WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if $\alpha : [a,b] \to \mathbb{C}$ is integrable and $B : [a,b] \to \mathcal{B}(H)$ is strongly differentiable and such that $B' \in L_2([a,b], \mathcal{B}(H))$, then for all $u \in [a,b]$

$$\left|\int_a^b \alpha(t) B(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) B(u)\right|^2 \leq \frac{1}{2}\left[\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right]^2 \int_a^b |\alpha(t)|^2 dt \int_a^b \left|B'(t)\right|^2 dt.$$

In particular, we have

$$\left|\int_a^b \alpha(t) B(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) B\left(\frac{a+b}{2}\right)\right|^2 \leq \frac{1}{8}(b-a)^2 \int_a^b |\alpha(t)|^2 dt \int_a^b \left|B'(t)\right|^2 dt.$$

Some examples for the inverse function are also provided.

1. Introduction

In 1998, Dragomir and Wang proved the following Ostrowski type inequality for p-norm [9].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then we have the inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q} (b-a)^{1/q} \|f'\|_{[a,b],p}, \tag{1.1}$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p dt\right)^{1/p}.$$

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From (1.1) we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

For $p=q=2$ we derive the $L_2[a,b]$-inequality

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2 &\\ \leq\frac{1}{3}\left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right](b-a)\|f'\|_{[a,b],2}^2,\end{aligned} \tag{1.3}$$

for all $x\in[a,b]$ and the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2\leq\frac{1}{12}(b-a)\|f'\|_{[a,b]}^2. \tag{1.4}$$

For a survey on scalar Ostrowski's inequality, see [8]. For recent papers on this inequality see also [1]-[2] and [10]-[12].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

In this paper we show among others that, if $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then for all $u\in[a,b]$,

$$\begin{aligned}&\left|\int_a^b\alpha(t)B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B(u)\right|^2\\ &\leq\frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2\int_a^b|\alpha(t)|^2\,dt\int_a^b|B'(t)|^2\,dt.\end{aligned}$$

In particular, we have

$$\begin{aligned}&\left|\int_a^b\alpha(t)B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B\left(\frac{a+b}{2}\right)\right|^2\\ &\leq\frac{1}{8}(b-a)^2\int_a^b|\alpha(t)|^2\,dt\int_a^b|B'(t)|^2\,dt.\end{aligned}$$

Some examples for the inverse function are also provided.

2. Main Results

In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$0\le\left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2=|\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)\,A(s)$$
$$-\alpha(t)\,\overline{\alpha(s)}A^*(s)\,A(t)+|\alpha(s)|^2\,|A(t)|^2,$$

which gives that

$$|\alpha(t)|^2\,|A(s)|^2+|\alpha(s)|^2\,|A(t)|^2\ge\alpha(s)\,\overline{\alpha(t)}A^*(t)\,A(s)+\alpha(t)\,\overline{\alpha(s)}A^*(s)\,A(t)$$

for all $s,\,t\in[a,b]$.

Now, multiply this with $w(s)\,w(t)\ge0$ to get

$$w(t)\,|\alpha(t)|^2\,w(s)\,|A(s)|^2+w(s)\,|\alpha(s)|^2\,w(t)\,|A(t)|^2$$
$$\ge w(t)\,\overline{\alpha(t)}A^*(t)\,w(s)\,\alpha(s)\,A(s)+w(s)\,\overline{\alpha(s)}A^*(s)\,w(t)\,\alpha(t)\,A(t)$$

for all $s,\,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\int_a^b w(t)\,|\alpha(t)|^2\,dt\int_a^b|A(s)|^2\,ds+\int_a^b|\alpha(s)|^2\,ds\int_a^b w(t)\,|A(t)|^2\,dt$$
$$\ge\int_a^b w(t)\,\overline{\alpha(t)}A^*(t)\,dt\int_a^b\alpha(s)\,A(s)\,ds$$
$$+\int_a^b w(s)\,\overline{\alpha(s)}A^*(s)\,ds\int_a^b\alpha(t)\,A(t)\,dt$$
$$=2\left|\int_a^b w(s)\,\alpha(s)\,A(s)\,ds\right|^2,$$

