

# OSTROWSKI TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR<sup>1,2</sup>

ABSTRACT. Let  $(H; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space. Denote by  $\mathcal{B}(H)$  the Banach  $C^*$ -algebra of bounded linear operators on  $H$ . For  $A \in \mathcal{B}(H)$  we define the modulus of  $A$  by  $|A| := (A^*A)^{1/2}$ . In this paper we show among others that, if  $s \in (a, b)$ , the function  $A : [a, b] \rightarrow \mathcal{B}(H)$  is continuous on  $[a, b]$  and strongly differentiable on  $(a, s) \cup (s, b)$  with  $A' \in L_2([a, b], \mathcal{B}(H))$ , then

$$\begin{aligned} & \left| (b-a)A(s) - \int_a^b A(t) dt \right|^2 \\ & \leq (b-a) \left[ \frac{1}{12} (b-a)^2 + \left( s - \frac{a+b}{2} \right)^2 \right] \int_a^b |A'(t)|^2 dt \end{aligned}$$

in the operator order of  $\mathcal{B}(H)$ . Some examples for the operator exponential and inverse functions are also provided.

## 1. INTRODUCTION

In 1998, Dragomir and Wang proved the following Ostrowski type inequality for  $p$ -norm [9].

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function on  $[a, b]$ . If  $f' \in L_p[a, b]$ , then we have the inequality*

$$(1.1) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{(q+1)^{1/q}} \left[ \left( \frac{x-a}{b-a} \right)^{q+1} + \left( \frac{b-x}{b-a} \right)^{q+1} \right]^{1/q} (b-a)^{1/q} \|f'\|_{[a,b],p},$$

for all  $x \in [a, b]$ , where  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  and  $\|\cdot\|_{[a,b],p}$  is the  $p$ -Lebesgue norm on  $L_p[a, b]$ , i.e., we recall it

$$\|g\|_{[a,b],p} := \left( \int_a^b |g(t)|^p dt \right)^{1/p}.$$

From (1.1) we get the following midpoint inequality

$$(1.2) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{2(q+1)^{1/q}} (b-a)^{1/q} \|f'\|_{[a,b],p},$$

and  $\frac{1}{2}$  is a best possible constant.

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For  $p = q = 2$  we derive the  $L_2[a, b]$ -inequality

$$(1.3) \quad \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right|^2 \\ \leq \frac{1}{3} \left[ \left( \frac{x-a}{b-a} \right)^3 + \left( \frac{b-x}{b-a} \right)^3 \right] (b-a) \|f'\|_{[a,b],2}^2,$$

for all  $x \in [a, b]$  and the midpoint inequality

$$(1.4) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right|^2 \leq \frac{1}{12} (b-a) \|f'\|_{[a,b]}^2.$$

For a survey on scalar Ostrowski's inequality, see [8]. For recent papers on this inequality see also [1]-[2] and [10]-[12].

Denote by  $\mathcal{B}(H)$  the Banach  $C^*$ -algebra of bounded linear operators on Hilbert space  $H$ . For  $A \in \mathcal{B}(H)$  we define the modulus of  $A$  by  $|A| := (A^*A)^{1/2}$ . It is well known that the modulus of operators does not satisfy, in general, the triangle inequality  $|A+B| \leq |A|+|B|$ , so the classical arguments using this inequality can not be used. In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that  $w : [a, b] \rightarrow (0, \infty)$  is continuous on  $[a, b]$  with  $\int_a^b w(s) ds = 1$ . We have for  $\alpha : [a, b] \rightarrow \mathbb{C}$  and  $A : [a, b] \rightarrow \mathcal{B}(H)$ ,

$$0 \leq \left| \overline{\alpha(t)} A(s) - \overline{\alpha(s)} A(t) \right|^2 = |\alpha(t)| |A(s)|^2 - \alpha(s) \overline{\alpha(t)} A^*(t) A(s) \\ - \alpha(t) \overline{\alpha(s)} A^*(s) A(t) + |\alpha(s)|^2 |A(t)|^2,$$

which gives that

$$|\alpha(t)|^2 |A(s)|^2 + |\alpha(s)|^2 |A(t)|^2 \geq \alpha(s) \overline{\alpha(t)} A^*(t) A(s) + \alpha(t) \overline{\alpha(s)} A^*(s) A(t)$$

for all  $s, t \in [a, b]$ .

Now, multiply this with  $w(s)w(t) \geq 0$  to get

$$w(t) |\alpha(t)|^2 w(s) |A(s)|^2 + w(s) |\alpha(s)|^2 w(t) |A(t)|^2 \\ \geq w(t) \overline{\alpha(t)} A^*(t) w(s) \alpha(s) A(s) + w(s) \overline{\alpha(s)} A^*(s) w(t) \alpha(t) A(t)$$

for all  $s, t \in [a, b]$ .

