OSTROWSKI AND TRAPEZOID TYPE RIEMANN-STIELTJES INTEGRAL INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. Let $u:[a,b]\to\mathbb{C}$ be a function of bounded variation and $B:[a,b]\to\mathcal{B}(H)$. We say that B is of *u-square-Lipschitz type* if there exists a selfadjoint operator K such that

$$|B(s)-B(t)|^2\le|u(s)-u(t)|^2K^2 \text{ for all } s,\ t\in[a,b].$$

In this paper we show among others that, if $B:[a,b]\to\mathcal{B}(H)$ is of *u-square-Lipschitz type* with u continuous, then for $v:[a,b]\to\mathbb{C}$ of bounded variation,

$$\left|[v(b)-v(a)]B(s)-\int_a^b B(t)\,dv(t)\right|^2\le\left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2\bigvee_a^b(u)\,K^2,$$

where $s\in(a,b)$. Applications for $B:[a,b]\to\mathcal{B}(H)$ strongly differentiable with $B'\in L_2([a,b],\mathcal{B}(H))$ are also provided.

1. Introduction

In 1998, Dragomir and Wang proved the following Ostrowski type inequality for p-norm [16].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$(1.1)\qquad \left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le\frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p},$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b|g(t)|^p\,dt\right)^{1/p}.$$

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From (1.1) we get the following midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{2(q+1)^{1/q}} (b-a)^{1/q} \|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

For $p = q = 2$ we derive the $L_2[a,b]$-inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^2 \leq \frac{1}{3}\left[\left(\frac{x-a}{b-a}\right)^3 + \left(\frac{b-x}{b-a}\right)^3\right](b-a)\|f'\|_{[a,b],2}^2, \tag{1.3}$$

for all $x \in [a,b]$ and the midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right|^2 \leq \frac{1}{12}(b-a)\|f'\|_{[a,b]}^2. \tag{1.4}$$

For a survey on scalar Ostrowski's inequality, see [15]. For recent papers on this inequality see also [2]-[4] and [17]-[20].

In 1999, Cerone and Dragomir proved the following *generalized trapezoid* type inequality for p-norm [7].

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f' \in L_p[a,b]$, then we have the inequality*

$$\left| (b-x) f(b) + (x-a) f(a) - \int_a^b f(t)\,dt \right| \leq \frac{1}{(q+1)^{1/q}}\left[(x-a)^{q+1} + (b-x)^{q+1}\right]^{1/q} \|f'\|_{[a,b],p}, \tag{1.5}$$

for all $x \in [a,b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

From (1.5) we get the following *trapezoid inequality*

$$\left| (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)\,dt \right| \leq \frac{1}{2(q+1)^{1/q}}(b-a)^{1+1/q}\|f'\|_{[a,b],p}, \tag{1.6}$$

and $\frac{1}{2}$ is a best possible constant.

For $p = q = 2$ we derive the $L_2[a,b]$-inequality

$$\left| (b-x) f(b) + (x-a) f(a) - \int_a^b f(t)\,dt \right|^2 \leq \frac{1}{3}\left[(x-a)^3 + (b-x)^3\right]\|f'\|_{[a,b],2}^2, \tag{1.7}$$

for all $x \in [a,b]$ and the trapezoid inequality

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|^2 \le \frac{1}{12}(b-a)^3\|f'\|_{[a,b]}^2 . \tag{1.8}$$

For a survey on scalar trapezoid inequality, see [7]. For recent papers on this inequality see also [1]-[6] and [8]-[23].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B| \le |A|+|B|$, so the classical arguments using this inequality can not be used. In order to obtain the corresponding version for the operator modulus we need the following preparations.

In this paper we provide upper bounds in the operator order for the quantities

$$\left|[v(b)-v(a)]B(s)-\int_a^b B(t)\,dv(t)\right|^2,\ s\in(a,b)$$

and

$$\left|[v(b)-v(s)]B(b)+[v(s)-v(a)]B(a)-\int_a^b B(t)\,dv(t)\right|^2,\ s\in(a,b)$$

in the case when $B:[a,b]\to\mathcal{B}(H)$ satisfies a Lipschitz type condition in terms of the operator modulus and $v:[a,b]\to\mathbb{C}$ is of bounded variation.

2. Main Results

Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$0 \le \left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2 = |\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s) \\ -\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2,$$

which gives that

$$|\alpha(t)|^2|A(s)|^2+|\alpha(s)|^2|A(t)|^2 \ge \alpha(s)\overline{\alpha(t)}A^*(t)A(s)+\alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s,\,t\in[a,b]$.

