WEIGHTED GENERALIZED TRAPEZOID TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we obtain among others the following result, if $\alpha : [a,b] \to \mathbb{C}$ is integrable and $B : [a,b] \to \mathcal{B}(H)$ is strongly differentiable and such that $B' \in L_2([a,b], \mathcal{B}(H))$, then

$$\left| \left(\int_u^b \alpha(s)\, ds \right) B(b) + \left(\int_a^u \alpha(s)\, ds \right) B(a) - \int_a^b \alpha(t) B(t)\, dt \right|^2$$
$$\leq \frac{1}{2} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right]^2 \int_a^b |\alpha(t)|^2 dt \int_a^b |B'(t)|^2 dt,$$

for all $u \in [a,b]$. In particular, we have

$$\left| \left(\int_{\frac{a+b}{2}}^b \alpha(s)\, ds \right) B(b) + \left(\int_a^{\frac{a+b}{2}} \alpha(s)\, ds \right) B(a) - \int_a^b \alpha(t) B(t)\, dt \right|^2$$
$$\leq \frac{1}{8}(b-a)^2 \int_a^b |\alpha(t)|^2 dt \int_a^b |B'(t)|^2 dt.$$

Some examples for the inverse function are also provided.

1. Introduction

In 1999, Cerone and Dragomir proved the following *generalized trapezoid* type inequality for p-norm [5].

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then we have the inequality*

$$\left| (b-x) f(b) + (x-a) f(a) - \int_a^b f(t)\, dt \right| \tag{1.1}$$
$$\leq \frac{1}{(q+1)^{1/q}} \left[(x-a)^{q+1} + (b-x)^{q+1} \right]^{1/q} \|f'\|_{[a,b],p},$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p dt \right)^{1/p}.$$

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From (1.1) we get the following *trapezoid inequality*

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|\leq\frac{1}{2(q+1)^{1/q}}(b-a)^{1+1/q}\|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

For $p=q=2$ we derive the $L_2[a,b]$-inequality

$$\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right|^2 \leq\frac{1}{3}\left[(x-a)^3+(b-x)^3\right]\|f'\|_{[a,b],2}^2, \tag{1.3}$$

for all $x\in[a,b]$ and the trapezoid inequality

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|^2\leq\frac{1}{12}(b-a)^3\|f'\|_{[a,b]}^2. \tag{1.4}$$

For a survey on scalar trapezoid inequality, see [5]. For recent papers on this inequality see also [1]-[4] and [6]-[15].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

In this paper we obtain among others the following result, if $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then

$$\left|\left(\int_u^b\alpha(s)\,ds\right)B(b)+\left(\int_a^u\alpha(s)\,ds\right)B(a)-\int_a^b\alpha(t)B(t)\,dt\right|^2 \leq\frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2\int_a^b|\alpha(t)|^2\,dt\int_a^b|B'(t)|^2\,dt, \tag{1.5}$$

for all $u\in[a,b]$. In particular, we have

$$\left|\left(\int_{\frac{a+b}{2}}^b\alpha(s)\,ds\right)B(b)+\left(\int_a^{\frac{a+b}{2}}\alpha(s)\,ds\right)B(a)-\int_a^b\alpha(t)B(t)\,dt\right|^2 \leq\frac{1}{8}(b-a)^2\int_a^b|\alpha(t)|^2\,dt\int_a^b|B'(t)|^2\,dt. \tag{1.6}$$

Some examples for the inverse function are also provided.

2. Main Results

In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$0\leq\left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^{2}=|\alpha(t)|\,|A(s)|^{2}-\alpha(s)\overline{\alpha(t)}A^{*}(t)A(s)$$
$$-\alpha(t)\overline{\alpha(s)}A^{*}(s)A(t)+|\alpha(s)|^{2}|A(t)|^{2},$$

which gives that

$$|\alpha(t)|^{2}|A(s)|^{2}+|\alpha(s)|^{2}|A(t)|^{2}\geq\alpha(s)\overline{\alpha(t)}A^{*}(t)A(s)+\alpha(t)\overline{\alpha(s)}A^{*}(s)A(t)$$

for all $s,t\in[a,b]$.

