SIMPLE WEIGHTED TRAPEZOID TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. In this paper we obtain among others the following result, if $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then

$$\left|\left(\int_a^b\alpha(s)\,ds\right)\frac{B(a)+B(b)}{2}-\int_a^b\alpha(t)B(t)\,dt\right|^2 \le\frac{1}{4}(b-a)\int_a^b|\alpha(s)|^2\,ds\int_a^b\left|B'(t)\right|^2dt.$$

In particular,

$$\left|(b-a)\frac{B(a)+B(b)}{2}-\int_a^b B(t)\,dt\right|^2\le\frac{1}{12}(b-a)^3\int_a^b\left|B'(t)\right|^2dt.$$

Some examples for the inverse and exponential functions are also provided.

1. Introduction

In 1999, Cerone and Dragomir proved the following *generalized trapezoid* type inequality for p-norm [5].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right| \le\frac{1}{(q+1)^{1/q}}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{1/q}\|f'\|_{[a,b],p}, \tag{1.1}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b|g(t)|^p\,dt\right)^{1/p}.$$

From (1.1) we get the following *trapezoid inequality*

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|\le\frac{1}{2(q+1)^{1/q}}(b-a)^{1+1/q}\|f'\|_{[a,b],p}, \tag{1.2}$$

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and $\frac{1}{2}$ is a best possible constant.

For $p=q=2$ we derive the $L_2[a,b]$-inequality

$$\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right|^2 \le \frac{1}{3}\left[(x-a)^3+(b-x)^3\right]\|f'\|_{[a,b],2}^2, \tag{1.3}$$

for all $x\in[a,b]$ and the trapezoid inequality

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|^2 \le \frac{1}{12}(b-a)^3\|f'\|_{[a,b]}^2. \tag{1.4}$$

For a survey on scalar trapezoid inequality, see [5]. For recent papers on this inequality see also [1]-[4] and [6]-[15].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\le|A|+|B|$, so the classical arguments using this inequality can not be used.

In this paper we obtain among others the following result, if $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then

$$\left|\left(\int_a^b\alpha(s)\,ds\right)\frac{B(a)+B(b)}{2}-\int_a^b\alpha(t)B(t)\,dt\right|^2 \le \frac{1}{4}(b-a)\int_a^b|\alpha(s)|^2\,ds\int_a^b|B'(t)|^2\,dt.$$

In particular,

$$\left|(b-a)\frac{B(a)+B(b)}{2}-\int_a^b B(t)\,dt\right|^2 \le \frac{1}{12}(b-a)^3\int_a^b|B'(t)|^2\,dt.$$

Some examples for the inverse and exponential functions are also provided.

2. Main Results

In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$0\le\left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2=|\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s) -\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2,$$

which gives that

$$|\alpha(t)|^2|A(s)|^2+|\alpha(s)|^2|A(t)|^2\ge\alpha(s)\overline{\alpha(t)}A^*(t)A(s)+\alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s,\,t\in[a,b]$.

Now, multiply this with $w(s)w(t)\geq 0$ to get

$$w(t)|\alpha(t)|^2 w(s)|A(s)|^2 + w(s)|\alpha(s)|^2 w(t)|A(t)|^2$$
$$\geq w(t)\overline{\alpha(t)}A^*(t)w(s)\alpha(s)A(s) + w(s)\overline{\alpha(s)}A^*(s)w(t)\alpha(t)A(t)$$

for all $s,\, t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\int_a^b w(t)|\alpha(t)|^2 dt \int_a^b |A(s)|^2 ds + \int_a^b |\alpha(s)|^2 ds \int_a^b w(t)|A(t)|^2 dt$$
$$\geq \int_a^b w(t)\overline{\alpha(t)}A^*(t)\,dt \int_a^b \alpha(s)A(s)\,ds$$
$$+ \int_a^b w(s)\overline{\alpha(s)}A^*(s)\,ds \int_a^b \alpha(t)A(t)\,dt$$
$$= 2\left|\int_a^b w(s)\alpha(s)A(s)\,ds\right|^2,$$

which proves that

$$\int_a^b w(t)|\alpha(t)|^2 dt \int_a^b w(t)|A(t)|^2 dt \geq \left|\int_a^b w(t)\alpha(t)A(t)\,dt\right|^2, \tag{2.1}$$

provided that $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and

$$A\in L_{2,w}([a,b],\mathcal{B}(H)) := \left\{A:[a,b]\to B(H),\ \int_a^b w(t)\|A(t)\|^2 dt<\infty\right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^n w_k|z_k|^2 \sum_{k=1}^n w_k|A_k|^2 \geq \left|\sum_{k=1}^n w_k z_k A_k\right|^2, \tag{2.2}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\geq 0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n w_k = 1$.