Now, multiply this with $w(s)w(t)\ge 0$ to get

$$w(t)|\alpha(t)|^2w(s)|A(s)|^2+w(s)|\alpha(s)|^2w(t)|A(t)|^2 \\ \ge w(t)\overline{\alpha(t)}A^*(t)w(s)\alpha(s)A(s)+w(s)\overline{\alpha(s)}A^*(s)w(t)\alpha(t)A(t)$$

for all $s,\,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned}&\int_a^b w(t)|\alpha(t)|^2dt\int_a^b|A(s)|^2ds+\int_a^b|\alpha(s)|^2ds\int_a^b w(t)|A(t)|^2dt\\&\geq\int_a^b w(t)\overline{\alpha(t)}A^*(t)\,dt\int_a^b\alpha(s)A(s)\,ds\\&+\int_a^b w(s)\overline{\alpha(s)}A^*(s)\,ds\int_a^b\alpha(t)A(t)\,dt\\&=2\left|\int_a^b w(s)\alpha(s)A(s)\,ds\right|^2,\end{aligned}$$

which proves that

$$\int_a^b w(t)|\alpha(t)|^2dt\int_a^b w(t)|A(t)|^2dt\geq\left|\int_a^b w(t)\alpha(t)A(t)\,dt\right|^2, \tag{2.1}$$

provided that $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and

$$A\in L_{2,w}([a,b],\mathcal{B}(H)):=\left\{A:[a,b]\to B(H),\ \int_a^b w(t)\|A(t)\|^2dt<\infty\right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^n w_k|z_k|^2\sum_{k=1}^n w_k|A_k|^2\geq\left|\sum_{k=1}^n w_kz_kA_k\right|^2, \tag{2.2}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\geq0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n w_k=1$.

Let $u:[a,b]\to\mathbb{C}$ be a function of bounded variation and $B:[a,b]\to\mathcal{B}(H)$. We say that B is of *u-square-Lipschitz type* if there exists a selfadjoint operator K such that

$$|B(s)-B(t)|^2\leq|u(s)-u(t)|^2K^2 \text{ for all } s,\ t\in[a,b]. \tag{2.3}$$

If $u(t)=t$, then the condition (2.3) becomes the following *square-Lipschitz type* condition

$$|B(s)-B(t)|^2\leq(s-t)^2K^2 \text{ for all } s,\ t\in[a,b]. \tag{2.4}$$

Theorem 3. *Assume that $u:[a,b]\to\mathbb{C}$ is of bounded variation, $v:[a,b]\to\mathbb{C}$ is continuous and $B:[a,b]\to\mathcal{B}(H)$ satisfies the condition (2.3), then*

$$\left|\int_a^b v(t)\,dB(t)\right|^2\leq\int_a^b|v(t)|^2d\left(\bigvee_a^t(u)\right)K^2, \tag{2.5}$$

where $\bigvee_a^t(u)$ is the total variation of u on $[a,t]$, $t\in(a,b]$.

Proof. By taking the square root in (2.3) and then the operator norm, we get

$$\|B(s)-B(t)\|\leq|u(s)-u(t)|\,\|K\| \tag{2.6}$$

for all s, $t\in[a,b]$.

Since $u:[a,b]\to\mathbb{C}$ is of bounded variation, hence by (2.6), B is of bounded variation in the norm of $\mathcal{B}(H)$ and since $v:[a,b]\to\mathbb{C}$ is continuous, it follows that the Riemann-Stieltjes integral $\int_a^b v(t)\,dB(t)$ exists.

Let $I_n : a = t_0 < t_1 < .... < t_n = b$ a division of $[a,b]$ with the norm $\delta(I_n) := \max_{i=0,...,n-1}(t_{i+1} - t_i)$ and the intermediate points $\xi_i \in [x_i, x_{i+1}]$, $i = 0, ..., n-1$. Using the definition of Riemann-Stieltjes integral, the continuity property of modulus of operators and the CBS discrete inequality (2.2) we have

