left|U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right|.\end{aligned}\tag{2.12}$$

We also have:

Theorem 2. *Let U be an unitary operator and $B:\Omega\to\mathcal{B}(H)$ strongly μ-measurable with $BU\in L_{2,w}(\Omega,\mu,\mathcal{B}(H))$ and such that either (2.2) or (2.3) is valid for some complex constants γ, Γ with $\gamma+\Gamma\neq 0$. Then*

$$\begin{aligned}&\int_{\Omega}w(s)\left|B(s)U\right|d\mu(s)\\&\leq\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{|\gamma+\Gamma|}U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right]+\frac{1}{4}\frac{|\Gamma-\gamma|^{2}}{|\gamma+\Gamma|}1_{H}\\&\leq\left|U^{*}\left(\int_{\Omega}w(s)B(s)\,d\mu(s)\right)U\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^{2}}{|\gamma+\Gamma|}1_{H}.\end{aligned}\tag{2.13}$$

Proof. The equivalence of the statements (2.2) and (2.3) follows by Corollary 1 for $A=B(s)$, $X=\Gamma U$ and $Y=\gamma U$ and taking into account that $|U|^{2}=1_{H}$.

By the properties of operator modulus, we have

$$|B(s)U|^{2}-2\operatorname{Re}\left[\left(\frac{\gamma+\Gamma}{2}U\right)^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}U\right|^{2}\leq\frac{1}{4}|\Gamma-\gamma|^{2}1_{H},$$

namely

$$|B(s)U|^{2}-2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\leq\frac{1}{4}|\Gamma-\gamma|^{2}1_{H},$$

or

$$|B(s)U|^{2}+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\leq 2\operatorname{Re}\left[\frac{\overline{\gamma}+\overline{\Gamma}}{2}U^{*}B(s)U\right]+\frac{1}{4}|\Gamma-\gamma|^{2}1_{H},\tag{2.14}$$

for μ-a.e. $s\in\Omega$.

By (2.7) we also have

$$2\left|\frac{\gamma+\Gamma}{2}\right||B(s)U|\leq|B(s)U|^{2}+\left|\frac{\gamma+\Gamma}{2}\right|^{2}1_{H}\tag{2.15}$$

for μ-a.e. $s\in\Omega$.

By (2.14) and (2.15) we get

$$|\gamma+\Gamma|\,|B(s)U|\leq\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)U^{*}B(s)U\right]+\frac{1}{4}|\Gamma-\gamma|^{2}1_{H}.$$

By dividing with $|\gamma+\Gamma|\neq 0$, we deduce the first inequality in (2.13).

The second inequality follows as above. □

Corollary 3. *Let U be an unitary operator and $B : \Omega \to \mathcal{B}(H)$ strongly μ-measurable with $BU \in L_{2,w}(\Omega, \mu, \mathcal{B}(H))$ and such that either (2.10) or (2.11) is valid for some real numbers $M > m > 0$. Then*

$$\begin{aligned} \int_\Omega w(s) |B(s) U| \, d\mu(s) &\le \operatorname{Re}\left[U^* \left(\int_\Omega w(s) B(s) \, d\mu(s)\right) U\right] + \frac{1}{4} \frac{(M-m)^2}{m+M} 1_H \\ &\le \left|U^* \left(\int_\Omega w(s) B(s) \, d\mu(s)\right) U\right| + \frac{1}{4} \frac{(M-m)^2}{m+M} 1_H. \end{aligned} \tag{2.16}$$

3. Discrete Inequalities

Assume that $B_k \in \mathcal{B}(H)$, $w_k \ge 0$, $k \in \{1, ..., n\}$ with $\sum_{k=1}^n = 1$ and U an unitary operator with the property

$$\left|B_k U - \frac{\gamma + \Gamma}{2} U\right|^2 \le \frac{1}{4} |\Gamma - \gamma|^2 1_H \text{ for } k \in \{1, ..., n\} \tag{3.1}$$

or, equivalently

$$\operatorname{Re}\left[\left(\overline{\Gamma} U^* - B_k^*\right)(B_k - \gamma U)\right] \ge 0 \text{ for } k \in \{1, ..., n\} \tag{3.2}$$

for some complex constants γ, Γ with $\operatorname{Re}(\Gamma \overline{\gamma}) > 0$. Then by (2.4),

$$\begin{aligned} \sum_{k=1}^n w_k |B_k U| &\le \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{2\sqrt{\operatorname{Re}(\Gamma \overline{\gamma})}} U^* \left(\sum_{k=1}^n w_k B_k\right) U\right] \\ &\le \frac{|\gamma + \Gamma|}{2\sqrt{\operatorname{Re}(\Gamma \overline{\gamma})}} \left|U^* \left(\sum_{k=1}^n w_k B_k\right) U\right|. \end{aligned} \tag{3.3}$$

Also, if either the condition (3.1) or (3.2) is valid for some complex constants γ, Γ with $\gamma + \Gamma \ne 0$, then by (2.16)

$$\begin{aligned} \sum_{k=1}^n w_k |B_k U| &\le \operatorname{Re}\left[\frac{\overline{\gamma} + \overline{\Gamma}}{|\gamma + \Gamma|} U^* \left(\sum_{k=1}^n w_k B_k\right) U\right] + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\gamma + \Gamma|} 1_H \\ &\le \left|U^* \left(\sum_{k=1}^n w_k B_k\right) U\right| + \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\gamma + \Gamma|} 1_H. \end{aligned} \tag{3.4}$$

References

[1] J. B. Diaz and F. T. Metcalf, A complementary triangle inequality in Hilbert and Banach spaces, *Proceedings Amer. Math. Soc.,* **17**(1) (1966), 88-97.

[2] S. S. Dragomir, Reverses of the continuous triangle inequality for Bochner integrals of vector-valued functions in Hilbert spaces, *J. Inequal. Pure Appl. Math.* **6**(2005), No.2, Article 46, 10 pp. `[ONLINE: http://jipam.vu.edu.au/article.php?sid=515]`

[3] J. Karamata, *Teorija i Praksa Stieltjesova Integrala* (Serbo-Coratian) (Stieltjes Integral, Theory and Practice), SANU, Posebna izdanja, 154, Beograd, 1949.

[4] M. Marden, The Geometry of the Zeros of a Polynomial in a Complex Variable, *Amer. Math. Soc. Math. Surveys*, **3**, New York, 1949.

[5] D. S. Mitrinović, J. E. Pečaric and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[6] M. Petrovich, Module d'une somme, *L' Ensignement Mathématique,* **19** (1917), 53-56.

[7] H. S. Wilf, Some applications of the inequality of arithmetic and geometric means to polynomial equations, *Proceedings Amer. Math. Soc.,* **14** (1963), 263-265.

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