Theorem 2. *Assume that $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then*

$$\left|\left(\int_a^b \alpha(s)\,ds\right)\frac{B(a)+B(b)}{2} - \int_a^b \alpha(t)B(t)\,dt\right|^2 \tag{2.3}$$
$$\leq \frac{1}{4}\int_a^b \left|\int_a^b \operatorname{sgn}(s-t)\alpha(s)\,ds\right|^2 dt \int_a^b |B'(t)|^2 dt$$
$$\leq \frac{1}{4}(b-a)\int_a^b |\alpha(s)|^2 ds \int_a^b |B'(t)|^2 dt.$$

In particular,

$$\left|(b-a)\frac{B(a)+B(b)}{2} - \int_a^b B(t)\,dt\right|^2 \leq \frac{1}{12}(b-a)^3\int_a^b |B'(t)|^2 dt. \tag{2.4}$$

The constant $\frac{1}{12}$ is best possible in (2.4).

Proof. Using the integration by parts formula, we have

$$\begin{aligned}
&\int_a^b \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] B'(t)\,dt \\
&= \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] B(t)\Big|_a^b \\
&- \int_a^b \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right]' B(t)\,dt \\
&= \left[\int_a^b \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] B(b) \\
&- \left[\int_a^a \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] B(a) - \int_a^b \alpha(t)\,B(t)\,dt \\
&= \left(\int_a^b \alpha(s)\,ds\right)\frac{B(a)+B(b)}{2} - \int_a^b \alpha(t)\,B(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] B'(t)\,dt \\
&= \frac{1}{2}\int_a^b \left(\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right) B'(t)\,dt.
\end{aligned}$$

Therefore we have the following identity of interest

$$\begin{aligned}
&\left(\int_a^b \alpha(s)\,ds\right)\frac{B(a)+B(b)}{2} - \int_a^b \alpha(t)\,B(t)\,dt \\
&= \frac{1}{2}\int_a^b \left(\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right) B'(t)\,dt.
\end{aligned} \tag{2.5}$$

Now, by using the CBS integral inequality, we get

$$\begin{aligned}
&\left|\left(\int_a^b \alpha(s)\,ds\right)\frac{B(a)+B(b)}{2} - \int_a^b \alpha(t)\,B(t)\,dt\right|^2 \\
&= \frac{1}{4}\left|\int_a^b \left(\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right) B'(t)\,dt\right|^2 \\
&\le \frac{1}{4}\int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right|^2 dt \int_a^b |B'(t)|^2\,dt \\
&= \frac{1}{4}\int_a^b \left|\int_a^b \operatorname{sgn}(s-t)\,\alpha(s)\,ds\right|^2 dt \int_a^b |B'(t)|^2\,dt.
\end{aligned} \tag{2.6}$$

Also, observe that by the CBS inequality,

$$\begin{aligned}\left|\int_a^b \operatorname{sgn}(s-t)\,\alpha(s)\,ds\right|^2 &\leq \int_a^b |\operatorname{sgn}(s-t)|^2\,ds \int_a^b |\alpha(s)|^2\,ds \\ &= (b-a)\int_a^b |\alpha(s)|^2\,ds,\end{aligned}$$

which proves the last part of (2.3).

If we take $\alpha(t)=1$, $t\in[a,b]$ in (2.6), then we get

$$\begin{aligned}&\left|(b-a)\frac{B(a)+B(b)}{2}-\int_a^b B(t)\,dt\right|^2 \\ &\leq \frac{1}{4}\int_a^b |t-a-b+t|^2\,dt\int_a^b |B'(t)|^2\,dt \\ &= \int_a^b \left(t-\frac{a+b}{2}\right)^2 dt\int_a^b |B'(t)|^2\,dt = \frac{1}{12}(b-a)^3\int_a^b |B'(t)|^2\,dt,\end{aligned}$$

which proves (2.4).