$$\begin{aligned}
(2.7)\qquad & \left| \int_a^b v(t)\, dB(t) \right|^2 \\
&= \left| \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} v(\xi_i)\left(B(t_{i+1}) - B(t_i)\right) \right|^2 \\
&= \left| \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} v(\xi_i) \left|u(t_{i+1}) - u(t_i)\right| \frac{B(t_{i+1}) - B(t_i)}{\left|u(t_{i+1}) - u(t_i)\right|} \right|^2 \\
&= \lim_{\delta(I_n)\to 0} \left| \sum_{i=0}^{n-1} v(\xi_i) \left|u(t_{i+1}) - u(t_i)\right| \frac{B(t_{i+1}) - B(t_i)}{\left|u(t_{i+1}) - u(t_i)\right|} \right|^2 \\
&\le \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|v(\xi_i)\right|^2 \left|u(t_{i+1}) - u(t_i)\right| \\
&\times \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|u(t_{i+1}) - u(t_i)\right| \left| \frac{B(t_{i+1}) - B(t_i)}{u(t_{i+1}) - u(t_i)} \right|^2 \\
&\le \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|v(\xi_i)\right|^2 \left|u(t_{i+1}) - u(t_i)\right| \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|u(t_{i+1}) - u(t_i)\right| K^2.
\end{aligned}$$

Now, observe that

$$\lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|u(t_{i+1}) - u(t_i)\right| = \bigvee_a^b (u)$$

and

$$\begin{aligned}
(2.8)\qquad & \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|v(\xi_i)\right|^2 \left|u(t_{i+1}) - u(t_i)\right| \\
&= \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|v(\xi_i)\right|^2 \left| \frac{u(t_{i+1}) - u(t_i)}{\bigvee_{t_i}^{t_{i+1}} (u)} \right| \bigvee_{t_i}^{t_{i+1}} (u) \\
&\le \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|v(\xi_i)\right|^2 \bigvee_{t_i}^{t_{i+1}} (u) \\
&= \lim_{\delta(I_n)\to 0} \sum_{i=0}^{n-1} \left|v(\xi_i)\right|^2 \left(\bigvee_a^{t_{i+1}} (u) - \bigvee_a^{t_i} (u) \right) = \int_a^b \left|v(t)\right|^2 d\left(\bigvee_a^t (u) \right).
\end{aligned}$$

By making use of (2.7) and (2.8) we derive the desired result (2.5). □

Remark 1. *If u in (2.3) is also continuous, then B will be continuous and we can assume that $v : [a,b] \to \mathbb{C}$ is of bounded variation in Theorem 3 for the inequality (2.5) to remain valid.*

Corollary 1. *Assume that $v:[a,b]\to\mathbb{C}$ is continuous and $B:[a,b]\to\mathcal{B}(H)$ satisfies the square-Lipschitz type condition for the operator K, then*

$$\left|\int_a^b v(t)\,dB(t)\right|^2 \le \int_a^b |v(t)|^2\,dtK^2. \tag{2.9}$$

Corollary 2. *Assume that $v:[a,b]\to\mathbb{C}$ is continuous and $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable with $B'\in L_2([a,b],\mathcal{B}(H))$, then*

$$\left|\int_a^b v(t)\,dB(t)\right|^2 \le \int_a^b |v(t)|^2\,dt\int_a^b |B'(\tau)|^2\,d\tau. \tag{2.10}$$

The proof follows by (2.9) for $K^2=\int_a^b |B'(\tau)|^2\,d\tau$.

Theorem 4. *Assume that $B:[a,b]\to\mathcal{B}(H)$ satisfies the condition (2.3) with u continuous and of bounded variation, then for $v:[a,b]\to\mathbb{C}$ of bounded variation, we have*

$$\begin{aligned}
&\left|[v(b)-v(a)]B(s)-\int_a^b B(t)\,dv(t)\right|^2 \\
&\le\left[\int_a^s |v(t)-v(a)|^2\,d\left(\bigvee_a^t(u)\right)+\int_s^b |v(b)-v(t)|^2\,d\left(\bigvee_s^t(u)\right)\right]K^2 \\
&\le\left[\int_a^s \left(\bigvee_a^t(v)\right)^2 d\left(\bigvee_a^t(u)\right)+\int_s^b \left(\bigvee_t^b(v)\right)^2 d\left(\bigvee_s^t(u)\right)\right]K^2 \\
&\le\left[\left(\bigvee_a^s(v)\right)^2\bigvee_a^s(u)+\left(\bigvee_s^b(v)\right)^2\bigvee_s^b(u)\right]K^2 \\
&\le\left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2\bigvee_a^b(u)\,K^2.
\end{aligned} \tag{2.11}$$

Proof. Using integration by parts for the Riemann-Stieltjes integral, we have for $s\in(a,b)$ that

$$\begin{aligned}
\int_a^s [v(t)-v(a)]\,dB(t) &= [v(t)-v(a)]B(t)\big|_a^s-\int_a^s B(t)\,dv(t) \\
&= [v(s)-v(a)]B(s)-\int_a^s B(t)\,dv(t)
\end{aligned}$$

and

$$\begin{aligned}
\int_s^b [v(t)-v(b)]\,dB(t) &= [v(t)-v(b)]B(t)\big|_s^b-\int_s^b B(t)\,dv(t) \\
&= [v(b)-v(s)]B(s)-\int_s^b B(t)\,dv(t).
\end{aligned}$$