Integrating over  $t$  and  $s$  on  $[a, b]$ , then we get

$$\int_a^b w(t) |\alpha(t)|^2 dt \int_a^b |A(s)|^2 ds + \int_a^b |\alpha(s)|^2 ds \int_a^b w(t) |A(t)|^2 dt \\ \geq \int_a^b w(t) \overline{\alpha(t)} A^*(t) dt \int_a^b \alpha(s) A(s) ds \\ + \int_a^b w(s) \overline{\alpha(s)} A^*(s) ds \int_a^b \alpha(t) A(t) dt \\ = 2 \left| \int_a^b w(s) \alpha(s) A(s) ds \right|^2,$$

which proves that

$$(1.5) \quad \int_a^b w(t) |\alpha(t)|^2 dt \int_a^b w(t) |A(t)|^2 dt \geq \left| \int_a^b w(t) \alpha(t) A(t) dt \right|^2,$$

provided that  $\alpha \in L_{2,w}([a, b], \mathbb{C})$  and

$$A \in L_{2,w}([a, b], \mathcal{B}(H)) := \left\{ A : [a, b] \rightarrow B(H), \int_a^b w(t) \|A(t)\|^2 dt < \infty \right\}.$$

In a similar way we can prove the following discrete inequality

$$(1.6) \quad \sum_{k=1}^n w_k |z_k|^2 \sum_{k=1}^n w_k |A_k|^2 \geq \left| \sum_{k=1}^n w_k z_k A_k \right|^2,$$

where  $z_k \in \mathbb{C}$ ,  $A_k \in \mathcal{B}(H)$ ,  $w_k \geq 0$  for  $k \in \{1, \dots, n\}$  and  $\sum_{k=1}^n w_k = 1$ .

## 2. MAIN RESULTS

We have the following Ostrowski type inequality for the modulus of operators:

**Theorem 2.** *Assume that for  $s \in (a, b)$ , the function  $A : [a, b] \rightarrow B(H)$  is continuous on  $[a, b]$  and strongly differentiable on  $(a, s) \cup (s, b)$  with  $A' \in L_2([a, b], \mathcal{B}(H))$ . Then*

$$(2.1) \quad \left| (b-a)A(s) - \int_a^b A(t) dt \right|^2 \leq (b-a) \left[ \frac{1}{12} (b-a)^2 + \left( s - \frac{a+b}{2} \right)^2 \right] \int_a^b |A'(t)|^2 dt.$$

*Proof.* Using integration by parts for Bochner's integral, we have

$$\int_a^s (t-a) A'(t) dt = (s-a)A(s) - \int_a^s A(t) ds$$

and

$$\int_s^b (t-b) A'(t) dt = (b-s)A(s) - \int_s^b A(t) ds,$$

and by adding this equalities we obtain the *Montgomery type identity* for the Bochner's integral

$$(2.2) \quad (b-a)A(s) - \int_a^b A(t) dt = \int_a^s (t-a) A'(t) dt + \int_s^b (t-b) A'(t) dt.$$

Consider the kernel

$$p_s(t) := \begin{cases} t-a & \text{if } t \in [a, s] \\ t-b & \text{if } t \in (s, b]. \end{cases}$$

Then by (2.2) we get the representation

$$(b-a)A(s) - \int_a^b A(t) dt = \int_a^b p_s(t) A'(t) dt.$$

Since  $p_s \in L_2([a, b], \mathbb{R})$  and  $A' \in L_2([a, b], \mathcal{B}(H))$ , then by (1.5) we get

$$\begin{aligned}
 (2.3) \quad & \left| (b-a)A(s) - \int_a^b A(t) ds \right|^2 \\
 &= \left| \int_a^b p_s(t) A'(t) dt \right|^2 \leq \int_a^b [p_s(t)]^2 dt \int_a^b |A'(t)|^2 dt \\
 &= \left[ \int_a^s (t-a)^2 dt + \int_s^b (t-b)^2 dt \right] \int_a^b |A'(t)|^2 dt \\
 &= \frac{1}{3} [(s-a)^3 + (b-s)^3] \int_a^b |A'(t)|^2 dt.
 \end{aligned}$$

Observe that, by simple calculations we get

$$\frac{1}{3} [(s-a)^3 + (b-s)^3] = (b-a) \left[ \frac{1}{12} (b-a)^2 + \left( s - \frac{a+b}{2} \right)^2 \right].$$

Therefore by (2.3) we derive the desired result (2.1).  $\square$

**Corollary 1.** *Assume that the function  $A : [a, b] \rightarrow B(H)$  is continuous on  $[a, b]$  and strongly differentiable on  $(a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b)$  with  $A' \in L_2([a, b], \mathcal{B}(H))$ . Then*

$$(2.4) \quad \left| (b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right|^2 \leq \frac{1}{12} (b-a)^3 \int_a^b |A'(t)|^2 dt.$$

The constant  $\frac{1}{12}$  is best possible in (2.4).

