CBS AND GRÜSS' TYPE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if $\alpha : [a,b] \to \mathbb{C}$, $B : [a,b] \to \mathcal{B}(H)$ are continuous and $v : [a,b] \to \mathbb{C}$ of bounded variation, then we have the Cauchy-Bunyakowsky-Schwarz (CBS) type inequality for Riemann-Stieltjes integral

$$\left|\int_a^b \alpha(t) B(t) dv(t)\right|^2 \leq \int_a^b |\alpha(t)|^2 dV(t) \int_a^b |B(t)|^2 dV(t),$$

where $V(t) := \bigvee_a^t (v)$ is the total variation of v on $[a,t]$, $t \in (0,b]$. Applications for Grüss' type inequalities are also provided.

1. Introduction

For two Lebesgue integrable functions f, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := \frac{1}{b-a}\int_a^b f(t) g(t) dt - \frac{1}{(b-a)^2}\int_a^b f(t) dt \int_a^b g(t) dt.$$

In 1934, G. Grüss [11] showed that

$$|D(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by the author in [2]:

Theorem 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If φ, ϕ, γ, $\Gamma \in \mathbb{K}$ and x, $y \in H$ are such that*

$$\operatorname{Re}\langle \phi e - x, x - \varphi e\rangle \geq 0 \quad \textit{and} \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e\rangle \geq 0 \tag{1.3}$$

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or, equivalently (see [4])

$$\left\| x-\frac{\varphi+\phi}{2}e\right\| \leq \frac{1}{2}\left|\phi-\varphi\right| \quad \text{and} \quad \left\| y-\frac{\gamma+\Gamma}{2}e\right\| \leq \frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.4}$$

then

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right| \leq \frac{1}{4}\left|\phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{1.5}$$

The constant $\frac{1}{4}$ *is best possible in (1.5).*

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by the author in 2001, [3].

Theorem 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex Hilbert space,* $\Omega\subset\mathbb{R}^n$ *be a Lebesgue measurable set and* $\rho:\Omega\to[0,\infty)$ *a Lebesgue measurable function with* $\int_\Omega \rho(s)\,ds=1$*. We denote by* $L_{2,\rho}(\Omega,H)$ *the set of all Bochner measurable functions* f *on* Ω *such that* $\|f\|_{2,\rho}^2:=\int_\Omega \rho(s)\|f(s)\|^2 ds<\infty$*. If* f*,* g *belong to* $L_{2,\rho}(\Omega,H)$ *and there exist the vectors* x*,* X*,* y*,* $Y\in H$ *such that*

$$\int_\Omega \rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle dt\geq 0, \tag{1.6}$$
$$\int_\Omega \rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle dt\geq 0,$$

then we have the inequality

$$\left|\int_\Omega \rho(t)\langle f(t),g(t)\rangle dt-\left\langle\int_\Omega \rho(t)f(t)\,dt,\int_\Omega \rho(t)g(t)\,dt\right\rangle\right| \tag{1.7}$$
$$\leq\frac{1}{4}\|X-x\|\|Y-y\|.$$

The constant $\frac{1}{4}$ *is sharp in the sense mentioned above.*

Remark 1. *A practical sufficient condition for (1.6) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle\geq 0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle\geq 0$$

or, equivalently

$$\left\|f(t)-\frac{X+x}{2}\right\|\leq\frac{1}{2}\|X-x\| \ \text{ and }\ \left\|g(t)-\frac{Y+y}{2}\right\|\leq\frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$*.*

For related results, see [1], [4]-[10] and [12]-[13].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

Motivated by the above results, in this paper we show among others that, if $\alpha:[a,b]\to\mathbb{C}$, $B:[a,b]\to\mathcal{B}(H)$ are continuous and $v:[a,b]\to\mathbb{C}$ of bounded variation, then

$$\left|\int_a^b \alpha(t)B(t)\,dv(t)\right|^2\leq\int_a^b|\alpha(t)|^2 dV(t)\int_a^b|B(t)|^2 dV(t),$$

where $V(t) := \bigvee_a^t(v)$ is the total variation of v on $[a,t]$, $t \in (0,b]$ and the integrals are taken in the Riemann-Stieltjes sense. Applications for Grüss' type inequalities are also provided.

2. General Results

We have the following general inequality for two monotonic nondecreasing integrators:

Lemma 1. *Assume that* $\alpha : [a,b] \to \mathbb{C}$, $B : [a,b] \to \mathcal{B}(H)$, $w : [a,b] \to [0,\infty)$ *are continuous and* u, $v : [a,b] \to \mathbb{R}$ *are monotonic nondecreasing, then*

$$\begin{aligned}(2.1)\quad &\int_a^b w(t)|\alpha(t)|^2 dv(t) \int_a^b w(s)|B(s)|^2 du(s) \\ &+ \int_a^b w(s)|\alpha(s)|^2 du(s) \int_a^b w(t)|B(t)|^2 dv(t) \\ &\geq \int_a^b w(t)\overline{\alpha(t)}B^*(t)\, dv(t) \int_a^b w(s)\alpha(s)B(s)\, du(s) \\ &+ \int_a^b w(s)\overline{\alpha(s)}B^*(s)\, du(s) \int_a^b w(t)\alpha(t)B(t)\, dv(t).\end{aligned}$$

In particular, for $u = v$ *we get*

$$(2.2)\quad \int_a^b w(t)|\alpha(t)|^2 dv(t) \int_a^b w(t)|B(t)|^2 dv(t) \geq \left|\int_a^b w(t)\alpha(t)B(t)\, dv(t)\right|^2.$$

Proof. We have

$$\begin{aligned}0 &\leq \left|\overline{\alpha(t)}B(s) - \overline{\alpha(s)}B(t)\right|^2 = |\alpha(t)||B(s)|^2 - \alpha(s)\overline{\alpha(t)}B^*(t)B(s) \\ &- \alpha(t)\overline{\alpha(s)}B^*(s)B(t) + |\alpha(s)|^2|B(t)|^2,\end{aligned}$$

which gives that

$$|\alpha(t)|^2|B(s)|^2 + |\alpha(s)|^2|B(t)|^2 \geq \alpha(s)\overline{\alpha(t)}B^*(t)B(s) + \alpha(t)\overline{\alpha(s)}B^*(s)B(t)$$

for all s, $t \in [a,b]$.

Now, multiply this with $w(s)w(t) \geq 0$ to get

$$\begin{aligned}&w(t)|\alpha(t)|^2 w(s)|B(s)|^2 + w(s)|\alpha(s)|^2 w(t)|B(t)|^2 \\ &\geq w(t)\overline{\alpha(t)}B^*(t)w(s)\alpha(s)B(s) + w(s)\overline{\alpha(s)}B^*(s)w(t)\alpha(t)B(t)\end{aligned}$$

for all s, $t \in \Omega$.

Integrating over $dv(t)$ and $du(s)$ on $[a,b]$, then we get

$$\begin{aligned}&\int_a^b w(t)|\alpha(t)|^2 dv(t) \int_a^b w(s)|B(s)|^2 du(s) \\ &+ \int_a^b w(s)|\alpha(s)|^2 du(s) \int_a^b w(t)|B(t)|^2 dv(t) \\ &\geq \int_a^b w(t)\overline{\alpha(t)}B^*(t)\, dv(t) \int_a^b w(s)\alpha(s)B(s)\, du(s) \\ &+ \int_a^b w(s)\overline{\alpha(s)}B^*(s)\, du(s) \int_a^b w(t)\alpha(t)B(t)\, dv(t),\end{aligned}$$

which proves (2.1).

If $u=v$, then

$$\begin{aligned}
&\int_a^b w(t)\overline{\alpha(t)}B^*(t)\,dv(t)\int_a^b w(s)\alpha(s)B(s)\,du(s)\\
&=\left(\int_a^b w(t)\alpha(t)B(t)\,dv(t)\right)^*\int_a^b w(s)\alpha(s)B(s)\,dv(s)\\
&=\left|\int_a^b w(s)\alpha(s)B(s)\,dv(s)\right|^2,
\end{aligned}$$

which proves (2.2). □

Remark 2. *If we take the square root in (2.2), we get*

$$\begin{aligned}
&\left(\int_a^b w(t)|\alpha(t)|^2\,dv(t)\right)^{1/2}\left(\int_a^b w(t)|B(t)|^2\,dv(t)\right)^{1/2} \qquad (2.3)\\
&\geq\left|\int_a^b w(t)\alpha(t)B(t)\,dv(t)\right|.
\end{aligned}$$

Lemma 2. *Let $B_k\in\mathcal{B}(H)$, $\alpha_k\in\mathbb{C}$ and $p_k\geq 0$ for $k\in\{1,...,n\}$. Then*

$$\begin{aligned}
&\sum_{k=1}^n p_k|\alpha_k|^2\sum_{k=1}^n p_k|B_k|^2-\left|\sum_{j=1}^n p_j\alpha_jB_j\right|^2 \qquad (2.4)\\
&=\sum_{k=1}^n p_k|\alpha_k|^2\sum_{j=1}^n p_j\left|B_j-\frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k|\alpha_k|^2}\sum_{k=1}^n p_k\alpha_kB_k\right|^2\geq 0.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\sum_{k=1}^n |\alpha_k|^2\sum_{k=1}^n |B_k|^2-\left|\sum_{j=1}^n \alpha_jB_j\right|^2 \qquad (2.5)\\
&=\sum_{k=1}^n |\alpha_k|^2\sum_{j=1}^n \left|B_j-\frac{\overline{\alpha_j}}{\sum_{k=1}^n |\alpha_k|^2}\sum_{k=1}^n \alpha_kB_k\right|^2\geq 0.
\end{aligned}$$

Proof. For $j\in\{1,...,n\}$ we have

$$\begin{aligned}
&\left|B_j-\frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k|\alpha_k|^2}\sum_{k=1}^n p_k\alpha_kB_k\right|^2\\
&=\left(B_j-\frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k|\alpha_k|^2}\sum_{k=1}^n p_k\alpha_kB_k\right)^*\left(B_j-\frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k|\alpha_k|^2}\sum_{k=1}^n p_k\alpha_kB_k\right)
\end{aligned}$$

$$= \left(B_j^* - \frac{\alpha_j}{\sum_{k=1}^n p_k |\alpha_k|^2} \left(\sum_{k=1}^n p_k \alpha_k B_k\right)^*\right) \left(B_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k B_k\right)$$
$$= |B_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \left(\sum_{k=1}^n p_k \alpha_k B_k\right)^* \alpha_j B_j$$
$$- \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} B_j^* \sum_{k=1}^n p_k \alpha_k B_k + \frac{|\alpha_j|^2}{\left(\sum_{k=1}^n p_k |\alpha_k|^2\right)^2} \left|\sum_{k=1}^n p_k \alpha_k B_k\right|^2.$$

If we multiply this equality with $p_j \geq 0$ and sum over j from 1 to n, we derive

$$\sum_{j=1}^n p_j \left|B_j - \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k B_k\right|^2$$
$$= \sum_{j=1}^n p_j |B_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \left(\sum_{k=1}^n p_k \alpha_k B_k\right)^* \sum_{j=1}^n p_j \alpha_j B_j$$
$$- \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{j=1}^n p_j \overline{\alpha_j} B_j^* \sum_{k=1}^n p_k \alpha_k B_k + \frac{\sum_{j=1}^n p_j |\alpha_j|^2}{\left(\sum_{k=1}^n p_k |\alpha_k|^2\right)^2} \left|\sum_{k=1}^n p_k \alpha_k B_k\right|^2$$

$$= \sum_{j=1}^n p_j |B_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \left|\sum_{k=1}^n p_k \alpha_k B_k\right|^2$$
$$- \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \left|\sum_{k=1}^n p_k \alpha_k B_k\right|^2 + \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \left|\sum_{k=1}^n p_k \alpha_k B_k\right|^2$$
$$= \sum_{j=1}^n p_j |B_j|^2 - \frac{1}{\sum_{k=1}^n p_k |\alpha_k|^2} \left|\sum_{k=1}^n p_k \alpha_k B_k\right|^2,$$

which is equivalent to (2.4). □

Theorem 3. *Let $B_k \in \mathcal{B}(H)$, $\alpha_k \in \mathbb{C}$ and $p_k \geq 0$ for $k \in \{1, ..., n\}$. Then*

$$\sum_{k=1}^n p_k |\alpha_k|^2 \sum_{k=1}^n p_k |B_k|^2 \geq \left|\sum_{j=1}^n p_j \alpha_j B_j\right|^2. \tag{2.6}$$

If $p_k > 0$ for $k \in \{1, ..., n\}$, then the equality holds in (2.3) if and only if

$$B_j = \frac{\overline{\alpha_j}}{\sum_{k=1}^n p_k |\alpha_k|^2} \sum_{k=1}^n p_k \alpha_k B_k \tag{2.7}$$

for all $j \in \{1, ..., n\}$ or if and only if

$$B_j = \overline{\alpha_j} B \tag{2.8}$$

for all $j \in \{1, ..., n\}$, where $B \in \mathcal{B}(H)$.

Remark 3. *Let $B_k \in \mathcal{B}(H)$ and $p_k \geq 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$. Then*

$$\sum_{k=1}^{n} p_k |B_k|^2 \geq \left| \sum_{j=1}^{n} p_j B_j \right|^2 . \tag{2.9}$$

The equality holds in (2.9) if and only if $B_j = \sum_{k=1}^{n} p_k B_k$ for all $j \in \{1,...,n\}$ or if and only if $B_j = B$ for all $j \in \{1,...,n\}$, where $B \in B$.

If we take the square root in (2.9), then we obtain

$$\left(\sum_{k=1}^{n} p_k |B_k|^2 \right)^{1/2} \geq \left| \sum_{j=1}^{n} p_j B_j \right| . \tag{2.10}$$

We have the following Cauchy-Bunyakowsky-Schwarz type inequality for the Riemann-Stieltjes integral of bounded variation integrators:

Theorem 4. *Assume that $\alpha : [a,b] \to \mathbb{C}$, $B : [a,b] \to \mathcal{B}(H)$ are continuous and $v : [a,b] \to \mathbb{C}$ of bounded variation, then*

$$\left| \int_a^b \alpha(t) B(t) dv(t) \right|^2 \leq \int_a^b |\alpha(t)|^2 dV(t) \int_a^b |B(t)|^2 dV(t), \tag{2.11}$$

where $V(t) := \bigvee_a^t (v)$ is the total variation of v on $[a,t]$, $t \in (0,b]$.

Proof. Let $I_n : a = t_0 < t_1 < .... < t_n = b$ a division of $[a,b]$ with the norm $\delta(I_n) := \max_{i=0,...,n-1}(t_{i+1} - t_i)$ and the intermediate points $\xi_i \in [x_i, x_{i+1}]$, $i = 0,...,n-1$. Using the definition of Riemann-Stieltjes integral, the continuity property of modulus of operators and the CBS inequality we have

$$\begin{aligned} &\left| \int_a^b \alpha(t) B(t) dv(t) \right|^2 \\ &= \left| \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} (v(t_{i+1}) - v(t_i)) \alpha(\xi_i) B(\xi_i) \right|^2 \\ &= \lim_{\delta(I_n)\to 0} \left| \sum_{i=0}^{n-1} (v(t_{i+1}) - v(t_i)) \alpha(\xi_i) B(\xi_i) \right|^2 \\ &= \lim_{\delta(I_n)\to 0} \left| \sum_{i=0}^{n-1} \frac{(v(t_{i+1}) - v(t_i))}{\bigvee_{t_i}^{t_{i+1}}(v)} \bigvee_{t_i}^{t_{i+1}}(v) \alpha(\xi_i) B(\xi_i) \right|^2 \\ &\leq \lim_{\delta(I_n)\to 0} \left[\sum_{i=0}^{n-1} \left| \frac{v(t_{i+1}) - v(t_i)}{\bigvee_{t_i}^{t_{i+1}}(v)} \right|^2 |\alpha(\xi_i)|^2 \bigvee_{t_i}^{t_{i+1}}(v) \right] \\ &\times \lim_{\delta(I_n)\to 0} \left[\sum_{i=0}^{n-1} \bigvee_{t_i}^{t_{i+1}}(v) |B(\xi_i)|^2 \right] \\ &= K. \end{aligned} \tag{2.12}$$

Now, since $|v(t_{i+1}) - v(t_i)| \le \bigvee_{t_i}^{t_{i+1}}(v)\,,\ i = 0, ..., n-1$, we observe that

$$\sum_{i=0}^{n-1}\left|\frac{v(t_{i+1}) - v(t_i)}{\bigvee_{t_i}^{t_{i+1}}(v)}\right|^2 |\alpha(\xi_i)|^2 \bigvee_{t_i}^{t_{i+1}}(v) \le \sum_{i=0}^{n-1}|\alpha(\xi_i)|^2\left(\bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v)\right),$$

which implies that

$$\begin{aligned}&\lim_{\delta(I_n)\to 0}\sum_{i=0}^{n-1}\left|\frac{v(t_{i+1}) - v(t_i)}{\bigvee_{t_i}^{t_{i+1}}(v)}\right|^2 |\alpha(\xi_i)|^2 \bigvee_{t_i}^{t_{i+1}}(v)\\ &\le \lim_{\delta(I_n)\to 0}\sum_{i=0}^{n-1}|\alpha(\xi_i)|^2\left(\bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v)\right) = \int_a^b |\alpha(t)|^2\, d\left(\bigvee_a^t(v)\right).\end{aligned}$$

Also

$$\begin{aligned}\lim_{\delta(I_n)\to 0}\left[\sum_{i=0}^{n-1}\bigvee_{t_i}^{t_{i+1}}(v)\,|B(\xi_i)|^2\right] &= \lim_{\delta(I_n)\to 0}\left[\sum_{i=0}^{n-1}\left(\bigvee_a^{t_{i+1}}(v) - \bigvee_a^{t_i}(v)\right)|B(\xi_i)|^2\right]\\ &= \int_a^b |B(t)|^2\, d\left(\bigvee_a^t(v)\right),\end{aligned}$$

which shows that

$$K \le \int_a^b |\alpha(t)|^2\, d\left(\bigvee_a^t(v)\right)\int_a^b |B(t)|^2\, d\left(\bigvee_a^t(v)\right). \tag{2.13}$$

By making use of (2.12) and (2.13) we derive (2.11). □

Corollary 1. *With the assumptions of Theorem 4, we have*

$$\left|\int_a^b \alpha(t)\, B(t)\, dv(t)\right| \le \left[\int_a^b |\alpha(t)|^2\, dV(t)\right]^{1/2}\left[\int_a^b |B(t)|^2\, dV(t)\right]^{1/2}. \tag{2.14}$$

3. Grüss Type Inequalities

We have the following Grüss type inequalities:

Theorem 5. *Assume that* $\alpha : [a,b] \to \mathbb{C},\ B : [a,b] \to \mathcal{B}(H)$ *are continuous and* $v : [a,b] \to \mathbb{C}$ *of bounded variation with* $v(b) \ne v(a)$, *then*

$$\begin{aligned}&\left|\int_a^b \beta(t)\, A(t)\, dv(t) - \frac{1}{v(b) - v(a)}\int_a^b \beta(s)\, dv(s)\int_a^b A(t)\, dv(t)\right|^2 \\ &\le \int_a^b \left|\beta(t) - \frac{1}{v(b) - v(a)}\int_a^b \beta(s)\, dv(s)\right|^2 dV(t)\\ &\times\left(\int_a^b |A(t)|^2\, dV(t) - \frac{1}{V(b)}\left|\int_a^b A(s)\, dV(s)\right|^2\right)\end{aligned} \tag{3.1}$$

and

$$\left|\int_a^b \beta(t) A(t)\, dv(t) - \frac{1}{v(b)-v(a)} \int_a^b \beta(s)\, dv(s) \int_a^b A(t)\, dv(t)\right|^2 \tag{3.2}$$
$$\leq \left(\int_a^b |\beta(t)|^2 dV(t) - \frac{1}{V(b)} \left|\int_a^b \beta(s)\, dV(s)\right|^2\right)$$
$$\times \int_a^b \left|A(t) - \frac{1}{v(b)-v(a)} \int_a^b A(s)\, dv(s)\right|^2 dV(t).$$

Also,

$$\left|\int_a^b \beta(t) A(t)\, dv(t) + \frac{v(b)-v(a)}{V^2(b)} \int_a^b \beta(s)\, dV(s) \int_a^b A(s)\, dV(s)\right. \tag{3.3}$$
$$- \frac{1}{V(b)} \int_a^b \beta(s)\, dV(s) \int_a^b A(t)\, dv(t)$$
$$\left. - \int_a^b \beta(t)\, dv(t) \frac{1}{V(b)} \int_a^b A(s)\, dV(s)\right|^2$$
$$\leq \left(\int_a^b |\beta(t)|^2 dV(t) - \frac{1}{V(b)} \left|\int_a^b \beta(s)\, dV(s)\right|^2\right)$$
$$\times \left(\int_a^b |A(t)|^2 dV(t) - \frac{1}{V(b)} \left|\int_a^b A(s)\, dV(s)\right|^2\right).$$

Proof. If we write the inequality (2.11) for $\alpha(t) = \beta(t) - \frac{1}{v(b)-v(a)} \int_a^b \beta(s)\, dv(s)$ and $B(t) = A(t) - \frac{1}{V(b)} \int_a^b A(s)\, dV(s)$, then we get

$$\left|\int_a^b \left(\beta(t) - \frac{1}{v(b)-v(a)} \int_a^b \beta(s)\, dv(s)\right)\right. \tag{3.4}$$
$$\left. \times \left(A(t) - \frac{1}{V(b)} \int_a^b A(s)\, dV(s)\right) dv(t)\right|^2$$
$$\leq \int_a^b \left|\beta(t) - \frac{1}{v(b)-v(a)} \int_a^b \beta(s)\, dv(s)\right|^2 dV(t)$$
$$\times \int_a^b \left|A(t) - \frac{1}{V(b)} \int_a^b A(s)\, dV(s)\right|^2 dV(t).$$

Observe that

$$\begin{aligned}
&\int_a^b \left(\beta(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right) \\
&\times \left(A(t) - \frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right) dv(t) \\
&= \int_a^b \beta(t)\,A(t)\,dv(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\int_a^b A(t)\,dv(t) \\
&- \frac{1}{V(b)}\int_a^b \left(\beta(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right) dv(t)\int_a^b A(s)\,dV(s) \\
&= \int_a^b \beta(t)\,A(t)\,dv(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\int_a^b A(t)\,dv(t)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left|A(t) - \frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right|^2 dV(t) \\
&= \int_a^b \left(A(t) - \frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right)^* \\
&\times \left(A(t) - \frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right) dV(t) \\
&= \int_a^b \left(A^*(t) - \frac{1}{V(b)}\left(\int_a^b A(s)\,dV(s)\right)^*\right) \\
&\times \left(A(t) - \frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right) dV(t)
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b \Bigg[|A(t)|^2 - \frac{1}{V(b)}\left(\int_a^b A(s)\,dV(s)\right)^* A(t) \\
&-A^*(t)\,\frac{1}{V(b)}\left(\int_a^b A(s)\,dV(s)\right) + \frac{1}{V^2(b)}\left|\int_a^b A(s)\,dV(s)\right|^2\Bigg] dV(t) \\
&= \int_a^b |A(t)|^2\,dV(t) - \frac{1}{V(b)}\left(\int_a^b A(s)\,dV(s)\right)^* \int_a^b A(t)\,dV(t) \\
&- \frac{1}{V(b)}\int_a^b A^*(t)\,dV(t)\left(\int_a^b A(s)\,dV(s)\right) \\
&+ \frac{1}{V^2(b)}\int_a^b dV(t)\left|\int_a^b A(s)\,dV(s)\right|^2 \\
&= \int_a^b |A(t)|^2\,dV(t) - \frac{1}{V(b)}\left|\int_a^b A(s)\,dV(s)\right|^2
\end{aligned}$$

and by (3.4) we derive (3.1).

Now, if we take $\alpha(t)=\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)$ and $B(t)=A(t)-\frac{1}{v(b)-v(a)}\int_a^b A(s)\,dv(s)$, then

$$\begin{aligned}&\left|\int_a^b\left(\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)\right)\right.\\&\left.\times\left(A(t)-\frac{1}{v(b)-v(a)}\int_a^b A(s)\,dv(s)\right)dv(t)\right|^2\\&\leq\int_a^b\left|\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)\right|^2dV(t)\\&\times\int_a^b\left|A(t)-\frac{1}{v(b)-v(a)}\int_a^b A(s)\,dv(s)\right|^2dV(t),\end{aligned}$$

which gives the desired result (3.1).

Further, by taking $\alpha(t)=\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)$ and $B(t)=A(t)-\frac{1}{V(b)}\int_a^b A(s)\,dV(s)$ in (2.11), we get

$$\begin{aligned}&\left|\int_a^b\left(\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)\right)\right.\\&\left.\times\left(A(t)-\frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right)dv(t)\right|^2\\&\leq\int_a^b\left|\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)\right|^2dV(t)\\&\times\int_a^b\left|A(t)-\frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right|^2dV(t).\end{aligned}\tag{3.5}$$

Since

$$\begin{aligned}&\int_a^b\left(\beta(t)-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)\right)\left(A(t)-\frac{1}{V(b)}\int_a^b A(s)\,dV(s)\right)dv(t)\\&=\int_a^b\beta(t)A(t)\,dv(t)+\frac{v(b)-v(a)}{V^2(b)}\int_a^b\beta(s)\,dV(s)\int_a^b A(s)\,dV(s)\\&-\frac{1}{V(b)}\int_a^b\beta(s)\,dV(s)\int_a^b A(t)\,dv(t)-\int_a^b\beta(t)\,dv(t)\frac{1}{V(b)}\int_a^b A(s)\,dV(s),\end{aligned}$$

hence by (3.5) we derive (3.3). □

Corollary 2. *Assume that* $\alpha : [a,b] \to \mathbb{C},\ B : [a,b] \to \mathcal{B}(H)$ *are continuous and* $v : [a,b] \to \mathbb{R}$ *monotonic nondecreasing with* $v(b) > v(a)$, *then*

$$\left| \int_a^b \beta(t) A(t) \, dv(t) - \frac{1}{v(b)-v(a)} \int_a^b \beta(s) \, dv(s) \int_a^b A(t) \, dv(t) \right|^2 \tag{3.6}$$
$$\leq \left(\int_a^b |\beta(t)|^2 \, dv(t) - \frac{1}{v(b)-v(a)} \left| \int_a^b \beta(s) \, dv(s) \right|^2 \right)$$
$$\times \left(\int_a^b |A(t)|^2 \, dv(t) - \frac{1}{v(b)-v(a)} \left| \int_a^b A(s) \, dv(s) \right|^2 \right).$$

For an operator $T \in \mathcal{B}(H)$, define

$$\operatorname{Re}(T) := \frac{1}{2}(T + T^*).$$

Lemma 3. *For any* $A,\ X,\ Y \in \mathcal{B}(H)$, *we have*

$$\left| A - \frac{X+Y}{2} \right|^2 - \frac{1}{4}|X-Y|^2 = \operatorname{Re}\left[(A^* - X^*)(A - Y) \right]. \tag{3.7}$$

Proof. We have

$$\begin{aligned}
&\left| A - \frac{X+Y}{2} \right|^2 - \frac{1}{4}|X-Y|^2 \\
&= |A|^2 - \frac{X^*+Y^*}{2}A - A^*\frac{X+Y}{2} + \frac{1}{4}\left(|X|^2 + X^*Y + Y^*X + |Y|^2 \right) \\
&- \frac{1}{4}\left(|X|^2 - X^*Y - Y^*X + |Y|^2 \right) \\
&= |A|^2 - \frac{X^*+Y^*}{2}A - A^*\frac{X+Y}{2} + \frac{1}{2}(X^*Y + Y^*X)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\left[(A^* - X^*)(A - Y) \right] \\
&= \operatorname{Re}\left[|A|^2 - X^*A - A^*Y + X^*Y \right] \\
&= |A|^2 - \operatorname{Re}(X^*A) - \operatorname{Re}(A^*Y) + \operatorname{Re}(X^*Y) \\
&= |A|^2 - \frac{1}{2}(X^*A + A^*X) - \frac{1}{2}(A^*Y + Y^*A) + \frac{1}{2}(X^*Y + Y^*X) \\
&= |A|^2 - \frac{1}{2}(X^* + Y^*)A - \frac{1}{2}A^*(X+Y) + \frac{1}{2}(X^*Y + Y^*X),
\end{aligned}$$

which proves the desired identity (3.7). □

Corollary 3. *Let* $A,\ X,\ Y \in \mathcal{B}(H)$. *The following statements are equivalent*

$$\left| A - \frac{X+Y}{2} \right|^2 \leq \frac{1}{4}|X-Y|^2$$

and

$$\operatorname{Re}\left[(X^* - A^*)(A - Y) \right] \geq 0.$$

Theorem 6. *Assume that* $w:[a,b]\to[0,\infty)$ *and* $B:[a,b]\to\mathcal{B}(H)$ *are continuous and* $v:[a,b]\to R$ *is monotonic nondecreasing with* $\int_a^b w(s)\,dv(s)=1$. *If*

$$\left|B(s)-\frac{X+Y}{2}\right|^2\le\frac{1}{4}|Y-X|^2 \text{ for all } s\in[a,b], \tag{3.8}$$

or, equivalently,

$$\operatorname{Re}\left[(B^*(s)-Y^*)(X-B(s))\right]\ge 0 \text{ for all } s\in[a,b], \tag{3.9}$$

then

$$\begin{aligned} 0&\le\int_a^b w(s)|B(s)|^2\,dv(s)-\left|\int_a^b w(s)B(s)\,dv(s)\right|^2 \\ &\le\operatorname{Re}\left[\left(\int_a^b w(s)B^*(s)\,dv(s)-Y^*\right)\left(X-\int_a^b w(s)B(s)\,dv(s)\right)\right] \\ &\le\frac{1}{4}|Y-X|^2. \end{aligned} \tag{3.10}$$

Proof. We have

$$\begin{aligned} K_1&:=\operatorname{Re}\left[\left(\int_a^b w(s)B(s)\,dv(s)-Y\right)^*\left(X-\int_a^b w(s)B(s)\,dv(s)\right)\right] \\ &=\operatorname{Re}\left[\left(\int_a^b w(s)B^*(s)\,dv(s)-Y^*\right)\left(X-\int_a^b w(s)B(s)\,dv(s)\right)\right] \\ &=\operatorname{Re}\left[\left(\int_a^b w(s)B^*(s)\,dv(s)\right)X\right]-\left|\int_a^b w(s)B(s)\,dv(s)\right|^2 \\ &-\operatorname{Re}(Y^*X)-\operatorname{Re}\left[Y^*\int_a^b w(s)B(s)\,dv(s)\right] \\ &=\left(\int_a^b w(s)\operatorname{Re}(B^*(s)X)\,dv(s)\right)-\left|\int_a^b w(s)B(s)\,dv(s)\right|^2 \\ &-\operatorname{Re}(Y^*X)-\int_a^b w(s)\operatorname{Re}(Y^*B(s))\,dv(s) \end{aligned}$$

and

$$\begin{aligned} K_2&:=\int_a^b w(s)\operatorname{Re}\left[(B(s)-Y)^*(X-B(s))\right]dv(s) \\ &=\int_a^b w(s)\left[\operatorname{Re}(B^*(s)X)-\operatorname{Re}(Y^*X)-|B(s)|^2+\operatorname{Re}(Y^*B(s))\right]dv(s) \\ &=\int_a^b w(s)\operatorname{Re}(B^*(s)X)\,dv(s)-\operatorname{Re}(Y^*X) \\ &-\int_a^b w(s)|B(s)|^2\,dv(s)+\int_a^b w(s)\operatorname{Re}(Y^*B(s))\,dv(s). \end{aligned}$$

Since

$$K_1 - K_2 = \int_a^b w(s)\left|B(s)\right|^2 dv(s) - \left|\int_a^b w(s) B(s) dv(s)\right|^2,$$

hence we derive the following identity of interest

$$\begin{aligned}(3.11)\qquad & \int_a^b w(s)\left|B(s)\right|^2 dv(s) - \left|\int_a^b w(s) B(s) dv(s)\right|^2 \\ & = \operatorname{Re}\left[\left(\int_a^b w(s) B^*(s) dv(s) - Y^*\right)\left(X - \int_a^b w(s) B(s) dv(s)\right)\right] \\ & - \int_a^b w(s) \operatorname{Re}\left[\left(B^*(s) - Y^*\right)\left(X - B(s)\right)\right] dv(s).\end{aligned}$$

Now, if condition (3.9) holds, then

$$\int_a^b w(s) \operatorname{Re}\left[\left(B^*(s) - Y^*\right)\left(X - B(s)\right)\right] dv(s) \geq 0,$$

which proves the first inequality in (3.4).

Observe that we have the following operator inequality

$$4\operatorname{Re}(C^*D) \leq \left|C + D\right|^2 \tag{3.12}$$

for all $C, D \in \mathcal{B}(H)$.

Indeed, we have

$$\begin{aligned} & \left|C + D\right|^2 - 4\operatorname{Re}(C^*D) \\ & = (C+D)^*(C+D) - 4\frac{C^*D + D^*C}{2} \\ & = (C^* + D^*)(C+D) - 2(C^*D + D^*C) \\ & = |C|^2 + D^*C + C^*D + |D|^2 - 2(C^*D + D^*C) \\ & = |C|^2 + |D|^2 - C^*D - D^*C = |C - D|^2 \geq 0.\end{aligned}$$

By utilising (3.12) we get

$$\begin{aligned} & \operatorname{Re}\left[\left(\int_a^b w(s) B(s) dv(s) - Y\right)^*\left(X - \int_a^b w(s) B(s) dv(s)\right)\right] \\ & \leq \frac{1}{4}\left|\int_a^b w(s) B(s) dv(s) - Y + X - \int_a^b w(s) B(s) dv(s)\right|^2 \\ & = \frac{1}{4}\left|Y - X\right|^2,\end{aligned}$$

which proves the last part of (3.4). □

Corollary 4. *Assume that $\alpha : [a,b] \to \mathbb{C}$, $B : [a,b] \to \mathcal{B}(H)$ are continuous and $v : [a,b] \to \mathbb{C}$ of bounded variation with $v(b) \neq v(a)$. If B satisfies either of the* *conditions (3.8) or (3.9) for some operators X and Y, then*

$$\left|\int_a^b \beta(t) B(t)\,dv(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\int_a^b B(t)\,dv(t)\right|^2 \tag{3.13}$$
$$\le \int_a^b \left|\beta(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right|^2 dV(t)$$
$$\times V(b)\left(\frac{1}{V(b)}\int_a^b |B(t)|^2\,dV(t) - \left|\frac{1}{V(b)}\int_a^b B(s)\,dV(s)\right|^2\right)$$
$$\le V(b)\int_a^b \left|\beta(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right|^2 dV(t)$$
$$\times \operatorname{Re}\left[\left(\frac{1}{V(b)}\int_a^b B^*(s)\,dV(s) - Y^*\right)\left(X - \frac{1}{V(b)}\int_a^b B(s)\,dV(s)\right)\right]$$
$$\le \frac{1}{4}V(b)\left(\int_a^b \left|\beta(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right|^2 dV(t)\right)|Y-X|^2.$$

The proof follows by (3.1) and (3.10) for $m \equiv 1$ and $\frac{1}{V(b)}dV(\cdot)$.

Remark 4. *Assume that $\alpha : [a,b] \to \mathbb{C}$, $B : [a,b] \to \mathcal{B}(H)$ are continuous and $v : [a,b] \to \mathbb{R}$ is monotonic nondecreasing with $v(b) > v(a)$. If B satisfies either of the conditions (3.8) or (3.9) for some operators X and Y, then*

$$\left|\int_a^b \beta(t) B(t)\,dv(t) - \frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\int_a^b B(t)\,dv(t)\right|^2 \tag{3.14}$$
$$\le [v(b)-v(a)]^2$$
$$\times\left(\frac{1}{v(b)-v(a)}\int_a^b |\beta(t)|^2\,dv(t) - \left|\frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right|^2\right)$$
$$\times\left(\frac{1}{v(b)-v(a)}\int_a^b |B(t)|^2\,dv(t) - \left|\frac{1}{v(b)-v(a)}\int_a^b B(s)\,dv(s)\right|^2\right)$$

$$\le [v(b)-v(a)]^2$$
$$\times\left(\frac{1}{v(b)-v(a)}\int_a^b |\beta(t)|^2\,dv(t) - \left|\frac{1}{v(b)-v(a)}\int_a^b \beta(s)\,dv(s)\right|^2\right)$$
$$\times \operatorname{Re}\left[\left(\frac{1}{v(b)-v(a)}\int_a^b B^*(s)\,dv(s) - Y^*\right)\right.$$
$$\left.\times\left(X - \frac{1}{v(b)-v(a)}\int_a^b B(s)\,dv(s)\right)\right]$$

$$\leq \frac{1}{4}\left[v\left(b\right)-v\left(a\right)\right]^{2}$$
$$\times\left(\frac{1}{v\left(b\right)-v\left(a\right)}\int_{a}^{b}\left|\beta\left(t\right)\right|^{2}dv\left(t\right)-\left|\frac{1}{v\left(b\right)-v\left(a\right)}\int_{a}^{b}\beta\left(s\right)dv\left(s\right)\right|^{2}\right)$$
$$\times\left|Y-X\right|^{2}.$$

Moreover, if there exist the constant $\delta,\ \gamma\in\mathbb{C}$ *such that*

$$\left|\beta\left(s\right)-\frac{\delta+\gamma}{2}\right|^{2}\leq\frac{1}{4}\left|\gamma-\delta\right|^{2}\text{ for a.e. }s\in\left[a,b\right],\tag{3.15}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\overline{\beta\left(s\right)}-\bar{\gamma}\right)\left(\delta-\beta\left(s\right)\right)\right]\geq0\text{ for a.e. }s\in\left[a,b\right],\tag{3.16}$$

then we have

$$\left|\int_{a}^{b}\beta\left(t\right)B\left(t\right)dv\left(t\right)-\frac{1}{v\left(b\right)-v\left(a\right)}\int_{a}^{b}\beta\left(s\right)dv\left(s\right)\int_{a}^{b}B\left(t\right)dv\left(t\right)\right|^{2}\tag{3.17}$$
$$\leq\frac{1}{16}\left[v\left(b\right)-v\left(a\right)\right]^{2}\left|\gamma-\delta\right|^{2}\left|Y-X\right|^{2}.$$

By taking the square root in this inequality we obtain the Grüss type inequality for modulus,

$$\left|\int_{a}^{b}\beta\left(t\right)B\left(t\right)dv\left(t\right)-\frac{1}{v\left(b\right)-v\left(a\right)}\int_{a}^{b}\beta\left(s\right)dv\left(s\right)\int_{a}^{b}B\left(t\right)dv\left(t\right)\right|\tag{3.18}$$
$$\leq\frac{1}{4}\left[v\left(b\right)-v\left(a\right)\right]\left|\gamma-\delta\right|\left|Y-X\right|.$$

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