which proves that

$$\int_a^b w(t)\,|\alpha(t)|^2\,dt\int_a^b w(t)\,|A(t)|^2\,dt\ge\left|\int_a^b w(t)\,\alpha(t)\,A(t)\,dt\right|^2, \tag{2.1}$$

provided that $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and

$$A\in L_{2,w}([a,b],\mathcal{B}(H)):=\left\{A:[a,b]\to B(H),\ \int_a^b w(t)\,\|A(t)\|^2\,dt<\infty\right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^n w_k\,|z_k|^2\sum_{k=1}^n w_k\,|A_k|^2\ge\left|\sum_{k=1}^n w_k z_k A_k\right|^2, \tag{2.2}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\ge0$ for $k\in\{1,\dots,n\}$ and $\sum_{k=1}^n w_k=1$.

Theorem 2. *Assume that $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then for all $u\in[a,b]$*

$$\left|\int_a^b\alpha(t)\,B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B(u)\right|^2 \tag{2.3}$$
$$\le\int_a^b\ell(u,t)\,|\alpha(t)|^2\,dt\int_a^b|B'(t)|^2\,dt,$$

where

$$0\leq\ell(u,t):=\begin{cases}(u-t)\left(\frac{u+t}{2}-a\right), & a\leq t\leq u,\\ \\ (t-u)\left(b-\frac{u+t}{2}\right), & u<t\leq b.\end{cases}$$

In particular, for $u=\frac{a+b}{2}$, *we get*

$$\left|\int_a^b\alpha(t)B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B\left(\frac{a+b}{2}\right)\right|^2 \leq\int_a^b\ell\left(\frac{a+b}{2},t\right)|\alpha(t)|^2\,dt\int_a^b|B'(t)|^2\,dt. \tag{2.4}$$

Proof. Let $u\in[a,b]$. Using the integration by parts formula for Bochner integral [**?**], we have

$$\begin{aligned}\int_u^b\left(\int_t^b\alpha(s)\,ds\right)B'(t)\,dt&=\left(\int_t^b\alpha(s)\,ds\right)B(t)\Big]_u^b+\int_u^b\alpha(t)B(t)\,dt\\&=-\left(\int_u^b\alpha(s)\,ds\right)B(u)+\int_u^b\alpha(t)B(t)\,dt\end{aligned}$$

and

$$\begin{aligned}\int_a^u\left(\int_a^t\alpha(s)\,ds\right)B'(t)\,dt&=\left(\int_a^t\alpha(s)\,ds\right)B(t)\Big]_a^u-\int_a^u\alpha(t)B(t)\,dt\\&=\left(\int_a^u\alpha(s)\,ds\right)B(u)-\int_a^u\alpha(t)B(t)\,dt.\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}&\int_u^b\left(\int_t^b\alpha(s)\,ds\right)B'(t)\,dt-\int_a^u\left(\int_a^t\alpha(s)\,ds\right)B'(t)\,dt\\&=\int_u^b\alpha(t)B(t)\,dt+\int_a^u\alpha(t)B(t)\,dt\\&-\left(\int_u^b\alpha(s)\,ds\right)B(u)-\left(\int_a^u\alpha(s)\,ds\right)B(u)\\&=\int_a^b\alpha(t)B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B(u).\end{aligned}$$

Therefore, we get the following identity of interest

$$\begin{aligned}&\int_a^b\alpha(t)B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B(u)\\&=\int_u^b\left(\int_t^b\alpha(s)\,ds\right)B'(t)\,dt-\int_a^u\left(\int_a^t\alpha(s)\,ds\right)B'(t)\,dt\\&=\int_a^b q(u,t)B'(t)\,dt\end{aligned} \tag{2.5}$$

for all $u \in [a,b]$, where

$$q(u,t) := \begin{cases} -\int_a^t \alpha(s)\,ds, & a \le t \le u, \\ \\ \int_t^b \alpha(s)\,ds, & u < t \le b. \end{cases}$$