*Proof.* It is enough to prove the best constant in the case of scalar case.

Consider the function

$$A_0(t) := \begin{cases} \frac{1}{2} (t-a)^2, & t \in [a, \frac{a+b}{2}], \\ \frac{1}{2} (t-b)^2, & t \in (\frac{a+b}{2}, b]. \end{cases}$$

Then

$$A_0\left(\frac{a+b}{2}\right) = \frac{1}{8} (b-a)^2$$

and

$$A'_0(t) := \begin{cases} t-a, & t \in (a, \frac{a+b}{2}), \\ t-b, & t \in (\frac{a+b}{2}, b). \end{cases}$$

We have

$$\int_a^b A_0(t) dt = \frac{1}{2} \int_a^{\frac{a+b}{2}} (t-a)^2 dt + \frac{1}{2} \int_{\frac{a+b}{2}}^b (t-b)^2 dt = \frac{(b-a)^3}{24},$$

$$\int_a^b |A'_0(t)|^2 dt = \int_a^{\frac{a+b}{2}} (t-a)^2 dt + \int_{\frac{a+b}{2}}^b (t-b)^2 dt = \frac{(b-a)^3}{12}$$

and

$$(b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt = \frac{1}{8} (b-a)^3 - \frac{(b-a)^3}{24} = \frac{(b-a)^3}{12}.$$

Therefore

$$\left| (b-a) A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right|^2 = \frac{(b-a)^6}{144}$$

and

$$\frac{1}{12} (b-a)^3 \int_a^b |A'(t)|^2 dt = \frac{(b-a)^6}{144},$$

which proves the sharpness of the constant  $\frac{1}{12}$ .  $\square$

**Remark 1.** Recall the Löwner–Heinz inequality which says that  $A \geq B \geq 0$  implies  $A^\alpha \geq B^\alpha$  for all  $\alpha \in [0, 1]$ . By taking the power  $1/2$  in (2.1) we derive

$$(2.5) \quad \left| (b-a) A(s) - \int_a^b A(t) dt \right| \leq (b-a)^{1/2} \left[ \frac{1}{12} (b-a)^2 + \left( s - \frac{a+b}{2} \right)^2 \right]^{1/2} \left( \int_a^b |A'(t)|^2 dt \right)^{1/2},$$

while from (2.4) we get

$$(2.6) \quad \left| (b-a) A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right| \leq \frac{\sqrt{3}}{6} (b-a)^{3/2} \left( \int_a^b |A'(t)|^2 dt \right)^{1/2}.$$

The constant  $\frac{\sqrt{3}}{6}$  is best possible in (2.6).

The following alternative result also holds:

**Theorem 3.** Assume that for  $s \in (a, b)$ , the function  $A : [a, b] \rightarrow \mathcal{B}(H)$  is continuous on  $[a, b]$  and strongly differentiable on  $(a, s) \cup (s, b)$  with  $A' \in L_2([a, b], \mathcal{B}(H))$ . Then

$$(2.7) \quad \begin{aligned} & \left| (b-a) A(s) - \int_a^b A(t) dt \right|^2 \\ & \leq \left[ (s-a)^4 + (b-s)^4 \right] \\ & \quad \times \left[ \left| \int_0^1 \tau A'((1-\tau)a + \tau s) d\tau \right|^2 + \left| \int_0^1 (1-\tau) A'((1-\tau)s + \tau b) d\tau \right|^2 \right] \\ & \leq \frac{1}{3} \left[ (s-a)^4 + (b-s)^4 \right] \\ & \quad \times \int_0^1 \left[ |A'((1-\tau)a + \tau s)|^2 + |A'((1-\tau)s + \tau b)|^2 \right] d\tau. \end{aligned}$$

*Proof.* We use the following change of variable  $t = (1-\tau)c + \tau d$ ,  $dt = (d-c) d\tau$  and

$$\int_c^d A(t) dt = (d-c) \int_0^1 A((1-\tau)c + \tau d) d\tau.$$

Therefore

$$\begin{aligned} \int_a^s (t-a) A'(t) dt &= (s-a) \int_0^1 ((1-\tau)a + \tau s - a) A'((1-\tau)a + \tau s) d\tau \\ &= (s-a)^2 \int_0^1 \tau A'((1-\tau)a + \tau s) d\tau \end{aligned}$$

and

$$\begin{aligned} \int_s^b (t-b) A'(t) dt &= (b-s) \int_0^1 ((1-\tau)s + \tau b - b) A'((1-\tau)s + \tau b) d\tau \\ &= -(b-s)^2 \int_0^1 (1-\tau) A'((1-\tau)s + \tau b) d\tau. \end{aligned}$$