Now, multiply this with $w(s)w(t)\geq0$ to get

$$w(t)|\alpha(t)|^{2}w(s)|A(s)|^{2}+w(s)|\alpha(s)|^{2}w(t)|A(t)|^{2}$$
$$\geq w(t)\overline{\alpha(t)}A^{*}(t)w(s)\alpha(s)A(s)+w(s)\overline{\alpha(s)}A^{*}(s)w(t)\alpha(t)A(t)$$

for all $s,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\int_a^b w(t)|\alpha(t)|^{2}dt\int_a^b|A(s)|^{2}ds+\int_a^b|\alpha(s)|^{2}ds\int_a^b w(t)|A(t)|^{2}dt$$
$$\geq\int_a^b w(t)\overline{\alpha(t)}A^{*}(t)\,dt\int_a^b\alpha(s)A(s)\,ds$$
$$+\int_a^b w(s)\overline{\alpha(s)}A^{*}(s)\,ds\int_a^b\alpha(t)A(t)\,dt$$
$$=2\left|\int_a^b w(s)\alpha(s)A(s)\,ds\right|^{2},$$

which proves that

$$\int_a^b w(t)|\alpha(t)|^{2}dt\int_a^b w(t)|A(t)|^{2}dt\geq\left|\int_a^b w(t)\alpha(t)A(t)\,dt\right|^{2},\tag{2.1}$$

provided that $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and

$$A\in L_{2,w}([a,b],\mathcal{B}(H)):=\left\{A:[a,b]\to B(H),\ \int_a^b w(t)\|A(t)\|^{2}dt<\infty\right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^{n}w_k|z_k|^{2}\sum_{k=1}^{n}w_k|A_k|^{2}\geq\left|\sum_{k=1}^{n}w_kz_kA_k\right|^{2},\tag{2.2}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\geq0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^{n}w_k=1$.

Theorem 2. *Assume that $\alpha:[a,b]\to\mathbb{C}$ is integrable and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable and such that $B'\in L_2([a,b],\mathcal{B}(H))$, then for all $u\in(a,b)$*

$$\left|\left(\int_u^b\alpha(s)\,ds\right)B(b)+\left(\int_a^u\alpha(s)\,ds\right)B(a)-\int_a^b\alpha(t)B(t)\,dt\right|^{2}\tag{2.3}$$
$$\leq\int_a^b k(u,t)|\alpha(t)|^{2}dt\int_a^b|B'(t)|^{2}dt,$$

where

$$0 \leq k(u,t) := \begin{cases} (t-a)\left(u - \frac{a+t}{2}\right), & a \leq t \leq u, \\ \\ (b-t)\left(\frac{b+t}{2} - u\right), & u < t \leq b. \end{cases}$$

In particular, for $u = \frac{a+b}{2}$*, we have*

$$\begin{aligned} (2.4) \qquad & \left| \left(\int_{\frac{a+b}{2}}^{b} \alpha(s)\, ds \right) B(b) + \left(\int_{a}^{\frac{a+b}{2}} \alpha(s)\, ds \right) B(a) - \int_a^b \alpha(t) B(t)\, dt \right|^2 \\ & \leq \frac{1}{2} \int_a^b (t-a)(b-t) |\alpha(t)|^2 dt \int_a^b |B'(t)|^2 dt \\ & \leq \frac{1}{8} (b-a)^2 \int_a^b |\alpha(t)|^2 dt \int_a^b |B'(t)|^2 dt. \end{aligned}$$