Consider

$$B(t)=\left(t-\frac{a+b}{2}\right)^2,\ t\in[a,b].$$

We have

$$\begin{aligned}&(b-a)\frac{B(a)+B(b)}{2}-\int_a^b B(t)\,dt \\ &=(b-a)\frac{\frac{(b-a)^2}{4}+\frac{(b-a)^2}{4}}{2}-\frac{1}{12}(b-a)^3=\frac{1}{6}(b-a)^3\end{aligned}$$

and

$$\begin{aligned}\frac{1}{12}(b-a)^3\int_a^b |B'(t)|^2\,dt &= \frac{1}{3}(b-a)^3\int_a^b \left|t-\frac{a+b}{2}\right|^2 dt \\ &=\frac{1}{3}(b-a)^3\frac{1}{12}(b-a)^3=\frac{1}{36}(b-a)^6.\end{aligned}$$

These give the same quantity $\frac{1}{36}(b-a)^6$ in both sides of (2.4). □

Remark 1. *If we take* $\alpha(t)=\left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$ *in (2.3) then we get*

$$\begin{aligned}&\left|(b-a)^2\frac{B(a)+B(b)}{8}-\int_a^b \left|t-\frac{a+b}{2}\right| B(t)\,dt\right|^2 \\ &\leq \frac{1}{48}(b-a)^4\int_a^b |B'(t)|^2\,dt.\end{aligned} \tag{2.7}$$

We can introduce the following concept:

Definition 1. *We say that the continuous function* $B:[a,b]\to\mathcal{B}(H)$ *is square modulus convex (concave) on* $[a,b]$ *if*

$$|B((1-t)u+tv)|^2 \leq (\geq)\,(1-t)\,|B(u)|^2+t\,|B(v)|^2 \tag{2.8}$$

in the operator order of $\mathcal{B}(H)$, *for all* $u,\ v\in[a,b]$ *and* $t\in[0,1]$.

Let A, $B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (2.2) we get

$$\begin{aligned}\left|(1-\alpha)A+\alpha B\right|^{2}&=\left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^{2}\\&\leq\left[\left((1-\alpha)^{1/2}\right)^{2}+\left(\alpha^{1/2}\right)^{2}\right]\left[\left|(1-\alpha)^{1/2}A\right|^{2}+\left|\alpha^{1/2}B\right|^{2}\right]\\&=(1-\alpha+\alpha)\left[(1-\alpha)|A|^{2}+\alpha|B|^{2}\right]\\&=(1-\alpha)|A|^{2}+\alpha|B|^{2}.\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let t_1, $t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}\left|C((1-\alpha)t_1+\alpha t_2)\right|^{2}&=\left|(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B\right|^{2}\\&=\left|(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)\right|^{2}\\&\leq(1-\alpha)\left|(1-t_1)A+t_1B\right|^{2}+\alpha\left|(1-t_2)A+t_2B\right|^{2}\\&=(1-\alpha)\left|C(t_1)\right|^{2}+\alpha\left|C(t_2)\right|^{2},\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\leq(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t):=\left[f((1-t)A+tB)\right]^{1/2},t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^{r}$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. Therefore for A, $B>0$, the function

$$B(t):=((1-t)A+tB)^{r/2},\ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\leq r\leq 2$ or $-1\leq r\leq 0$.

Corollary 1. *With the assumptions of Theorem 2 and if $B':[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$, then*

$$\begin{aligned}&\left|\left(\int_a^b\alpha(s)\,ds\right)\frac{B(a)+B(b)}{2}-\int_a^b\alpha(t)B(t)\,dt\right|^{2}\\&\leq\frac{1}{4}\int_a^b\left|\int_a^b\operatorname{sgn}(s-t)\alpha(s)\,ds\right|^{2}dt\frac{\left|B'(a)\right|^{2}+\left|B'(b)\right|^{2}}{2}\\&\leq\frac{1}{4}(b-a)\int_a^b|\alpha(s)|^{2}ds\frac{\left|B'(a)\right|^{2}+\left|B'(b)\right|^{2}}{2}.\end{aligned}\tag{2.9}$$

In particular,

$$\left|(b-a)\frac{B(a)+B(b)}{2}-\int_a^b B(t)\,dt\right|^2 \leq \frac{1}{24}(b-a)^3\left[|B'(a)|^2+|B'(b)|^2\right]. \tag{2.10}$$

Proof. It follows by (2.3) on observing that

$$\begin{aligned}\int_a^b |B'(t)|^2\,dt &= (b-a)\int_0^1 |B'((1-s)a+sb)|^2\,ds \\ &\leq (b-a)\int_0^1 \left[(1-s)|B'(a)|^2+s|B'(b)|^2\right]ds \\ &= (b-a)\frac{|B'(a)|^2+|B'(b)|^2}{2}.\end{aligned}$$

□

We also have:

Corollary 2. *With the assumptions of Theorem 2 and if $B':[a,b]\to\mathcal{B}(H)$ is square modulus concave on $[a,b]$, then*

$$\begin{aligned}&\left|\left(\int_a^b \alpha(s)\,ds\right)\frac{B(a)+B(b)}{2}-\int_a^b \alpha(t)B(t)\,dt\right|^2 \\ &\leq \frac{1}{4}\int_a^b\left|\int_a^b \operatorname{sgn}(s-t)\alpha(s)\,ds\right|^2 dt\left|B'\left(\frac{a+b}{2}\right)\right|^2 \\ &\leq \frac{1}{4}(b-a)\int_a^b|\alpha(s)|^2\,ds\left|B'\left(\frac{a+b}{2}\right)\right|^2\end{aligned} \tag{2.11}$$

In particular,

$$\left|(b-a)\frac{B(a)+B(b)}{2}-\int_a^b B(t)\,dt\right|^2 \leq \frac{1}{12}(b-a)^3\left|B'\left(\frac{a+b}{2}\right)\right|^2. \tag{2.12}$$

Proof. Since $B':[a,b]\to\mathcal{B}(H)$ is square modulus concave, then

$$\left|B'\left(\frac{u+v}{2}\right)\right|^2 \geq \frac{|B'(u)|^2+|B'(v)|^2}{2}$$

for all $u,v\in[a,b]$.

By taking $u=(1-s)a+sb$ and $v=sa+(1-s)b$, $s\in[0,1]$ we get

$$\left|B'\left(\frac{a+b}{2}\right)\right|^2 \geq \frac{|B'((1-s)a+sb)|^2+|B'(sa+(1-s)b)|^2}{2}$$

$s\in[0,1]$.

If we take the integral over we $s\in[0,1]$ get

$$\begin{aligned}\left|B'\left(\frac{a+b}{2}\right)\right|^2 &\geq \frac{1}{2}\int_0^1\left[\left|B'\left((1-s)a+sb\right)\right|^2+\left|B'\left(sa+(1-s)b\right)\right|^2\right]ds\\ &= \int_0^1\left|B'\left((1-s)a+sb\right)\right|^2 ds.\end{aligned}$$

The results follow now by Theorem 2. □

3. Some Examples

Further, let $A,\ B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $A,\ B>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t):=((1-t)A+tB)^{-1}$, $t\in[0,1]$ and observe that

$$B'(t)=-\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1},\ t\in[0,1].$$

Assume that $\alpha:[0,1]\to\mathbb{C}$ is integrable, then from (2.3) we have for all $u\in(0,1)$

$$\begin{aligned}&\left|\left(\int_a^b\alpha(s)\,ds\right)\frac{A^{-1}+B^{-1}}{2}-\int_a^b\alpha(t)\left((1-t)A+tB\right)^{-1}dt\right|^2\\ &\leq\frac{1}{4}\int_a^b\left|\int_a^b\operatorname{sgn}(s-t)\,\alpha(s)\,ds\right|^2dt\\ &\times\int_a^b\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^2dt\\ &\leq\frac{1}{4}(b-a)\int_a^b|\alpha(s)|^2ds\\ &\times\int_a^b\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^2dt.\end{aligned}\tag{3.1}$$

In particular,

$$\begin{aligned}&\left|(b-a)\frac{A^{-1}+B^{-1}}{2}-\int_a^b\left((1-t)A+tB\right)^{-1}dt\right|^2\\ &\leq\frac{1}{12}(b-a)^3\\ &\times\int_a^b\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^2dt.\end{aligned}\tag{3.2}$$

Since for any operator $V\in\mathcal{B}(H)$ we have $|V|^2\leq\|V\|^2$, then

$$\begin{aligned}&\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^2\\ &\leq\left\|\left((1-t)A+tB\right)^{-1}\right\|^4\|B-A\|^2\end{aligned}$$

for all $t \in [0,1]$, which implies that

$$\left| \left(\int_a^b \alpha(s)\,ds \right) \frac{A^{-1}+B^{-1}}{2} - \int_a^b \alpha(t)\left((1-t)A+tB\right)^{-1} dt \right|^2 \leq \frac{1}{4}(b-a)\|B-A\|^2 \int_a^b |\alpha(s)|^2 ds \int_a^b \left\| \left((1-t)A+tB\right)^{-1} \right\|^4 dt. \tag{3.3}$$

In particular,

$$\left| (b-a)\frac{A^{-1}+B^{-1}}{2} - \int_a^b \left((1-t)A+tB\right)^{-1} dt \right|^2 \leq \frac{1}{12}(b-a)^3 \|B-A\|^2 \int_a^b \left\| \left((1-t)A+tB\right)^{-1} \right\|^4 dt. \tag{3.4}$$