By (2.2) we derive the following operator identity of interest

$$(2.8) \quad (b-a)A(s) - \int_a^b A(t) ds = (s-a)^2 \int_0^1 \tau A'((1-\tau)a + \tau s) d\tau - (b-s)^2 \int_0^1 (1-\tau) A'((1-\tau)s + \tau b) d\tau.$$

If we take the modulus and use the elementary inequality which follows by (1.6)

$$\left(|z_1|^2 + |z_2|^2\right) \left(|A_1|^2 + |A_2|^2\right) \geq |z_1 A_1 + z_2 A_2|^2,$$

then we get

$$\begin{aligned} &\left| (b-a)A(s) - \int_a^b A(t) dt \right|^2 \\ &= \left| (s-a)^2 \int_0^1 \tau A'((1-\tau)a + \tau s) d\tau - (b-s)^2 \int_0^1 (1-\tau) A'((1-\tau)s + \tau b) d\tau \right|^2 \\ &\leq \left[ (s-a)^4 + (b-s)^4 \right] \\ &\quad \times \left[ \left| \int_0^1 \tau A'((1-\tau)a + \tau s) d\tau \right|^2 + \left| \int_0^1 (1-\tau) A'((1-\tau)s + \tau b) d\tau \right|^2 \right], \end{aligned}$$

which proves the first inequality in (2.7).

By (1.5) we have

$$\begin{aligned} \left| \int_0^1 \tau A'((1-\tau)a + \tau s) d\tau \right|^2 &\leq \int_0^1 \tau^2 d\tau \int_0^1 |A'((1-\tau)a + \tau s)|^2 d\tau \\ &= \frac{1}{3} \int_0^1 |A'((1-\tau)a + \tau s)|^2 d\tau \end{aligned}$$

and

$$\begin{aligned} \left| \int_0^1 (1-\tau)^2 A'((1-\tau)s + \tau b) d\tau \right|^2 &\leq \int_0^1 (1-\tau)^2 d\tau \int_0^1 |A'((1-\tau)s + \tau b)|^2 d\tau \\ &= \frac{1}{3} \int_0^1 |A'((1-\tau)s + \tau b)|^2 d\tau, \end{aligned}$$

which proves the last part of (2.7).  $\square$

The following representation result holds.

**Lemma 1.** *Let  $B : [a, b] \rightarrow \mathcal{B}(H)$  be a Bochner integrable on  $[a, b]$ . Then for any  $\lambda \in [0, 1]$  we have the representation*

$$(2.9) \quad \int_0^1 B[(1-t)a + tb] dt = (1-\lambda) \int_0^1 B[(1-t)((1-\lambda)a + \lambda b) + tb] dt \\ + \lambda \int_0^1 B[(1-t)a + t((1-\lambda)a + \lambda b)] dt.$$

In particular,

$$(2.10) \quad \int_0^1 B[(1-t)a + tb] dt \\ = \frac{1}{2} \int_0^1 \left( B \left[ (1-t) \frac{a+b}{2} + tb \right] + B \left[ (1-t)a + t \frac{a+b}{2} \right] \right) dt.$$

*Proof.* For  $\lambda = 0$  and  $\lambda = 1$  the equality (2.9) is obvious.

Let  $\lambda \in (0, 1)$ . Observe that

$$\int_0^1 B[(1-t)(\lambda b + (1-\lambda)a) + tb] dt \\ = \int_0^1 B[((1-t)\lambda + t)b + (1-t)(1-\lambda)a] dt$$

and

$$\int_0^1 B[t(\lambda b + (1-\lambda)a) + (1-t)a] dt = \int_0^1 B[t\lambda b + (1-\lambda t)a] dt.$$

If we make the change of variable  $u := (1-t)\lambda + t$  then we have  $1-u = (1-t)(1-\lambda)$  and  $du = (1-\lambda) dt$ . Then

$$\int_0^1 B[((1-t)\lambda + t)b + (1-t)(1-\lambda)a] dt = \frac{1}{1-\lambda} \int_\lambda^1 B[ub + (1-u)a] du.$$

If we make the change of variable  $u := \lambda t$  then we have  $du = \lambda dt$  and

$$\int_0^1 B[t\lambda b + (1-\lambda t)a] dt = \frac{1}{\lambda} \int_0^\lambda B[ub + (1-u)a] du.$$