By taking the operator modulus and using the Cauchy-Bunyakowsky-Schwarz integral inequality we get

$$\begin{aligned}
&\left|\int_a^b \alpha(t)\,B(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) B(u)\right|^2 \\
&= \left|\int_a^b q(u,t)\,B'(t)\,dt\right|^2 \\
&\le \int_a^b |q(u,t)|^2\,dt \int_a^b |B'(t)|^2\,dt \\
&= \left[\int_a^u \left(\left|\int_a^t \alpha(s)\,ds\right|^2\right) dt + \int_u^b \left(\left|\int_t^b \alpha(s)\,ds\right|^2\right) dt\right] \int_a^b |B'(t)|^2\,dt \\
&\le \left[\int_a^u \left((t-a)\int_a^t |\alpha(s)|^2\,ds\right) dt + \int_u^b \left((b-t)\int_t^b |\alpha(s)|^2\,ds\right) dt\right] \\
&\times \int_a^b |B'(t)|^2\,dt \\
&= K.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_a^u \left((t-a)\int_a^t |\alpha(s)|^2\,ds\right) dt \\
&= \frac{1}{2}\int_a^u \left(\int_a^t |\alpha(s)|^2\,ds\right) d\left((t-a)^2\right) \\
&= \frac{1}{2}\left(\int_a^t |\alpha(s)|^2\,ds\right)(t-a)^2\bigg|_a^u - \frac{1}{2}\int_a^u (t-a)^2\,|\alpha(t)|^2\,dt \\
&= \frac{1}{2}\left(\int_a^u |\alpha(s)|^2\,ds\right)(u-a)^2 - \frac{1}{2}\int_a^u (t-a)^2\,|\alpha(t)|^2\,dt \\
&= \frac{1}{2}\int_a^u \left[(u-a)^2 - (t-a)^2\right] |\alpha(t)|^2\,dt = \int_a^u (u-t)\left(\frac{u+t}{2} - a\right) |\alpha(t)|^2\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_u^b \left((b-t) \int_t^b |\alpha(s)|^2 ds \right) dt \\
&= -\frac{1}{2} \int_u^b \left(\int_t^b |\alpha(s)|^2 ds \right) d\left((b-t)^2 \right) \\
&= -\frac{1}{2} \left[\left(\int_t^b |\alpha(s)|^2 ds \right) (b-t)^2 \Big|_u^b + \int_u^b (b-t)^2 |\alpha(t)|^2 dt \right] \\
&= -\frac{1}{2} \left[\int_u^b (b-t)^2 |\alpha(t)|^2 dt - \left(\int_u^b |\alpha(s)|^2 ds \right) (b-u)^2 \right] \\
&= \frac{1}{2} \int_u^b \left[(b-u)^2 - (b-t)^2 \right] |\alpha(t)|^2 dt = \int_u^b (t-u) \left(b - \frac{u+t}{2} \right) |\alpha(t)|^2 dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
K = &\left[\int_a^u (u-t) \left(\frac{u+t}{2} - a \right) |\alpha(t)|^2 dt + \int_u^b (t-u) \left(b - \frac{u+t}{2} \right) |\alpha(t)|^2 dt \right] \\
&\times \int_a^b |B'(t)|^2 dt
\end{aligned}$$

and the inequality (2.3) is thus proved. □

Corollary 1. *Assume that $\alpha : [a,b] \to \mathbb{C}$ is integrable and $B : [a,b] \to \mathcal{B}(H)$ is strongly differentiable and such that $B' \in L_2([a,b], \mathcal{B}(H))$, then for all $u \in [a,b]$*

$$\begin{aligned}
&\left| \int_a^b \alpha(t) B(t) dt - \left(\int_a^b \alpha(s) ds \right) B(u) \right|^2 \\
&\leq \frac{1}{2} \left[(u-a)^2 \int_a^u |\alpha(t)|^2 dt + (b-u)^2 \int_u^b |\alpha(t)|^2 dt \right] \int_a^b |B'(t)|^2 dt \\
&\leq \frac{1}{2} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right]^2 \int_a^b |\alpha(t)|^2 dt \int_a^b |B'(t)|^2 dt.
\end{aligned} \tag{2.6}$$