Now, if $A \geq m > 0$ and $B \geq m > 0$, then $\left((1-t)A+tB\right)^{-1} \leq m^{-1}$ for $t \in [0,1]$, which implies $\left\| \left((1-t)A+tB\right)^{-1} \right\|^4 \leq m^{-4}$ and by (3.3) we get

$$\left| \left(\int_a^b \alpha(s)\,ds \right) \frac{A^{-1}+B^{-1}}{2} - \int_a^b \alpha(t)\left((1-t)A+tB\right)^{-1} dt \right|^2 \leq \frac{1}{4}(b-a)\frac{\|B-A\|^2}{m^4} \int_a^b |\alpha(s)|^2 ds, \tag{3.5}$$

while from (3.4)

$$\left| (b-a)\frac{A^{-1}+B^{-1}}{2} - \int_a^b \left((1-t)A+tB\right)^{-1} dt \right|^2 \leq \frac{1}{12}(b-a)^3 \frac{\|B-A\|^2}{m^4}. \tag{3.6}$$

Consider the function $B(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $B'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. By making use of (2.3) we get

$$\begin{aligned} &\left| \left(\int_a^b \alpha(s)\,ds \right) \frac{\exp(aT)+\exp(bT)}{2} - \int_a^b \alpha(t)\exp(tT)\,dt \right|^2 \\ &\leq \frac{1}{4}\int_a^b \left| \int_a^b \operatorname{sgn}(s-t)\,\alpha(s)\,ds \right|^2 dt \int_a^b |T\exp(tT)|^2 dt \\ &\leq \frac{1}{4}(b-a)\int_a^b |\alpha(s)|^2 ds \int_a^b |T\exp(tT)|^2 dt. \end{aligned} \tag{3.7}$$

In particular,

$$\left| (b-a)\frac{\exp(aT)+\exp(bT)}{2} - \int_a^b \exp(tT)\,dt \right|^2 \leq \frac{1}{12}(b-a)^3 \int_a^b |T\exp(tT)|^2 dt. \tag{3.8}$$

Since for any operator $V \in \mathcal{B}(H)$ we have $|V|^2 \leq \|V\|^2$ and $\|\exp(tT)\| \leq \exp(|t| \|T\|)$, $t \in \mathbb{R}$, $T \in \mathcal{B}(H)$, then by (3.7) we get

$$\left| \left(\int_a^b \alpha(s)\, ds \right) \frac{\exp(aT) + \exp(bT)}{2} - \int_a^b \alpha(t) \exp(tT)\, dt \right|^2 \tag{3.9}$$
$$\leq \frac{1}{4}(b-a) \int_a^b |\alpha(s)|^2 ds \int_a^b \|T \exp(tT)\|^2 dt$$
$$\leq \frac{1}{4}(b-a) \|T\|^2 \int_a^b |\alpha(s)|^2 ds \int_a^b \|\exp(tT)\|^2 dt$$
$$\leq \frac{1}{4}(b-a) \|T\|^2 \int_a^b |\alpha(s)|^2 ds \int_a^b \exp(2\|T\| |t|)\, dt.$$

Observe that, if $0 \leq a \leq b$, then

$$\int_a^b \exp(2\|T\| |t|)\, dt = \int_a^b \exp(2\|T\| t)\, dt = \frac{\exp(2\|T\| b) - \exp(2\|T\| a)}{2\|T\|}$$

and by (3.9) we get

$$\left| \left(\int_a^b \alpha(s)\, ds \right) \frac{\exp(aT) + \exp(bT)}{2} - \int_a^b \alpha(t) \exp(tT)\, dt \right|^2 \tag{3.10}$$
$$\leq \frac{1}{8}(b-a) \|T\| \int_a^b |\alpha(s)|^2 ds \left[\exp(2\|T\| b) - \exp(2\|T\| a)\right]$$

for any $T \in \mathcal{B}(H)$.

In particular, we have

$$\left| (b-a) \frac{\exp(aT) + \exp(bT)}{2} - \int_a^b \exp(tT)\, dt \right|^2 \tag{3.11}$$
$$\leq \frac{1}{8}(b-a)^2 \|T\| \left[\exp(2\|T\| b) - \exp(2\|T\| a)\right]$$

for any $T \in \mathcal{B}(H)$.

If T is invertible, then [2]

$$\int_a^b \exp(tT)\, dt = T^{-1}\left[\exp(bT) - \exp(aT)\right] \tag{3.12}$$

and by (3.12) we get

$$\left| (b-a) \frac{\exp(aT) + \exp(bT)}{2} - T^{-1}\left[\exp(bT) - \exp(aT)\right] \right|^2 \tag{3.13}$$
$$\leq \frac{1}{8}(b-a)^2 \|T\| \left[\exp(2\|T\| b) - \exp(2\|T\| a)\right]$$

for $0 \leq a \leq b$.

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