Therefore

$$(1-\lambda) \int_0^1 B[(1-t)(\lambda b + (1-\lambda)a) + tb] dt \\ + \lambda \int_0^1 B[t(\lambda b + (1-\lambda)a) + (1-t)a] dt \\ = \int_\lambda^1 B[ub + (1-u)a] du + \int_0^\lambda B[ub + (1-u)a] du \\ = \int_0^1 B[ub + (1-u)a] du$$

and the identity (2.9) is proved.  $\square$

**Corollary 2.** *Assume that the function  $A : [a, b] \rightarrow \mathcal{B}(H)$  is continuous on  $[a, b]$  and strongly differentiable on  $(a, \frac{a+b}{2}) \cup (\frac{a+b}{2}, b)$  with  $A' \in L_2([a, b], \mathcal{B}(H))$ . Then*

$$\begin{aligned}
(2.11) \quad & \left| (b-a) A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right|^2 \\
& \leq \frac{1}{8} (b-a)^4 \left[ \left| \int_0^1 \tau A' \left( (1-\tau)a + \tau \frac{a+b}{2} \right) d\tau \right|^2 \right. \\
& \quad \left. + \left| \int_0^1 (1-\tau) A' \left( (1-\tau) \frac{a+b}{2} + \tau b \right) d\tau \right|^2 \right] \\
& \leq \frac{1}{12} (b-a)^4 \int_0^1 |A'((1-\tau)a + \tau b)|^2 dt = \frac{1}{12} (b-a)^3 \int_0^1 |A'(t)|^2 dt.
\end{aligned}$$

The constants  $\frac{1}{8}$  and  $\frac{1}{12}$  are best possible in (2.4).

*Proof.* From (2.7) we have for  $s = \frac{a+b}{2}$  that

$$\begin{aligned}
(2.12) \quad & \left| (b-a) A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right|^2 \\
& \leq \frac{1}{8} (b-a)^4 \left[ \left| \int_0^1 \tau A' \left( (1-\tau)a + \tau \frac{a+b}{2} \right) d\tau \right|^2 \right. \\
& \quad \left. + \left| \int_0^1 (1-\tau) A' \left( (1-\tau) \frac{a+b}{2} + \tau b \right) d\tau \right|^2 \right] \\
& \leq \frac{1}{24} (b-a)^4 \\
& \quad \times \int_0^1 \left[ \left| A' \left( (1-\tau)a + \tau \frac{a+b}{2} \right) \right|^2 + \left| A' \left( (1-\tau) \frac{a+b}{2} + \tau b \right) \right|^2 \right] d\tau.
\end{aligned}$$

By (2.10) we also have

$$\begin{aligned}
& \frac{1}{2} \int_0^1 \left[ \left| A' \left( (1-\tau)a + \tau \frac{a+b}{2} \right) \right|^2 + \left| A' \left( (1-\tau) \frac{a+b}{2} + \tau b \right) \right|^2 \right] d\tau \\
& = \int_0^1 |A'((1-\tau)a + \tau b)|^2 dt = \frac{1}{b-a} \int_0^1 |A'(t)|^2 dt
\end{aligned}$$

and the inequality (2.11) is thus proved.  $\square$

We can introduce the following concept:

**Definition 1.** *We say that the continuous function  $B : [a, b] \rightarrow \mathcal{B}(H)$  is square modulus convex on  $[a, b]$  if*

$$(2.13) \quad |B((1-t)u + tv)|^2 \leq (1-t)|B(u)|^2 + t|B(v)|^2$$

*in the operator order of  $\mathcal{B}(H)$ , for all  $u, v \in [a, b]$  and  $t \in [0, 1]$ .*



Let  $A, B \in \mathcal{B}(H)$  and  $\alpha \in [0, 1]$ . Then by (1.6) we get

$$\begin{aligned} |(1-\alpha)A + \alpha B|^2 &= \left| (1-\alpha)^{1/2} (1-\alpha)^{1/2} A + \alpha^{1/2} \alpha^{1/2} B \right|^2 \\ &\leq \left[ \left( (1-\alpha)^{1/2} \right)^2 + \left( \alpha^{1/2} \right)^2 \right] \left[ \left| (1-\alpha)^{1/2} A \right|^2 + \left| \alpha^{1/2} B \right|^2 \right] \\ &= (1-\alpha + \alpha) \left[ (1-\alpha) |A|^2 + \alpha |B|^2 \right] \\ &= (1-\alpha) |A|^2 + \alpha |B|^2. \end{aligned}$$

Consider the function  $C : [0, 1] \rightarrow \mathcal{B}(H)$ ,  $C(t) = |(1-t)A + tB|$ . Let  $t_1, t_2 \in [0, 1]$  and  $\alpha \in [0, 1]$ . Then

$$\begin{aligned} |C((1-\alpha)t_1 + \alpha t_2)|^2 &= |(1 - (1-\alpha)t_1 - \alpha t_2)A + ((1-\alpha)t_1 + \alpha t_2)B|^2 \\ &= |(1-\alpha)((1-t_1)A + t_1B) + \alpha((1-t_2)A + t_2B)|^2 \\ &\leq (1-\alpha)|((1-t_1)A + t_1B)|^2 + \alpha|(1-t_2)A + t_2B|^2 \\ &= (1-\alpha)|C(t_1)|^2 + \alpha|C(t_2)|^2, \end{aligned}$$

which shows that  $C$  is *square modulus convex* on  $[0, 1]$ .