In particular,

$$\begin{aligned}
&\left| \int_a^b \alpha(t) B(t) dt - \left(\int_a^b \alpha(s) ds \right) B\left(\frac{a+b}{2} \right) \right|^2 \\
&\leq \frac{1}{8} (b-a)^2 \int_a^b |\alpha(t)|^2 dt \int_a^b |B'(t)|^2 dt.
\end{aligned} \tag{2.7}$$

Proof. We have

$$\begin{aligned}
&\int_a^b \ell(u,t)\,|\alpha(t)|^2\,dt \\
&= \int_a^u \ell(u,t)\,|\alpha(t)|^2\,dt + \int_u^b \ell(u,t)\,|\alpha(t)|^2\,dt \\
&= \int_a^u (u-t)\left(\frac{u+t}{2}-a\right)|\alpha(t)|^2\,dt + \int_u^b (t-u)\left(b-\frac{u+t}{2}\right)|\alpha(t)|^2\,dt \\
&\le \sup_{t\in[a,u]}\left[(u-t)\left(\frac{u+t}{2}-a\right)\right]\int_a^u |\alpha(t)|^2\,dt \\
&+ \sup_{t\in[u,b]}\left[(t-u)\left(b-\frac{u+t}{2}\right)\right]\int_u^b |\alpha(t)|^2\,dt \\
&= \frac{1}{2}(u-a)^2\int_a^u |\alpha(t)|^2\,dt + \frac{1}{2}(b-u)^2\int_u^b |\alpha(t)|^2\,dt,
\end{aligned}$$

which proves the first part of (2.6).

Observe also that

$$\begin{aligned}
&\frac{1}{2}(u-a)^2\int_a^u |\alpha(t)|^2\,dt + \frac{1}{2}(b-u)^2\int_u^b |\alpha(t)|^2\,dt \\
&\le \frac{1}{2}\max\left\{(u-a)^2,(b-u)^2\right\}\int_a^b |\alpha(t)|^2\,dt \\
&= \frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2\int_a^b |\alpha(t)|^2\,dt,
\end{aligned}$$

for $u\in[a,b]$, which proves the last part of (2.6). □

Corollary 2. *Assume that $\alpha:[a,b]\to\mathbb{C}$ is essentially bounded on $[a,b]$ and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then for all $u\in[a,b]$*

$$\begin{aligned}
&\left|\int_a^b \alpha(t)\,B(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right)B(u)\right|^2 \\
&\le \frac{1}{3}\left[(u-a)^3\operatorname*{essup}_{t\in[a,u]}|\alpha(t)|^2 + (b-u)^3\operatorname*{essup}_{t\in[u,b]}|\alpha(t)|^2\right]\int_a^b |B'(t)|^2\,dt \\
&\le (b-a)\left[\frac{1}{12}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\operatorname*{essup}_{t\in[a,b]}|\alpha(t)|^2\int_a^b |B'(t)|^2\,dt.
\end{aligned} \tag{2.8}$$

In particular,

$$\left|\int_a^b \alpha(t) B(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) B\left(\frac{a+b}{2}\right)\right|^2 \tag{2.9}$$
$$\leq \frac{1}{24}(b-a)^3 \left[\operatorname*{essup}_{t\in[a,u]} |\alpha(t)|^2 + \operatorname*{essup}_{t\in[u,b]} |\alpha(t)|^2\right] \int_a^b |B'(t)|^2\,dt$$
$$\leq \frac{1}{12}(b-a)^3 \operatorname*{essup}_{t\in[a,b]} |\alpha(t)|^2 \int_a^b |B'(t)|^2\,dt.$$