Proof. Let $u \in [a,b]$. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned} (2.5) \qquad & \int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B'(t)\, dt \\ & = \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B(t) \Big|_a^b \\ & - \int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right)' B(t)\, dt \\ & = \left(\int_a^b \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B(b) \\ & - \left(\int_a^a \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B(a) \\ & - \int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right)' B(t)\, dt \\ & = \left(\int_u^b \alpha(s)\, ds \right) B(b) + \left(\int_a^u \alpha(s)\, ds \right) B(a) - \int_a^b \alpha(t) B(t)\, dt. \end{aligned}$$

Also,

$$\begin{aligned} (2.6) \qquad & \int_a^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B'(t)\, dt \\ & = \int_a^u \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B'(t)\, dt \\ & + \int_u^b \left(\int_a^t \alpha(s)\, ds - \int_a^u \alpha(s)\, ds \right) B'(t)\, dt \\ & = - \int_a^u \left(\int_t^u \alpha(s)\, ds \right) B'(t)\, dt + \int_u^b \left(\int_u^t \alpha(s)\, ds \right) B'(t)\, dt. \end{aligned}$$

By utilising (2.5) and (2.6) we derive the following identity of interest

$$\left(\int_u^b \alpha(s)\,ds\right) B(b) + \left(\int_a^u \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt = \int_u^b \left(\int_u^t \alpha(s)\,ds\right) B'(t)\,dt - \int_a^u \left(\int_t^u \alpha(s)\,ds\right) B'(t)\,dt \tag{2.7}$$

for all $u \in [a,b]$.

Now if we take $u \in (a,b)$, then we can define the kernel

$$p_\alpha(u,t) := \begin{cases} -\int_t^u \alpha(s)\,ds, & a \le t \le u, \\ \int_u^t \alpha(s)\,ds, & u < t \le b. \end{cases}$$

By making use of (2.7) we get the equality of interest

$$\left(\int_u^b \alpha(s)\,ds\right) B(b) + \left(\int_a^u \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt = \int_a^b p_\alpha(u,t) B'(t)\,dt. \tag{2.8}$$

If we take the modulus and use the CBS integral inequality, then we get

$$\begin{aligned} &\left|\left(\int_u^b \alpha(s)\,ds\right) B(b) + \left(\int_a^u \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt\right|^2 \\ &= \left|\int_a^b p_\alpha(u,t) B'(t)\,dt\right|^2 \le \int_a^b |p_\alpha(u,t)|^2\,dt \int_a^b |B'(t)|^2\,dt \\ &= \int_a^b \left|\int_u^t \alpha(s)\,ds\right|^2 dt \int_a^b |B'(t)|^2\,dt \\ &\le \int_a^b \left((t-u)\int_u^t |\alpha(s)|^2\,ds\right) dt \int_a^b |B'(t)|^2\,dt. \end{aligned} \tag{2.9}$$

Using the integration by parts formula, we have

$$\begin{aligned}
&\int_a^b \left((t-u)\int_u^t |\alpha(s)|^2 ds\right) dt \\
&= \int_a^b \left(\int_u^t |\alpha(s)|^2 ds\right) d\left(\frac{(t-u)^2}{2}\right) \\
&= \frac{(t-u)^2}{2}\int_u^t |\alpha(s)|^2 ds\Bigg|_a^b - \int_a^b \frac{(t-u)^2}{2}|\alpha(t)|^2 dt \\
&= \frac{(b-u)^2}{2}\int_u^b |\alpha(s)|^2 ds - \frac{(a-u)^2}{2}\int_u^a |\alpha(s)|^2 ds \\
&- \int_a^b \frac{(t-u)^2}{2}|\alpha(t)|^2 dt \\
&= \frac{(b-u)^2}{2}\int_u^b |\alpha(s)|^2 ds + \frac{(u-a)^2}{2}\int_a^u |\alpha(s)|^2 ds \\
&- \int_a^u \frac{(t-u)^2}{2}|\alpha(t)|^2 dt - \int_u^b \frac{(t-u)^2}{2}|\alpha(t)|^2 dt
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&= \frac{(b-u)^2}{2}\int_u^b |\alpha(s)|^2 ds - \int_u^b \frac{(t-u)^2}{2}|\alpha(t)|^2 dt \\
&+ \frac{(u-a)^2}{2}\int_a^u |\alpha(s)|^2 ds - \int_a^u \frac{(u-t)^2}{2}|\alpha(t)|^2 dt \\
&= \int_u^b \left[\frac{(b-u)^2}{2} - \frac{(t-u)^2}{2}\right]|\alpha(t)|^2 dt \\
&+ \int_a^u \left[\frac{(u-a)^2}{2} - \frac{(u-t)^2}{2}\right]|\alpha(t)|^2 dt \\
&= \int_u^b (b-t)\left(\frac{b+t}{2} - u\right)|\alpha(t)|^2 dt + \int_a^u (t-a)\left(u - \frac{a+t}{2}\right)|\alpha(t)|^2 dt.
\end{aligned}$$