Assume that  $f$  is *nonnegative* on  $I$  and *operator convex*, namely

$$f((1-\alpha)A + \alpha B) \leq (1-\alpha)f(A) + \alpha f(B)$$

for all  $\alpha \in [0, 1]$  and selfadjoint operators  $A, B$  with spectra in  $I$ .

For such function and  $A, B$ , we consider

$$D(t) := [f((1-t)A + tB)]^{1/2}, t \in [0, 1].$$

Then, using a similar proof as above for the modulus function, we conclude that  $D$  is *square modulus convex* on  $[0, 1]$ .

The function  $f(t) = t^r$  is operator convex on  $(0, \infty)$  if either  $1 \leq r \leq 2$  or  $-1 \leq r \leq 0$  and is operator concave on  $(0, \infty)$  if  $0 \leq r \leq 1$ . Therefore for  $A, B > 0$ , the function

$$B(t) := ((1-t)A + tB)^{r/2}, t \in [0, 1]$$

is *square modulus convex* on  $[0, 1]$  for  $1 \leq r \leq 2$  or  $-1 \leq r \leq 0$ .

**Proposition 1.** *With the assumption of Theorem 2 and if  $A'$  is square modulus convex on  $(a, b)$ , then*

$$\begin{aligned} (2.14) \quad &\left| (b-a)A(s) - \int_a^b A(t) dt \right|^2 \\ &\leq \frac{1}{2} (b-a)^2 \left[ \frac{1}{12} (b-a)^2 + \left( s - \frac{a+b}{2} \right)^2 \right] \left[ |A'(a)|^2 + |A'(b)|^2 \right]. \end{aligned}$$

*In particular,*

$$(2.15) \quad \left| (b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right|^2 \leq \frac{1}{24} (b-a)^4 \left[ |A'(a)|^2 + |A'(b)|^2 \right].$$

*Proof.* We have

$$\begin{aligned} \int_a^b |A'(t)|^2 dt &= (b-a) \int_0^1 |A'((1-s)a + sb)|^2 ds \\ &\leq (b-a) \int_0^1 \left[ (1-s)|A'(a)|^2 + s|A'(b)|^2 \right] ds \\ &= (b-a) \frac{|A'(a)|^2 + |A'(b)|^2}{2}, \end{aligned}$$

and by (2.1) we get (2.14).  $\square$

We also have:

**Proposition 2.** *With the assumption of Theorem 3 and if  $A'$  is square modulus convex on  $(a, b)$  then*

$$(2.16) \quad \left| (b-a)A(s) - \int_a^b A(t) dt \right|^2 \leq \frac{1}{3} \left[ (s-a)^4 + (b-s)^4 \right] \left[ |A'(s)|^2 + \frac{|A'(b)|^2 + |A'(a)|^2}{2} \right].$$

*In particular,*

$$(2.17) \quad \left| (b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t) dt \right|^2 \leq \frac{1}{24} (b-a)^2 \left[ \left| A'\left(\frac{a+b}{2}\right) \right|^2 + \frac{|A'(b)|^2 + |A'(a)|^2}{2} \right] \leq \frac{1}{12} (b-a)^4 \left[ |A'(a)|^2 + |A'(b)|^2 \right].$$

*Proof.* By the convexity

$$\int_0^1 |A'((1-\tau)s + \tau b)|^2 d\tau \leq \frac{1}{2} \left[ |A'(s)|^2 + |A'(b)|^2 \right]$$

and

$$\int_0^1 |A'((1-\tau)a + \tau s)|^2 d\tau \leq \frac{1}{2} \left[ |A'(a)|^2 + |A'(s)|^2 \right]$$

and by (2.7) we derive

$$\begin{aligned} &\left| (b-a)A(s) - \int_a^b A(t) dt \right|^2 \\ &\leq \frac{1}{3} \left[ (s-a)^4 + (b-s)^4 \right] \\ &\quad \times \frac{1}{2} \left[ |A'(s)|^2 + |A'(b)|^2 + |A'(a)|^2 + |A'(s)|^2 \right] \\ &= \frac{1}{3} \left[ (s-a)^4 + (b-s)^4 \right] \left[ |A'(s)|^2 + \frac{|A'(b)|^2 + |A'(a)|^2}{2} \right], \end{aligned}$$

which proves (2.16).  $\square$

## 3. SOME EXAMPLES

Consider the function  $A(t) = \exp(tT)$ , where  $t \in \mathbb{R}$  and  $T \in \mathcal{B}(H)$ . Then  $A'(t) = T \exp(tT)$ , for  $t \in \mathbb{R}$  and  $T \in \mathcal{B}(H)$ . By making use of (2.4) we get

$$(3.1) \quad \left| (b-a) \exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT) dt \right|^2 \leq \frac{1}{12} (b-a)^3 \int_a^b |T \exp(tT)|^2 dt.$$