If we add these two equalities, then we get

$$\begin{aligned}
&\int_a^s [v(t)-v(a)]\,dB(t)+\int_s^b [v(t)-v(b)]\,dB(t)\\
&=[v(s)-v(a)]\,B(s)+[v(b)-v(s)]\,B(s)\\
&-\int_a^s B(t)\,dv(t)-\int_s^b B(t)\,dv(t)\\
&=[v(b)-v(a)]\,B(s)-\int_a^b B(t)\,dv(t),
\end{aligned}$$

namely

$$[v(b)-v(a)]\,B(s)-\int_a^b B(t)\,dv(t)=\int_a^b p_v(t,s)\,dB(t),$$

where for $s\in(a,b)$,

$$p_v(t,s):=\begin{cases} v(t)-v(a) & \text{for } a\le t\le s,\\ \\ v(t)-v(b) & \text{for } s<t\le b.\end{cases}$$

Observe that B is continuous and $p_v(\cdot,s)$ is of bounded variation for $s\in(a,b)$, then by (2.5) and Remark 1 we get

$$\left|\int_a^b p_v(t,s)\,dB(t)\right|^2 \le \left(\int_a^b |p_v(t,s)|^2\, d\left(\bigvee_a^t(u)\right)\right)K^2 \tag{2.12}$$

$$\begin{aligned}
&=\left[\int_a^s |v(t)-v(a)|^2\, d\left(\bigvee_a^t(u)\right)+\int_s^b |v(b)-v(t)|^2\, d\left(\bigvee_a^t(u)\right)\right]K^2\\
&=\left[\int_a^s |v(t)-v(a)|^2\, d\left(\bigvee_a^t(u)\right)+\int_s^b |v(b)-v(t)|^2\, d\left(\bigvee_s^t(u)\right)\right]K^2
\end{aligned}$$

which proves the first inequality in (2.10).

The rest is obvious. □

Remark 2. *If $p\in(a,b)$ is such that $\bigvee_p^b(u)=\bigvee_a^p(u)=\frac{1}{2}\bigvee_a^b(u)$, then by (2.11)*

$$\begin{aligned}
&\left|[v(b)-v(a)]\,B(p)-\int_a^b B(t)\,dv(t)\right|^2\\
&\le\frac{1}{2}\left[\left(\bigvee_a^p(v)\right)^2+\left(\bigvee_p^b(v)\right)^2\right]\bigvee_a^b(u)\,K^2.
\end{aligned}\tag{2.13}$$

If $q\in(a,b)$ is such that $\bigvee_a^q(v)=\bigvee_q^b(v)=\frac{1}{2}\bigvee_a^b(v)$, then by (2.11)

$$\left|[v(b)-v(a)]\,B(q)-\int_a^b B(t)\,dv(t)\right|^2\le\frac{1}{4}\left(\bigvee_a^b(v)\right)^2\bigvee_a^b(u)\,K^2. \tag{2.14}$$

Corollary 3. *Assume that $B:[a,b]\to\mathcal{B}(H)$ satisfies the square-Lipschitz type condition for the operator K, then for $v:[a,b]\to\mathbb{C}$ of bounded variation,*

$$\begin{aligned}
&\left|[v(b)-v(a)]B(s)-\int_a^b B(t)\,dv(t)\right|^2 \\
&\le\left[\int_a^s|v(t)-v(a)|^2\,dt+\int_s^b|v(b)-v(t)|^2\,dt\right]K^2 \\
&\le\left[\int_a^s\left(\bigvee_a^t(v)\right)^2dt+\int_s^b\left(\bigvee_t^b(v)\right)^2dt\right]K^2 \\
&\le\left[\left(\bigvee_a^s(v)\right)^2(s-a)+\left(\bigvee_s^b(v)\right)^2(b-s)\right]K^2 \\
&\le\left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2(b-a)\,K^2.
\end{aligned}\tag{2.15}$$

Remark 3. *If we take $s=\frac{a+b}{2}$ in (2.15), then*

$$\begin{aligned}
&\left|[v(b)-v(a)]B(s)-\int_a^b B(t)\,dv(t)\right|^2 \\
&\le\frac{1}{2}(b-a)\left[\left(\bigvee_a^{\frac{a+b}{2}}(v)\right)^2+\left(\bigvee_{\frac{a+b}{2}}^b(v)\right)^2\right]K^2.
\end{aligned}\tag{2.16}$$

If $q\in(a,b)$ is as in Remark 2, then we get from (2.15) that

$$\left|[v(b)-v(a)]B(q)-\int_a^b B(t)\,dv(t)\right|^2\le\frac{1}{4}(b-a)\left(\bigvee_a^b(v)\right)^2K^2.$$

Corollary 4. *Assume that $B:[a,b]\to\mathcal{B}(H)$ is strongly differentiable with $B'\in L_2([a,b],\mathcal{B}(H))$, then for $v:[a,b]\to\mathbb{C}$ of bounded variation,*

$$\begin{aligned}
&\left|[v(b)-v(a)]B(s)-\int_a^b B(t)\,dv(t)\right|^2 \\
&\le\left[\int_a^s|v(t)-v(a)|^2\,dt+\int_s^b|v(b)-v(t)|^2\,dt\right]\int_a^b|B'(\tau)|^2\,d\tau \\
&\le\left[\int_a^s\left(\bigvee_a^t(v)\right)^2dt+\int_s^b\left(\bigvee_t^b(v)\right)^2dt\right]\int_a^b|B'(\tau)|^2\,d\tau \\
&\le\left[\left(\bigvee_a^s(v)\right)^2(s-a)+\left(\bigvee_s^b(v)\right)^2(b-s)\right]\int_a^b|B'(\tau)|^2\,d\tau \\
&\le\left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2(b-a)\int_a^b|B'(\tau)|^2\,d\tau.
\end{aligned}\tag{2.17}$$

Remark 4. *If we take $s=\frac{a+b}{2}$ in (2.17), then*

$$\left|[v(b)-v(a)]B\left(\frac{a+b}{2}\right)-\int_a^b B(t)\,dv(t)\right|^2 \le \frac{1}{2}(b-a)\left[\left(\bigvee_a^{\frac{a+b}{2}}(v)\right)^2+\left(\bigvee_{\frac{a+b}{2}}^b(v)\right)^2\right]\int_a^b |B'(\tau)|^2\,d\tau. \tag{2.18}$$

If $q\in(a,b)$ is as in Remark 2, then we get from (2.17) that

$$\left|[v(b)-v(a)]B(q)-\int_a^b B(t)\,dv(t)\right|^2 \le \frac{1}{4}(b-a)\left(\bigvee_a^b(v)\right)^2\int_a^b |B'(\tau)|^2\,d\tau. \tag{2.19}$$

Theorem 5. *Assume that $B:[a,b]\to\mathcal{B}(H)$ satisfies the condition (2.3) with u continuous and of bounded variation, then for $v:[a,b]\to\mathbb{C}$ of bounded variation, we have*

$$\begin{aligned}\left|[v(b)-v(s)]B(b)+[v(s)-v(a)]B(a)-\int_a^b B(t)\,dv(t)\right|^2 &\le \left[\int_a^s |v(t)-v(s)|^2\,d\left(\bigvee_a^t(u)\right)+\int_s^b |v(t)-v(s)|^2\,d\left(\bigvee_s^t(u)\right)\right]K^2\\ &\le \left[\int_a^s \left(\bigvee_t^s(v)\right)^2 d\left(\bigvee_a^t(u)\right)+\int_s^b \left(\bigvee_s^t(v)\right)^2 d\left(\bigvee_s^t(u)\right)\right]K^2\\ &\le \left[\left(\bigvee_a^s(v)\right)^2\bigvee_a^s(u)+\left(\bigvee_s^b(v)\right)^2\bigvee_s^b(u)\right]K^2\\ &\le \left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2\bigvee_a^b(u)\,K^2.\end{aligned} \tag{2.20}$$

Proof. Using integration by parts for the Riemann-Stieltjes integral, we have for $s\in(a,b)$ that

$$\begin{aligned}\int_a^b [v(t)-v(s)]\,dB(t) &= [v(t)-v(s)]B(t)\big|_a^b-\int_a^b B(t)\,dv(t)\\ &= [v(b)-v(s)]B(b)+[v(s)-v(a)]B(a)-\int_a^b B(t)\,dv(t)\end{aligned} \tag{2.21}$$

By (2.5) we get

$$\left|\int_a^b [v(t)-v(s)]\,dB(t)\right|^2 \leq \int_a^b |v(t)-v(s)|^2\, d\left(\bigvee_a^t(u)\right)K^2 \tag{2.22}$$
$$= \int_a^s |v(t)-v(s)|^2\, d\left(\bigvee_a^t(u)\right)K^2 + \int_s^b |v(t)-v(s)|^2\, d\left(\bigvee_a^t(u)\right)K^2$$
$$= \int_a^s |v(t)-v(s)|^2\, d\left(\bigvee_a^t(u)\right)K^2 + \int_s^b |v(t)-v(s)|^2\, d\left(\bigvee_s^t(u)\right)K^2,$$

which proves the first inequality in (2.20).

The rest is obvious. □

Remark 5. *If $p\in(a,b)$ is as in Remark 2, then we have by (2.20)*

$$\left|[v(b)-v(p)]\,B(b)+[v(p)-v(a)]\,B(a)-\int_a^b B(t)\,dv(t)\right|^2 \tag{2.23}$$
$$\leq \frac{1}{2}\left[\left(\bigvee_a^p(v)\right)^2+\left(\bigvee_p^b(v)\right)^2\right]\bigvee_a^b(u)\,K^2.$$

If $q\in(a,b)$ is as in Remark 2, then we get from (2.20)

$$\left|[v(b)-v(q)]\,B(b)+[v(q)-v(a)]\,B(a)-\int_a^b B(t)\,dv(t)\right|^2 \tag{2.24}$$
$$\leq \frac{1}{4}\left(\bigvee_a^b(v)\right)^2\bigvee_a^b(u)\,K^2.$$

Corollary 5. *Assume that $B:[a,b]\to\mathcal{B}(H)$ satisfies the square-Lipschitz type condition for the operator K, then for $v:[a,b]\to\mathbb{C}$ of bounded variation,*

$$\left|[v(b)-v(s)]\,B(b)+[v(s)-v(a)]\,B(a)-\int_a^b B(t)\,dv(t)\right|^2 \tag{2.25}$$
$$\leq \left[\left(\bigvee_a^s(v)\right)^2(s-a)+\left(\bigvee_s^b(v)\right)^2(b-s)\right]K^2$$
$$\leq \left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2(b-a)\,K^2.$$

Remark 6. *If we take* $s=\frac{a+b}{2}$ *in (2.25), then*

$$\left|\left[v(b)-v\left(\frac{a+b}{2}\right)\right]B(b)+\left[v\left(\frac{a+b}{2}\right)-v(a)\right]B(a)-\int_a^b B(t)\,dv(t)\right|^2 \le \frac{1}{2}(b-a)\left[\left(\bigvee_a^{\frac{a+b}{2}}(v)\right)^2+\left(\bigvee_{\frac{a+b}{2}}^b(v)\right)^2\right]K^2. \tag{2.26}$$

If $q\in(a,b)$ *is as in Remark 2, then we get from (2.25) that*

$$\left|[v(b)-v(q)]B(b)+[v(q)-v(a)]B(a)-\int_a^b B(t)\,dv(t)\right|^2 \le \frac{1}{4}(b-a)\left(\bigvee_a^b(v)\right)^2K^2. \tag{2.27}$$

Corollary 6. *Assume that* $B:[a,b]\to\mathcal{B}(H)$ *is strongly differentiable with* $B'\in L_2([a,b],\mathcal{B}(H))$*, then for* $v:[a,b]\to\mathbb{C}$ *of bounded variation,*

$$\begin{aligned}&\left|[v(b)-v(s)]B(b)+[v(s)-v(a)]B(a)-\int_a^b B(t)\,dv(t)\right|^2 \\ &\le\left[\int_a^s|v(t)-v(s)|^2dt+\int_s^b|v(t)-v(s)|^2dt\right]\int_a^b|B'(\tau)|^2d\tau \\ &\le\left[\int_a^s\left(\bigvee_t^s(v)\right)^2dt+\int_s^b\left(\bigvee_s^t(v)\right)^2dt\right]\int_a^b|B'(\tau)|^2d\tau \\ &\le\left[\left(\bigvee_a^s(v)\right)^2(s-a)+\left(\bigvee_s^b(v)\right)^2(b-s)\right]\int_a^b|B'(\tau)|^2d\tau \\ &\le\left[\max\left\{\bigvee_a^s(v),\bigvee_s^b(v)\right\}\right]^2(b-a)\int_a^b|B'(\tau)|^2d\tau.\end{aligned} \tag{2.28}$$

Remark 7. *If we take* $s=\frac{a+b}{2}$ *in (2.28), then we derive*

$$\left|\left[v(b)-v\left(\frac{a+b}{2}\right)\right]B(b)+\left[v\left(\frac{a+b}{2}\right)-v(a)\right]B(a)\right. \tag{2.29}$$

$$\left.-\int_a^b B(t)\,dv(t)\right|^2 \le \frac{1}{2}(b-a)\left[\left(\bigvee_a^{\frac{a+b}{2}}(v)\right)^2+\left(\bigvee_{\frac{a+b}{2}}^b(v)\right)^2\right]\int_a^b|B'(\tau)|^2d\tau. \tag{2.30}$$

If $q \in (a,b)$ is as in Remark 2, then we get from (2.28)

$$\left| [v(b) - v(q)] B(b) + [v(q) - v(a)] B(a) - \int_a^b B(t)\, dv(t) \right|^2 \leq \frac{1}{4} \left(\bigvee_a^b (v) \right)^2 (b-a) \int_a^b |B'(\tau)|^2 d\tau. \tag{2.31}$$

3. Some Examples

Let $A,\ B \in \mathcal{B}(H)$ such that $(1-t)A + tB$ is invertible for all $t \in [0,1]$. For this to happen, it is enough to assume that $A,\ B > 0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t) := ((1-t)A + tB)^{-1}$, $t \in [0,1]$ and observe that

$$B'(t) = -((1-t)A + tB)^{-1} (B - A) ((1-t)A + tB)^{-1},\ t \in (0,1).$$

By (2.17) we get for the function of bounded variation $v : [0,1] \to \mathbb{C}$ that

$$\begin{aligned} & \left| [v(1) - v(0)] ((1-s)A + sB)^{-1} - \int_0^1 ((1-t)A + tB)^{-1} dv(t) \right|^2 \\ & \leq \left[\left(\bigvee_0^s (v) \right)^2 s + \left(\bigvee_s^1 (v) \right)^2 (1-s) \right] \\ & \times \int_0^1 \left| ((1-\tau)A + \tau B)^{-1} (B-A) ((1-\tau)A + \tau B)^{-1} \right|^2 d\tau \\ & \leq \left[\max \left\{ \bigvee_0^s (v), \bigvee_s^1 (v) \right\} \right]^2 \\ & \times \int_0^1 \left| ((1-\tau)A + \tau B)^{-1} (B-A) ((1-\tau)A + \tau B)^{-1} \right|^2 d\tau \end{aligned} \tag{3.1}$$

for $s \in (0,1)$.

By utilising (2.18) we then get the midpoint inequality

$$\begin{aligned} & \left| [v(1) - v(0)] \left(\frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t)A + tB)^{-1} dv(t) \right|^2 \\ & \leq \frac{1}{2} \left[\left(\bigvee_0^{\frac{1}{2}} (v) \right)^2 + \left(\bigvee_{\frac{1}{2}}^1 (v) \right)^2 \right] \\ & \times \int_0^1 \left| ((1-\tau)A + \tau B)^{-1} (B-A) ((1-\tau)A + \tau B)^{-1} \right|^2 d\tau, \end{aligned} \tag{3.2}$$

where $v : [0,1] \to \mathbb{C}$ is of bounded variation.

If $q \in (0,1)$ is such that $\bigvee_0^q (v) = \bigvee_q^1 (v) = \frac{1}{2} \bigvee_0^1 (v)$, then we get from (2.17) that

$$\left| [v(1) - v(0)] ((1-q) A + qB)^{-1} - \int_0^1 ((1-t) A + tB)^{-1} dv(t) \right|^2 \le \frac{1}{4} \left(\bigvee_0^1 (v) \right)^2 \int_0^1 \left| ((1-\tau) A + \tau B)^{-1} (B - A) ((1-\tau) A + \tau B)^{-1} \right|^2 d\tau, \tag{3.3}$$

for $s \in (0,1)$.

Since for any operator $V \in \mathcal{B}(H)$ we have $|V| \le \|V\|$, hence

$$\left| ((1-t) A + tB)^{-1} (B - A) ((1-t) A + tB)^{-1} \right| \le \left\| ((1-t) A + tB)^{-1} \right\|^2 \|B - A\|,$$

which implies that

$$\int_0^1 \left| ((1-\tau) A + \tau B)^{-1} (B - A) ((1-\tau) A + \tau B)^{-1} \right|^2 d\tau \le \|B - A\|^2 \int_0^1 \left\| ((1-t) A + tB)^{-1} \right\|^4 dt.$$

Therefore by (3.2) we derive

$$\left| [v(1) - v(0)] \left(\frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t) A + tB)^{-1} dv(t) \right|^2 \le \frac{1}{2} \left[\left(\bigvee_0^{\frac{1}{2}} (v) \right)^2 + \left(\bigvee_{\frac{1}{2}}^1 (v) \right)^2 \right] \|B - A\|^2 \int_0^1 \left\| ((1-t) A + tB)^{-1} \right\|^4 dt, \tag{3.4}$$

where $v : [0,1] \to \mathbb{C}$ is of bounded variation.

Now, if $A \ge m > 0$ and $B \ge m > 0$, then $((1-t) A + tB)^{-1} \le m^{-1}$ for $t \in [0,1]$ and by (3.4) we obtain the simpler inequality

$$\left| [v(1) - v(0)] \left(\frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t) A + tB)^{-1} dv(t) \right|^2 \le \frac{1}{2} \left[\left(\bigvee_0^{\frac{1}{2}} (v) \right)^2 + \left(\bigvee_{\frac{1}{2}}^1 (v) \right)^2 \right] \frac{\|B - A\|^2}{m^4}. \tag{3.5}$$

From (2.28) we derive that

$$\begin{aligned}
&\left|[v(1)-v(s)]B^{-1}+[v(s)-v(0)]A^{-1}-\int_0^1((1-t)A+tB)^{-1}dv(t)\right|^2 \\
&\leq\left[\left(\bigvee_0^s(v)\right)^2 s+\left(\bigvee_s^1(v)\right)^2(1-s)\right] \\
&\times\int_0^1\left|((1-\tau)A+\tau B)^{-1}(B-A)((1-\tau)A+\tau B)^{-1}\right|^2 d\tau \\
&\leq\left[\max\left\{\bigvee_0^s(v),\bigvee_s^1(v)\right\}\right]^2 \\
&\times\int_0^1\left|((1-\tau)A+\tau B)^{-1}(B-A)((1-\tau)A+\tau B)^{-1}\right|^2 d\tau,
\end{aligned}\tag{3.6}$$

for $s\in(a,b)$.

This implies that

$$\begin{aligned}
&\left|\left[v(1)-v\left(\frac{1}{2}\right)\right]B^{-1}+\left[v\left(\frac{1}{2}\right)-v(0)\right]A^{-1}\right. \\
&\left.-\int_0^1((1-t)A+tB)^{-1}dv(t)\right|^2 \\
&\leq\frac{1}{2}\left[\left(\bigvee_0^s(v)\right)^2+\left(\bigvee_s^1(v)\right)^2\right] \\
&\times\int_0^1\left|((1-\tau)A+\tau B)^{-1}(B-A)((1-\tau)A+\tau B)^{-1}\right|^2 d\tau.
\end{aligned}\tag{3.7}$$

If $q\in(0,1)$ is such that $\bigvee_0^q(v)=\bigvee_q^1(v)=\frac{1}{2}\bigvee_0^1(v)$, then we get

$$\begin{aligned}
&\left|[v(1)-v(q)]B^{-1}+[v(q)-v(0)]A^{-1}-\int_0^1((1-t)A+tB)^{-1}dv(t)\right|^2 \\
&\leq\frac{1}{4}\left(\bigvee_0^1(v)\right)^2 \\
&\times\int_0^1\left|((1-\tau)A+\tau B)^{-1}(B-A)((1-\tau)A+\tau B)^{-1}\right|^2 d\tau.
\end{aligned}\tag{3.8}$$

From (3.7) we also derive

$$\begin{aligned}
&\left|\left[v(1)-v\left(\frac{1}{2}\right)\right]B^{-1}+\left[v\left(\frac{1}{2}\right)-v(0)\right]A^{-1}\right. \\
&\left.-\int_0^1\left((1-t)A+tB\right)^{-1}dv(t)\right|^2 \\
&\le \frac{1}{2}\left[\left(\bigvee_0^s(v)\right)^2+\left(\bigvee_s^1(v)\right)^2\right]\|B-A\|^2 \\
&\times\int_0^1\left\|\left((1-t)A+tB\right)^{-1}\right\|^4 dtd\tau.
\end{aligned} \tag{3.9}$$

Moreove, if if $A \geq m > 0$ and $B \geq m > 0$, then we obtain the simpler bound

$$\begin{aligned}
&\left|\left[v(1)-v\left(\frac{1}{2}\right)\right]B^{-1}+\left[v\left(\frac{1}{2}\right)-v(0)\right]A^{-1}\right. \\
&\left.-\int_0^1\left((1-t)A+tB\right)^{-1}dv(t)\right|^2 \\
&\le \frac{1}{2}\left[\left(\bigvee_0^s(v)\right)^2+\left(\bigvee_s^1(v)\right)^2\right]\frac{\|B-A\|^2}{m^4}.
\end{aligned} \tag{3.10}$$

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