Proof. Observe that

$$\int_a^b \ell(u,t)\,|\alpha(t)|^2\,dt$$
$$= \int_a^u \ell(u,t)\,|\alpha(t)|^2\,dt + \int_u^b \ell(u,t)\,|\alpha(t)|^2\,dt$$
$$\leq \operatorname*{essup}_{t\in[a,u]} |\alpha(t)|^2 \int_a^u \ell(u,t)\,dt + \operatorname*{essup}_{t\in[u,b]} |\alpha(t)|^2 \int_u^b \ell(u,t)\,dt$$
$$= \operatorname*{essup}_{t\in[a,u]} |\alpha(t)|^2 \int_a^u (u-t)\left(\frac{u+t}{2} - a\right) dt$$
$$+ \operatorname*{essup}_{t\in[u,b]} |\alpha(t)|^2 \int_u^b (t-u)\left(b - \frac{u+t}{2}\right) dt$$
$$= \frac{1}{3}(u-a)^3 \operatorname*{essup}_{t\in[a,u]} |\alpha(t)|^2 + \frac{1}{3}(b-u)^3 \operatorname*{essup}_{t\in[u,b]} |\alpha(t)|^2.$$

We also have

$$\frac{1}{3}(u-a)^3 \operatorname*{essup}_{t\in[a,u]} |\alpha(t)|^2 + \frac{1}{3}(b-u)^3 \operatorname*{essup}_{t\in[u,b]} |\alpha(t)|^2$$
$$\leq \frac{1}{3}\left[(u-a)^3 + (b-u)^3\right] \operatorname*{essup}_{t\in[a,b]} |\alpha(t)|^2$$
$$= (b-a)\left[\frac{1}{12}(b-a)^2 + \left(u - \frac{a+b}{2}\right)^2\right] \operatorname*{essup}_{t\in[a,b]} |\alpha(t)|^2,$$

which proves the last part of (2.8). □

We can introduce the following concept:

Definition 1. *We say that the continuous function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$ if*

$$|B((1-t)u + tv)|^2 \leq (\geq)\,(1-t)\,|B(u)|^2 + t\,|B(v)|^2 \tag{2.10}$$

in the operator order of $\mathcal{B}(H)$, for all u, $v \in [a,b]$ and $t \in [0,1]$.

Let A, $B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (2.2) we get

$$\begin{aligned}
|(1-\alpha)A+\alpha B|^2 &= \left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^2\\
&\le \left[\left((1-\alpha)^{1/2}\right)^2+\left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2+\left|\alpha^{1/2}B\right|^2\right]\\
&=(1-\alpha+\alpha)\left[(1-\alpha)|A|^2+\alpha|B|^2\right]\\
&=(1-\alpha)|A|^2+\alpha|B|^2.
\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let t_1, $t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}
|C((1-\alpha)t_1+\alpha t_2)|^2 &= |(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B|^2\\
&=|(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)|^2\\
&\le(1-\alpha)|(1-t_1)A+t_1B|^2+\alpha|(1-t_2)A+t_2B|^2\\
&=(1-\alpha)|C(t_1)|^2+\alpha|C(t_2)|^2,
\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\le(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t):=[f((1-t)A+tB)]^{1/2},t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$. Therefore for A, $B>0$, the function

$$B(t):=((1-t)A+tB)^{r/2},\ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\le r\le 2$ or $-1\le r\le 0$.

Corollary 3. *With the assumptions of Theorem 2 and if $B':[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\int_a^b\alpha(t)B(t)\,dt-\left(\int_a^b\alpha(s)\,ds\right)B(u)\right|^2\\
&\le\int_a^b\ell(u,t)|\alpha(t)|^2\,dt\frac{|B'(a)|^2+|B'(b)|^2}{2}\\
&\le\frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2\int_a^b|\alpha(t)|^2\,dt\\
&\times\frac{|B'(a)|^2+|B'(b)|^2}{2}.
\end{aligned}\tag{2.11}$$

In particular

$$\left|\int_a^b \alpha(t) B(t) dt - \left(\int_a^b \alpha(s) ds\right) B\left(\frac{a+b}{2}\right)\right|^2 \tag{2.12}$$
$$\leq \int_a^b \ell\left(\frac{a+b}{2}, t\right) |\alpha(t)|^2 dt \frac{|B'(a)|^2 + |B'(b)|^2}{2}$$
$$\leq \frac{1}{8}(b-a)^2 \int_a^b |\alpha(t)|^2 dt \frac{|B'(a)|^2 + |B'(b)|^2}{2}.$$

Proof. It follows by (2.3) on observing that

$$\begin{aligned}\int_a^b |B'(t)|^2 dt &= (b-a)\int_0^1 |B'((1-s)a+sb)|^2 ds \\ &\leq (b-a)\int_0^1 \left[(1-s)|B'(a)|^2 + s|B'(b)|^2\right] ds \\ &= (b-a)\frac{|B'(a)|^2 + |B'(b)|^2}{2}.\end{aligned}$$

□

We also have:

Corollary 4. *With the assumptions of Theorem 2 and if $B' : [a,b] \to \mathcal{B}(H)$ is square modulus concave on $[a,b]$, then*

$$\left|\int_a^b \alpha(t) B(t) dt - \left(\int_a^b \alpha(s) ds\right) B(u)\right|^2 \tag{2.13}$$
$$\leq \int_a^b \ell(u,t) |\alpha(t)|^2 \left|B'\left(\frac{a+b}{2}\right)\right|^2$$
$$\leq \frac{1}{2}\left[\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right]^2 \int_a^b |\alpha(t)|^2 dt$$
$$\times \left|B'\left(\frac{a+b}{2}\right)\right|^2$$

In particular,

$$\left|\int_a^b \alpha(t) B(t) dt - \left(\int_a^b \alpha(s) ds\right) B(u)\right|^2 \tag{2.14}$$
$$\leq \int_a^b \ell\left(\frac{a+b}{2}, t\right) |\alpha(t)|^2 dt \left|B'\left(\frac{a+b}{2}\right)\right|^2$$
$$\leq \frac{1}{8}(b-a)^2 \int_a^b |\alpha(t)|^2 dt \left|B'\left(\frac{a+b}{2}\right)\right|^2$$

Proof. Since $B' : [a,b] \to \mathcal{B}(H)$ is square modulus concave, then

$$\left|B'\left(\frac{u+v}{2}\right)\right|^2 \geq \frac{|B'(u)|^2 + |B'(v)|^2}{2}$$

for all $u, v \in [a,b]$.

By taking $u=(1-s)a+sb$ and $v=sa+(1-s)b$, $s\in[0,1]$ we get

$$\left|B'\left(\frac{a+b}{2}\right)\right|^2\geq\frac{|B'((1-s)a+sb)|^2+|B'(sa+(1-s)b)|^2}{2}$$

$s\in[0,1]$.

If we take the integral over we $s\in[0,1]$ get

$$\begin{aligned}\left|B'\left(\frac{a+b}{2}\right)\right|^2 &\geq \frac{1}{2}\int_0^1\left[|B'((1-s)a+sb)|^2+|B'(sa+(1-s)b)|^2\right]ds\\ &= \int_0^1|B'((1-s)a+sb)|^2\,ds.\end{aligned}$$

The results follow now by Theorem 2. □

3. Some Examples

Further, let A, $B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that A, $B>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t):=((1-t)A+tB)^{-1}$, $t\in[0,1]$ and observe that

$$B'(t)=-((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1},\ t\in[0,1].$$

Assume that $\alpha:[0,1]\to\mathbb{C}$ is integrable, then from (2.3) we have for all $u\in(0,1)$

$$\begin{aligned}&\left|\int_0^1\alpha(t)((1-t)A+tB)^{-1}dt-\left(\int_0^1\alpha(s)\,ds\right)((1-u)A+uB)^{-1}\right|^2\\ &\leq\int_0^1\ell(u,t)|\alpha(t)|^2dt\\ &\times\int_0^1\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2dt,\end{aligned}\tag{3.1}$$

where

$$0\leq\ell(u,t):=\begin{cases}(u-t)\frac{u+t}{2},\ 0\leq t\leq u,\\ \\ (t-u)\left(1-\frac{u+t}{2}\right),\ u<t\leq 1.\end{cases}$$

In particular, for $u=\frac{1}{2}$, we have

$$\begin{aligned}&\left|\int_0^1\alpha(t)((1-t)A+tB)^{-1}dt-\left(\int_0^1\alpha(s)\,ds\right)\left(\frac{A+B}{2}\right)^{-1}\right|^2\\ &\leq\int_0^1\ell\left(\frac{1}{2},t\right)|\alpha(t)|^2dt\\ &\times\int_0^1\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2dt,\end{aligned}\tag{3.2}$$

where

$$0\leq\ell\left(\frac{1}{2},t\right):=\begin{cases}\left(\frac{1}{2}-t\right)\frac{1+2t}{4},\ 0\leq t\leq\frac{1}{2},\\ \\ \left(t-\frac{1}{2}\right)\frac{3-2t}{4},\ \frac{1}{2}<t\leq 1.\end{cases}$$

Since for any operator $V \in \mathcal{B}(H)$ we have $|V|^2 \le \|V\|^2$, then

$$\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2 \le \left\|((1-t)A+tB)^{-1}\right\|^4 \|B-A\|^2$$

for all $t \in [0,1]$, which implies that

$$\int_0^1 \left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2 dt \le \|B-A\|^2 \int_0^1 \left\|((1-t)A+tB)^{-1}\right\|^4 dt.$$

Therefore, by (3.1) we get

$$\left|\int_0^1 \alpha(t)((1-t)A+tB)^{-1} dt - \left(\int_0^1 \alpha(s)\,ds\right)((1-u)A+uB)^{-1}\right|^2 \le \|B-A\|^2 \int_0^1 \ell(u,t)|\alpha(t)|^2 dt \int_0^1 \left\|((1-t)A+tB)^{-1}\right\|^4 dt, \tag{3.3}$$

while from (3.2) we derive

$$\left|\int_0^1 \alpha(t)((1-t)A+tB)^{-1} dt - \left(\int_0^1 \alpha(s)\,ds\right)\left(\frac{A+B}{2}\right)^{-1}\right|^2 \le \|B-A\|^2 \int_0^1 \ell\left(\frac{1}{2},t\right)|\alpha(t)|^2 dt \int_0^1 \left\|((1-t)A+tB)^{-1}\right\|^4 dt. \tag{3.4}$$

Now, if $A \ge m > 0$ and $B \ge m > 0$, then $((1-t)A+tB)^{-1} \le m^{-1}$ for $t \in [0,1]$, which implies $\left\|((1-t)A+tB)^{-1}\right\|^4 \le m^{-4}$ and by (3.3) we get

$$\begin{aligned}\left|\int_0^1 \alpha(t)((1-t)A+tB)^{-1} dt - \left(\int_0^1 \alpha(s)\,ds\right)((1-u)A+uB)^{-1}\right|^2 &\le \frac{\|B-A\|^2}{m^4}\int_0^1 \ell(u,t)|\alpha(t)|^2 dt \\ &\le \frac{1}{2}\frac{\|B-A\|^2}{m^4}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2 \int_a^b |\alpha(t)|^2 dt\end{aligned} \tag{3.5}$$

while from (3.4) we get

$$\begin{aligned}\left|\int_0^1 \alpha(t)((1-t)A+tB)^{-1} dt - \left(\int_0^1 \alpha(s)\,ds\right)\left(\frac{A+B}{2}\right)^{-1}\right|^2 &\le \frac{\|B-A\|^2}{m^4}\int_0^1 \ell\left(\frac{1}{2},t\right)|\alpha(t)|^2 dt \\ &\le \frac{1}{8}\frac{\|B-A\|^2}{m^4}\int_0^1 |\alpha(t)|^2 dt.\end{aligned} \tag{3.6}$$

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