If  $T$  is invertible, then [3]

$$(3.2) \quad \int_a^b \exp(tT) dt = T^{-1} [\exp(bT) - \exp(aT)].$$

From (3.1) we derive

$$(3.3) \quad \left| (b-a) \exp\left(\frac{a+b}{2}T\right) - T^{-1} [\exp(bT) - \exp(aT)] \right|^2 \leq \frac{1}{12} (b-a)^3 \int_a^b |T \exp(tT)|^2 dt.$$

For  $T$  invertible, if we consider  $B(t) = T \exp(tT)$ , then  $B'(t) = T^2 \exp(tT)$  and

$$\int_a^b B(t) dt = \exp(bT) - \exp(aT).$$

By (2.17) we derive

$$(3.4) \quad \left| (b-a) T \exp\left(\frac{a+b}{2}T\right) - \exp(bT) + \exp(aT) \right|^2 \leq \frac{1}{12} (b-a)^3 \int_a^b |T^2 \exp(tT)|^2 dt.$$

Since for any operator  $V \in \mathcal{B}(H)$  we have  $|V|^2 \leq \|V\|^2$  and  $\|\exp(tT)\| \leq \exp(t\|T\|)$ ,  $t \in \mathbb{R}$ ,  $T \in \mathcal{B}(H)$ , then by (3.1) we get

$$(3.5) \quad \left| (b-a) \exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT) dt \right|^2 \leq \frac{1}{12} (b-a)^3 \int_a^b \|T \exp(tT)\|^2 dt \leq \frac{1}{12} \|T\|^2 (b-a)^3 \int_a^b \|\exp(tT)\|^2 dt \leq \frac{1}{12} \|T\|^2 (b-a)^3 \int_a^b \exp(2\|T\||t|) dt.$$

If  $0 \leq a \leq b$ , then

$$\int_a^b \exp(2\|T\||t|) dt = \int_a^b \exp(2\|T\|t) dt = \frac{\exp(2\|T\|b) - \exp(2\|T\|a)}{2\|T\|}$$

and by (3.5) we get

$$(3.6) \quad \left| (b-a) \exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT) dt \right|^2 \leq \frac{1}{24} (b-a)^3 \|T\| [\exp(2\|T\|b) - \exp(2\|T\|a)]$$

for any  $T \in \mathcal{B}(H)$ .

Moreover, if  $T$  is invertible, then we also have the exponential inequality

$$(3.7) \quad \left| (b-a) \exp\left(\frac{a+b}{2}T\right) - T^{-1} [\exp(bT) - \exp(aT)] \right|^2 \\ \leq \frac{1}{24} (b-a)^3 \|T\| [\exp(2\|T\|b) - \exp(2\|T\|a)].$$

Consider the function  $A(t) = \exp[(1-t)A](B-A)\exp(tB)$ ,  $t \in [0, 1]$ . Then, integrating by parts

$$\begin{aligned} & \int_0^1 f(t) dt \\ &= \int_0^1 (\exp[(1-t)A]B\exp(tB) - \exp[(1-t)A]A\exp(tB)) dt \\ &= \int_0^1 \exp[(1-t)A](\exp(tB))' dt + \int_0^1 (\exp[(1-t)A])' \exp(tB) dt \\ &= \exp[(1-t)A]\exp(tB)|_0^1 + A \int_0^1 \exp[(1-t)A]\exp(tB) dt \\ &+ \exp[(1-t)A]\exp(tB)|_0^1 - \int_0^1 (\exp[(1-t)A])B\exp(tB) dt \\ &= 2(\exp B - \exp A) - \int_0^1 \exp[(1-t)A](B-A)\exp(tB) dt \\ &= 2(\exp B - \exp A) - \int_0^1 f(t) dt, \end{aligned}$$

which gives the following identity of interest [4]

$$\int_0^1 \exp[(1-t)A](B-A)\exp(tB) dt = \exp B - \exp A$$

for all  $A, B \in \mathcal{B}(H)$ .

Also

$$\begin{aligned} A'(t) &= -A \exp[(1-t)A](B-A)\exp(tB) \\ &+ \exp[(1-t)A](B-A)B\exp(tB) dt \\ &= \exp[(1-t)A](B-A)B\exp(tB) dt \\ &- \exp[(1-t)A]A(B-A)\exp(tB) \\ &= \exp[(1-t)A][(B-A)B - A(B-A)]\exp(tB) \\ &= \exp[(1-t)A](B^2 - 2AB + A^2)\exp(tB). \end{aligned}$$

By utilising (2.4) we get

$$(3.8) \quad \left| \exp\left(\frac{1}{2}A\right)(B-A)\exp\left(\frac{1}{2}B\right) - \exp B + \exp A \right|^2 \\ \leq \frac{1}{12} \int_a^b |\exp[(1-t)A](B^2 - 2AB + A^2)\exp(tB)|^2 dt,$$

for all  $A, B \in \mathcal{B}(H)$ .