By utilising (2.9) and (2.10) we derive (2.3).

For $u = \frac{a+b}{2}$ we have

$$k\left(\frac{a+b}{2}, t\right) := \begin{cases} (t-a)\left(\frac{a+b}{2} - \frac{a+t}{2}\right), & a \le t \le \frac{a+b}{2}, \\ (b-t)\left(\frac{b+t}{2} - \frac{a+b}{2}\right), & \frac{a+b}{2} < t \le b \end{cases}$$
$$= \frac{1}{2}(t-a)(b-t),$$

which gives the first inequality in (2.4).

The last part of (2.4) follows by the elementary inequality

$$\gamma\delta \le \frac{1}{4}(\gamma+\delta)^2, \ \gamma, \ \delta \ge 0.$$

□

Corollary 1. *With the assumptions of Theorem 2, we have*

$$\left|\left(\int_u^b \alpha(s)\,ds\right)B(b)+\left(\int_a^u \alpha(s)\,ds\right)B(a)-\int_a^b \alpha(t)\,B(t)\,dt\right|^2 \tag{2.11}$$
$$\leq \frac{1}{2}\left[(u-a)^2\int_a^u |\alpha(t)|^2\,dt+(b-u)^2\int_u^b |\alpha(t)|^2\,dt\right]\int_a^b |B'(t)|^2\,dt$$
$$\leq \frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2\int_a^b |\alpha(t)|^2\,dt\int_a^b |B'(t)|^2\,dt,$$

for all $u\in[a,b]$.

In particular,

$$\left|\left(\int_{\frac{a+b}{2}}^b \alpha(s)\,ds\right)B(b)+\left(\int_a^{\frac{a+b}{2}} \alpha(s)\,ds\right)B(a)-\int_a^b \alpha(t)\,B(t)\,dt\right|^2 \tag{2.12}$$
$$\leq \frac{1}{8}(b-a)^2\int_a^b |\alpha(t)|^2\,dt\int_a^b |B'(t)|^2\,dt.$$

Proof. We observe that

$$k(u,t):=\begin{cases}(t-a)\left(u-\frac{a+t}{2}\right), & a\leq t\leq u,\\[2mm] (b-t)\left(\frac{b+t}{2}-u\right), & u<t\leq b.\end{cases}$$
$$\leq\begin{cases}\frac{1}{2}(u-a)^2, & a\leq t\leq u,\\[2mm] \frac{1}{2}(b-u)^2, & u<t\leq b.\end{cases}\leq\frac{1}{2}\max\left\{(u-a)^2,(b-u)^2\right\}$$
$$=\frac{1}{2}\left[\max\{u-a,b-u\}\right]^2=\frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2$$

for all $t\in[a,b]$.

Then

$$\int_a^b k(u,t)\,|\alpha(t)|^2\,dt=\int_a^u k(u,t)\,|\alpha(t)|^2\,dt+\int_u^b k(u,t)\,|\alpha(t)|^2\,dt$$
$$\leq\frac{1}{2}(u-a)^2\int_a^u |\alpha(t)|^2\,dt+\frac{1}{2}(b-u)^2\int_u^b |\alpha(t)|^2\,dt$$
$$\leq\frac{1}{2}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]^2\int_a^b |\alpha(t)|^2\,dt$$

and by (2.3) we derive (2.11). □

Corollary 2. *With the assumptions of Theorem 2, we have*

$$\left|\left(\int_u^b \alpha(s)\,ds\right)B(b)+\left(\int_a^u \alpha(s)\,ds\right)B(a)-\int_a^b \alpha(t)\,B(t)\,dt\right|^2 \tag{2.13}$$
$$\leq \frac{1}{3}\left[(u-a)^3\operatorname*{essup}_{t\in[a,u]}|\alpha(t)|^2+(b-u)^3\operatorname*{essup}_{t\in[u,b]}|\alpha(t)|^2\right]\int_a^b |B'(t)|^2\,dt$$
$$\leq (b-a)\operatorname*{essup}_{t\in[a,b]}|\alpha(t)|^2\left[\frac{1}{12}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\int_a^b |B'(t)|^2\,dt,$$

for all $u\in[a,b]$.

In particular, we have

$$\left|\left(\int_{\frac{a+b}{2}}^b \alpha(s)\,ds\right)B(b)+\left(\int_a^{\frac{a+b}{2}} \alpha(s)\,ds\right)B(a)-\int_a^b \alpha(t)\,B(t)\,dt\right|^2 \tag{2.14}$$
$$\frac{1}{24}(b-a)^3\left[\operatorname*{essup}_{t\in[a,u]}|\alpha(t)|^2+\operatorname*{essup}_{t\in[u,b]}|\alpha(t)|^2\right]\int_a^b |B'(t)|^2\,dt$$
$$\leq \frac{1}{12}(b-a)^3\operatorname*{essup}_{t\in[a,b]}|\alpha(t)|^2\int_a^b |B'(t)|^2\,dt.$$

Proof. Observe that

$$\int_a^u (t-a)\left(u-\frac{a+t}{2}\right)dt=\frac{1}{3}(u-a)^3$$

and

$$\int_u^b (b-t)\left(\frac{b+t}{2}-u\right)dt=\frac{1}{3}(b-u)^3$$

for $u\in(a,b)$.

Then

$$\int_a^b k(u,t)\,|\alpha(t)|^2\,dt$$
$$=\int_a^u k(u,t)\,|\alpha(t)|^2\,dt+\int_u^b k(u,t)\,|\alpha(t)|^2\,dt$$
$$\leq \operatorname*{essup}_{t\in[a,u]}|\alpha(t)|^2\int_a^u k(u,t)\,dt+\operatorname*{essup}_{t\in[u,b]}|\alpha(t)|^2\int_u^b k(u,t)$$
$$=\operatorname*{essup}_{t\in[a,u]}|\alpha(t)|^2\int_a^u (t-a)\left(u-\frac{a+t}{2}\right)dt$$
$$+\operatorname*{essup}_{t\in[u,b]}|\alpha(t)|^2\int_u^b (b-t)\left(\frac{b+t}{2}-u\right)dt$$

$$= \frac{1}{3}(u-a)^3 \operatorname*{essup}_{t\in[a,u]} |\alpha(t)|^2 + \frac{1}{3}(b-u)^3 \operatorname*{essup}_{t\in[u,b]} |\alpha(t)|^2$$
$$\leq \operatorname*{essup}_{t\in[a,b]} |\alpha(t)|^2 \left[\frac{1}{3}(u-a)^3 + \frac{1}{3}(b-u)^3\right]$$
$$= (b-a) \operatorname*{essup}_{t\in[a,b]} |\alpha(t)|^2 \left[\frac{1}{12}(b-a)^2 + \left(u - \frac{a+b}{2}\right)^2\right]$$

and by (2.3) we derive (2.13). □

We can introduce the following concept:

Definition 1. *We say that the continuous function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$ if*

$$|B((1-t)u + tv)|^2 \leq (\geq)\,(1-t)|B(u)|^2 + t|B(v)|^2 \tag{2.15}$$

in the operator order of $\mathcal{B}(H)$, for all u, $v \in [a,b]$ and $t \in [0,1]$.

Let A, $B \in \mathcal{B}(H)$ and $\alpha \in [0,1]$. Then by (2.2) we get

$$|(1-\alpha)A + \alpha B|^2 = \left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A + \alpha^{1/2}\alpha^{1/2}B\right|^2$$
$$\leq \left[\left((1-\alpha)^{1/2}\right)^2 + \left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2 + \left|\alpha^{1/2}B\right|^2\right]$$
$$= (1-\alpha+\alpha)\left[(1-\alpha)|A|^2 + \alpha|B|^2\right]$$
$$= (1-\alpha)|A|^2 + \alpha|B|^2.$$

Consider the function $C : [0,1] \to \mathcal{B}(H)$, $C(t) = |(1-t)A + tB|$. Let t_1, $t_2 \in [0,1]$ and $\alpha \in [0,1]$. Then

$$|C((1-\alpha)t_1 + \alpha t_2)|^2 = |(1-(1-\alpha)t_1 - \alpha t_2)A + ((1-\alpha)t_1 + \alpha t_2)B|^2$$
$$= |(1-\alpha)((1-t_1)A + t_1B) + \alpha((1-t_2)A + t_2B)|^2$$
$$\leq (1-\alpha)|(1-t_1)A + t_1B|^2 + \alpha|(1-t_2)A + t_2B|^2$$
$$= (1-\alpha)|C(t_1)|^2 + \alpha|C(t_2)|^2,$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A + \alpha B) \leq (1-\alpha)f(A) + \alpha f(B)$$

for all $\alpha \in [0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t) := [f((1-t)A + tB)]^{1/2}, t \in [0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. Therefore for A, $B > 0$, the function

$$B(t) := ((1-t)A + tB)^{r/2},\ t \in [0,1]$$

is *square modulus convex* on $[0,1]$ for $1 \leq r \leq 2$ or $-1 \leq r \leq 0$.

Corollary 3. *With the assumptions of Theorem 2 and if* $B' : [a,b] \to \mathcal{B}(H)$ *is square modulus convex on* $[a,b]$, *then*

$$\left|\left(\int_u^b \alpha(s)\,ds\right) B(b) + \left(\int_a^u \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt\right|^2 \tag{2.16}$$
$$\leq (b-a)\left(\int_a^b k(u,t)\,|\alpha(t)|^2\,dt\right)\frac{|B'(a)|^2 + |B'(b)|^2}{2}.$$

In particular,

$$\left|\left(\int_{\frac{a+b}{2}}^b \alpha(s)\,ds\right) B(b) + \left(\int_a^{\frac{a+b}{2}} \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt\right|^2 \tag{2.17}$$
$$\leq \frac{1}{2}\left(\int_a^b (t-a)(b-t)\,|\alpha(t)|^2\,dt\right)\frac{|B'(a)|^2 + |B'(b)|^2}{2}$$
$$\leq \frac{1}{8}(b-a)^2\left(\int_a^b |\alpha(t)|^2\,dt\right)\frac{|B'(a)|^2 + |B'(b)|^2}{2}.$$

Proof. It follows by (2.3) on observing that

$$\int_a^b |B'(t)|^2\,dt = (b-a)\int_0^1 |B'((1-s)a + sb)|^2\,ds$$
$$\leq (b-a)\int_0^1 \left[(1-s)|B'(a)|^2 + s|B'(b)|^2\right] ds$$
$$= (b-a)\frac{|B'(a)|^2 + |B'(b)|^2}{2}.$$

□

We also have:

Corollary 4. *With the assumptions of Theorem 2 and if* $B' : [a,b] \to \mathcal{B}(H)$ *is square modulus concave on* $[a,b]$, *then*

$$\left|\left(\int_u^b \alpha(s)\,ds\right) B(b) + \left(\int_a^u \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt\right|^2 \tag{2.18}$$
$$\leq (b-a)\left(\int_a^b k(u,t)\,|\alpha(t)|^2\,dt\right)\left|B'\left(\frac{a+b}{2}\right)\right|^2.$$

In particular,

$$\left|\left(\int_{\frac{a+b}{2}}^b \alpha(s)\,ds\right) B(b) + \left(\int_a^{\frac{a+b}{2}} \alpha(s)\,ds\right) B(a) - \int_a^b \alpha(t) B(t)\,dt\right|^2 \tag{2.19}$$
$$\leq \frac{1}{2}\left(\int_a^b (t-a)(b-t)\,|\alpha(t)|^2\,dt\right)\left|B'\left(\frac{a+b}{2}\right)\right|^2$$
$$\leq \frac{1}{8}(b-a)^2\left(\int_a^b |\alpha(t)|^2\,dt\right)\left|B'\left(\frac{a+b}{2}\right)\right|^2.$$

Proof. Since $B':[a,b]\to\mathcal{B}(H)$ is square modulus concave, then

$$\left|B'\left(\frac{u+v}{2}\right)\right|^2\geq\frac{|B'(u)|^2+|B'(v)|^2}{2}$$

for all $u,v\in[a,b]$.

By taking $u=(1-s)a+sb$ and $v=sa+(1-s)b$, $s\in[0,1]$ we get

$$\left|B'\left(\frac{a+b}{2}\right)\right|^2\geq\frac{|B'((1-s)a+sb)|^2+|B'(sa+(1-s)b)|^2}{2}$$

$s\in[0,1]$.

If we take the integral over we $s\in[0,1]$ get

$$\begin{aligned}\left|B'\left(\frac{a+b}{2}\right)\right|^2 &\geq \frac{1}{2}\int_0^1\left[|B'((1-s)a+sb)|^2+|B'(sa+(1-s)b)|^2\right]ds\\ &= \int_0^1|B'((1-s)a+sb)|^2\,ds.\end{aligned}$$

The results follow now by Theorem 2. □

3. Some Examples

Further, let $A,\,B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $A,\,B>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t):=((1-t)A+tB)^{-1}$, $t\in[0,1]$ and observe that

$$B'(t)=-((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1},\ t\in[0,1].$$

Assume that $\alpha:[0,1]\to\mathbb{C}$ is integrable, then from (2.3) we have for all $u\in(0,1)$

$$\begin{aligned}&\left|\left(\int_u^1\alpha(s)\,ds\right)B^{-1}+\left(\int_0^u\alpha(s)\,ds\right)A^{-1}\right.\\ &\left.-\int_0^1\alpha(t)((1-t)A+tB)^{-1}dt\right|^2\\ &\leq\int_0^1k(u,t)|\alpha(t)|^2dt\\ &\times\int_0^1\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2dt,\end{aligned}\tag{3.1}$$

where

$$0\leq k(u,t):=\begin{cases}t\left(u-\frac{t}{2}\right),\ 0\leq t\leq u,\\ (1-t)\left(\frac{1+t}{2}-u\right),\ u<t\leq 1.\end{cases}$$

In particular, for $u=\frac{1}{2}$, we have

$$\begin{aligned}(3.2)\quad & \left|\left(\int_{\frac{1}{2}}^{1}\alpha(s)\,ds\right)B^{-1}+\left(\int_{0}^{\frac{1}{2}}\alpha(s)\,ds\right)A^{-1}\right.\\ & \left.-\int_{0}^{1}\alpha(t)\left((1-t)A+tB\right)^{-1}dt\right|^{2}\\ & \leq\frac{1}{2}\int_{0}^{1}t(1-t)\left|\alpha(t)\right|^{2}dt\\ & \times\int_{0}^{1}\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^{2}dt,\\ & \leq\frac{1}{8}\int_{0}^{1}\left|\alpha(t)\right|^{2}dt\int_{0}^{1}\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^{2}dt.\end{aligned}$$

Since for any operator $V\in\mathcal{B}(H)$ we have $|V|^{2}\leq\|V\|^{2}$, then

$$\begin{aligned}& \left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^{2}\\ & \leq\left\|\left((1-t)A+tB\right)^{-1}\right\|^{4}\|B-A\|^{2}\end{aligned}$$

for all $t\in[0,1]$, which implies that

$$\begin{aligned}& \int_{0}^{1}\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^{2}dt\\ & \leq\|B-A\|^{2}\int_{0}^{1}\left\|\left((1-t)A+tB\right)^{-1}\right\|^{4}dt.\end{aligned}$$

Therefore, by (3.1) we get

$$\begin{aligned}(3.3)\quad & \left|\left(\int_{u}^{1}\alpha(s)\,ds\right)B^{-1}+\left(\int_{0}^{u}\alpha(s)\,ds\right)A^{-1}\right.\\ & \left.-\int_{0}^{1}\alpha(t)\left((1-t)A+tB\right)^{-1}dt\right|^{2}\\ & \leq\|B-A\|^{2}\int_{0}^{1}k(u,t)\left|\alpha(t)\right|^{2}dt\int_{0}^{1}\left\|\left((1-t)A+tB\right)^{-1}\right\|^{4}dt,\end{aligned}$$

while from (3.2) we derive

$$\begin{aligned}(3.4)\quad & \left|\left(\int_{\frac{1}{2}}^{1}\alpha(s)\,ds\right)B^{-1}+\left(\int_{0}^{\frac{1}{2}}\alpha(s)\,ds\right)A^{-1}\right.\\ & \left.-\int_{0}^{1}\alpha(t)\left((1-t)A+tB\right)^{-1}dt\right|^{2}\\ & \leq\frac{1}{2}\|B-A\|^{2}\int_{0}^{1}t(1-t)\left|\alpha(t)\right|^{2}dt\int_{0}^{1}\left\|\left((1-t)A+tB\right)^{-1}\right\|^{4}dt\\ & \leq\frac{1}{8}\|B-A\|^{2}\int_{0}^{1}\left|\alpha(t)\right|^{2}dt\int_{0}^{1}\left\|\left((1-t)A+tB\right)^{-1}\right\|^{4}dt.\end{aligned}$$

Now, if $A \geq m > 0$ and $B \geq m > 0$, then $((1-t)A + tB)^{-1} \leq m^{-1}$ for $t \in [0,1]$, which implies $\left\| ((1-t)A + tB)^{-1} \right\|^4 \leq m^{-4}$ and by (3.3) we get

$$\begin{aligned}
&\left| \left(\int_u^1 \alpha(s)\, ds \right) B^{-1} + \left(\int_0^u \alpha(s)\, ds \right) A^{-1} \right. \\
&\left. - \int_0^1 \alpha(t) ((1-t)A + tB)^{-1} dt \right|^2 \\
&\leq \int_0^1 k(u,t) |\alpha(t)|^2 dt \\
&\leq \frac{1}{2} \frac{\|B-A\|^2}{m^4} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right]^2 \int_0^1 |\alpha(t)|^2 dt,
\end{aligned} \tag{3.5}$$

while from (3.4) we get

$$\begin{aligned}
&\left| \left(\int_{\frac{1}{2}}^1 \alpha(s)\, ds \right) B^{-1} + \left(\int_0^{\frac{1}{2}} \alpha(s)\, ds \right) A^{-1} \right. \\
&\left. - \int_0^1 \alpha(t) ((1-t)A + tB)^{-1} dt \right|^2 \\
&\leq \frac{1}{2} \frac{\|B-A\|^2}{m^4} \int_0^1 t(1-t) |\alpha(t)|^2 dt \leq \frac{1}{8} \frac{\|B-A\|^2}{m^4} \int_0^1 |\alpha(t)|^2 dt.
\end{aligned} \tag{3.6}$$

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