Since

$$\begin{aligned}
& \left| \exp [(1-t)A] (B^2 - 2AB + A^2) \exp (tB) \right|^2 \\
& \leq \left\| \exp [(1-t)A] (B^2 - 2AB + A^2) \exp (tB) \right\|^2 \\
& \leq \left\| \exp [(1-t)A] \right\|^2 \left\| B^2 - 2AB + A^2 \right\|^2 \left\| \exp (tB) \right\|^2 \\
& \leq \exp [2(1-t)\|A\|] \left\| B^2 - 2AB + A^2 \right\|^2 \exp (2t\|B\|) \\
& = \left\| B^2 - 2AB + A^2 \right\|^2 \exp [2[(1-t)\|A\| + t\|B\|]] \\
& = \left\| B^2 - 2AB + A^2 \right\|^2 \exp \{2[(1-t)\|A\| + t\|B\|]\},
\end{aligned}$$

hence by (3.8) we get

$$\begin{aligned}
(3.9) \quad & \left| \exp \left( \frac{1}{2}A \right) (B - A) \exp \left( \frac{1}{2}B \right) - \exp B + \exp A \right|^2 \\
& \leq \frac{1}{12} \left\| B^2 - 2AB + A^2 \right\|^2 \int_a^b \exp \{2[(1-t)\|A\| + t\|B\|]\} dt \\
& = \frac{1}{12} \left\| B^2 - 2AB + A^2 \right\|^2 \begin{cases} \exp (2\|A\|) & \text{if } \|B\| = \|A\|, \\ \frac{\exp (2\|B\|) - \exp (2\|A\|)}{2(\|B\| - \|A\|)} & \text{if } \|B\| \neq \|A\|. \end{cases}
\end{aligned}$$

Further, let  $A, B \in \mathcal{B}(H)$  such that  $(1-t)A + tB$  is invertible for all  $t \in [0, 1]$ . For this to happen, it is enough to assume that  $A, B > 0$  in the operator order of  $\mathcal{B}(H)$ . Consider the function  $A(t) := ((1-t)A + tB)^{-1}$ ,  $t \in [0, 1]$  and observe that

$$A'(t) = -((1-t)A + tB)^{-1} (B - A) ((1-t)A + tB)^{-1}, \quad t \in [0, 1].$$

By utilising (2.4) we then get

$$\begin{aligned}
(3.10) \quad & \left| \left( \frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t)A + tB)^{-1} dt \right|^2 \\
& \leq \frac{1}{12} \int_0^1 \left| ((1-t)A + tB)^{-1} (B - A) ((1-t)A + tB)^{-1} \right|^2 dt
\end{aligned}$$

for  $A, B \in \mathcal{B}(H)$  such that  $(1-t)A + tB$  is invertible for all  $t \in [0, 1]$ .

Since

$$\begin{aligned}
& \left| ((1-t)A + tB)^{-1} (B - A) ((1-t)A + tB)^{-1} \right| \\
& \leq \left\| ((1-t)A + tB)^{-1} \right\|^2 \|B - A\|,
\end{aligned}$$

hence by (3.10) we derive

$$\begin{aligned}
(3.11) \quad & \left| \left( \frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t)A + tB)^{-1} dt \right|^2 \\
& \leq \frac{1}{12} \|B - A\|^2 \int_0^1 \left\| ((1-t)A + tB)^{-1} \right\|^4 dt.
\end{aligned}$$

Now, if  $A \geq m > 0$  and  $B \geq m > 0$ , then  $((1-t)A + tB)^{-1} \leq m^{-1}$  for  $t \in [0, 1]$  and by (3.11) we obtain the simpler inequality

$$\left| \left( \frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t)A + tB)^{-1} dt \right|^2 \leq \frac{1}{12} \frac{\|B-A\|^2}{m^4}.$$

Since  $f(u) = u^{-1}$  is operator convex on  $(0, \infty)$ , then by taking the square root and using the Hermite-Hadamard operator inequality [6], we derive

$$(3.12) \quad 0 \leq \int_0^1 ((1-t)A + tB)^{-1} dt - \left( \frac{A+B}{2} \right)^{-1} \leq \frac{\sqrt{3}}{6} \frac{\|B-A\|}{m^2},$$

provided that  $A \geq m > 0$  and  $B \geq m > 0$ .

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<sup>1</sup>MATHEMATICS, COLLEGE OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO BOX 14428, MELBOURNE CITY, MC 8001, AUSTRALIA.

*E-mail address:* sever.dragomir@vu.edu.au

*URL:* <http://rgmia.org/dragomir>

<sup>2</sup>DST-NRF CENTRE OF EXCELLENCE IN THE MATHEMATICAL, AND STATISTICAL SCIENCES, SCHOOL OF COMPUTER SCIENCE, & APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND,, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA