SIMPLE GRÜSS' TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if $B : [a,b] \to \mathcal{B}(H)$ is a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ an integrable function, then

$$\left| \int_a^b \alpha(t) B(t) \, dt - \frac{1}{b-a} \int_a^b \alpha(t) \, dt \int_a^b B(t) \, dt \right|^2$$
$$\leq \frac{1}{6} (b-a)^4 \left[\frac{1}{b-a} \int_a^b |\alpha(t)|^2 \, dt - \left| \frac{1}{b-a} \int_a^b \alpha(s) \, ds \right|^2 \right] \int_a^b \left| B'(s) \right|^2 ds.$$

Applications for the inverse and exponential functions are also given.

1. Introduction

For two Lebesgue integrable functions f, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := \frac{1}{b-a} \int_a^b f(t) g(t) \, dt - \frac{1}{(b-a)^2} \int_a^b f(t) \, dt \int_a^b g(t) \, dt.$$

In 1934, G. Grüss [12] showed that

$$|D(f,g)| \leq \frac{1}{4} (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by the author in [3]:

Theorem 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If φ, ϕ, γ, $\Gamma \in \mathbb{K}$ and x, $y \in H$ are such that*

$$\operatorname{Re} \langle \phi e - x, x - \varphi e \rangle \geq 0 \quad \textit{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0 \tag{1.3}$$

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or, equivalently (see [5])

$$\left\| x-\frac{\varphi+\phi}{2}e\right\| \leq \frac{1}{2}\left|\phi-\varphi\right| \quad \text{and} \quad \left\| y-\frac{\gamma+\Gamma}{2}e\right\| \leq \frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.4}$$

then

$$\left|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle\right| \leq \frac{1}{4}\left|\phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{1.5}$$

The constant $\frac{1}{4}$ is best possible in (1.5).

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by the author in 2001, [4].

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space, $\Omega\subset\mathbb{R}^n$ be a Lebesgue measurable set and $\rho:\Omega\to[0,\infty)$ a Lebesgue measurable function with $\int_\Omega \rho(s)\,ds = 1$. We denote by $L_{2,\rho}(\Omega,H)$ the set of all Bochner measurable functions f on Ω such that $\|f\|_{2,\rho}^2 := \int_\Omega \rho(s)\|f(s)\|^2 ds<\infty$. If f, g belong to $L_{2,\rho}(\Omega,H)$ and there exist the vectors x, X, y, $Y\in H$ such that*

$$\begin{aligned} &\int_\Omega \rho(t)\operatorname{Re}\langle X-f(t), f(t)-x\rangle\, dt \geq 0, \\ &\int_\Omega \rho(t)\operatorname{Re}\langle Y-g(t), g(t)-y\rangle\, dt \geq 0, \end{aligned} \tag{1.6}$$

then we have the inequality

$$\begin{aligned} &\left|\int_\Omega \rho(t)\langle f(t),g(t)\rangle\, dt - \left\langle \int_\Omega \rho(t) f(t)\, dt, \int_\Omega \rho(t) g(t)\, dt\right\rangle\right| \\ &\leq \frac{1}{4}\|X-x\|\,\|Y-y\|. \end{aligned} \tag{1.7}$$

The constant $\frac{1}{4}$ is sharp in the sense mentioned above.

Remark 1. *A practical sufficient condition for (1.6) to hold is*

$$\operatorname{Re}\langle X-f(t), f(t)-x\rangle \geq 0, \ \operatorname{Re}\langle Y-g(t), g(t)-y\rangle \geq 0$$

or, equivalently

$$\left\| f(t)-\frac{X+x}{2}\right\| \leq \frac{1}{2}\|X-x\| \ \text{ and } \ \left\| g(t)-\frac{Y+y}{2}\right\| \leq \frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$.

For related results, see [2], [5]-[11] and [13]-[14].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

2. Main Results

In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that $p:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b p(s)\,ds=1$. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$\begin{aligned}0 &\le \left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2=|\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s)\\&-\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2,\end{aligned}$$

which gives that

$$|\alpha(t)|^2|A(s)|^2+|\alpha(s)|^2|A(t)|^2\ge\alpha(s)\overline{\alpha(t)}A^*(t)A(s)+\alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s,\,t\in[a,b]$.

Now, multiply this with $p(s)p(t)\ge 0$ to get

$$\begin{aligned}&p(t)|\alpha(t)|^2p(s)|A(s)|^2+p(s)|\alpha(s)|^2p(t)|A(t)|^2\\&\ge p(t)\overline{\alpha(t)}A^*(t)p(s)\alpha(s)A(s)+p(s)\overline{\alpha(s)}A^*(s)p(t)\alpha(t)A(t)\end{aligned}$$

for all $s,\,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned}&\int_a^b p(t)|\alpha(t)|^2dt\int_a^b|A(s)|^2ds+\int_a^b|\alpha(s)|^2ds\int_a^b p(t)|A(t)|^2dt\\&\ge\int_a^b p(t)\overline{\alpha(t)}A^*(t)\,dt\int_a^b\alpha(s)A(s)\,ds\\&+\int_a^b p(s)\overline{\alpha(s)}A^*(s)\,ds\int_a^b\alpha(t)A(t)\,dt\\&=2\left|\int_a^b p(s)\alpha(s)A(s)\,ds\right|^2,\end{aligned}$$

which proves that

$$\int_a^b p(t)|\alpha(t)|^2dt\int_a^b p(t)|A(t)|^2dt\ge\left|\int_a^b p(t)\alpha(t)A(t)\,dt\right|^2, \tag{2.1}$$

provided that $\alpha\in L_{2,p}([a,b],\mathbb{C})$ and

$$A\in L_{2,p}([a,b],\mathcal{B}(H)):=\left\{A:[a,b]\to B(H),\ \int_a^b p(t)\|A(t)\|^2dt<\infty\right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^n p_k|z_k|^2\sum_{k=1}^n p_k|A_k|^2\ge\left|\sum_{k=1}^n p_kz_kA_k\right|^2, \tag{2.2}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $p_k\ge 0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n p_k=1$.

We have the following equality:

Lemma 1. *Let $B:[a,b]\to\mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\to\mathbb{C}$ an integrable function, then for all $\beta\in\mathbb{C}$*

$$(2.3)\qquad (b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt$$
$$=\int_a^b (\alpha(t)-\beta)\left(\int_a^t (s-a)\,B'(s)\,ds+\int_t^b (s-b)\,B'(s)\,ds\right)dt.$$

Proof. We start to the Montgomery identity for a strongly differentiable function $B:[a,b]\to\mathcal{B}(H)$

$$(2.4)\qquad B(t)(b-a)-\int_a^b B(s)\,ds=\int_a^t (s-a)\,B'(s)\,ds+\int_t^b (s-b)\,B'(s)\,ds$$

that holds for all $t\in[a,b]$.

Indeed, integrating by parts, we have

$$\int_a^t (s-a)\,B'(s)\,ds=(t-a)\,B(t)-\int_a^t B(s)\,ds$$

and

$$\int_t^b (s-b)\,B'(s)\,ds=(b-t)\,B(t)-\int_t^b B(s)\,ds$$

which by addition gives (2.4).

If we multiply this identity by $\alpha(t)$ and integrate over t in $[a,b]$, then we get

$$(2.5)\qquad (b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(s)\,ds$$
$$=\int_a^b \alpha(t)\left(\int_a^t (s-a)\,B'(s)\,ds+\int_t^b (s-b)\,B'(s)\,ds\right)dt.$$

Now, if we replace $\alpha(t)$ by $\alpha(t)-\beta$, then we get

$$(b-a)\int_a^b [\alpha(t)-\beta]\,B(t)\,dt-\int_a^b [\alpha(t)-\beta]\,dt\int_a^b B(t)\,dt$$
$$=(b-a)\int_a^b \alpha(t)\,B(t)\,dt-(b-a)\,\beta\int_a^b B(t)\,dt$$
$$-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt+(b-a)\,\beta\int_a^b B(t)\,dt$$
$$=(b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(s)\,ds$$

and by (2.5) we derive (2.3). □

Theorem 3. *Let $B:[a,b]\to\mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\to\mathbb{C}$ an integrable function, then for all $\beta\in\mathbb{C}$*

$$(2.6)\qquad \left|\int_a^b \alpha(t)\,B(t)\,dt-\frac{1}{b-a}\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right|^2$$
$$\le\frac{1}{6}(b-a)^3\int_a^b |\alpha(t)-\beta|^2\,dt\int_a^b |B'(s)|^2\,ds.$$

In particular,

$$\left|\int_a^b \alpha(t) B(t)\,dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t)|^2\,dt \int_a^b |B'(s)|^2\,ds. \tag{2.7}$$

Proof. By taking the modulus and using the CBS integral inequality, we get

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \\ &= \left|\int_a^b (\alpha(t)-\beta)\left(\int_a^t (s-a) B'(s)\,ds + \int_t^b (s-b) B'(s)\,ds\right) dt\right|^2 \\ &\leq \int_a^b |\alpha(t)-\beta|^2\,dt \int_a^b \left|\int_a^t (s-a) B'(s)\,ds + \int_t^b (s-b) B'(s)\,ds\right|^2 dt \\ &=: K. \end{aligned} \tag{2.8}$$

If we consider the kernel for $t \in [a,b]$,

$$L(t,s) := \begin{cases} s-a, & a \leq s \leq t, \\ s-b, & t < s \leq b, \end{cases}$$

then we have

$$\int_a^t (s-a) B'(s)\,ds + \int_t^b (s-b) B'(s)\,ds = \int_a^b L(t,s) B'(s)\,ds,$$

and by the CBS integral inequality we get

$$\begin{aligned} &\left|\int_a^t (s-a) B'(s)\,ds + \int_t^b (s-b) B'(s)\,ds\right|^2 \\ &= \left|\int_a^b L(t,s) B'(s)\,ds\right|^2 \leq \int_a^b |L(t,s)|^2\,ds \int_a^b |B'(s)|^2\,ds. \end{aligned} \tag{2.9}$$

Now, observe that

$$\begin{aligned} \int_a^b |L(t,s)|^2\,ds &= \int_a^t |L(t,s)|^2\,ds + \int_t^b |L(t,s)|^2\,ds \\ &= \int_a^t (s-a)^2\,ds + \int_t^b (s-b)^2\,ds \\ &= \frac{1}{3}\left[(t-a)^3 + (b-t)^3\right] \\ &= (b-a)\left[\frac{1}{12}(b-a)^2 + \left(t - \frac{a+b}{2}\right)^2\right], \end{aligned}$$

for $t \in [a,b]$.

This implies that

$$\begin{aligned}
&\int_a^b \left| \int_a^t (s-a) B'(s)\, ds + \int_t^b (s-b) B'(s)\, ds \right|^2 dt \\
&\leq \int_a^b \left(\int_a^b |L(t,s)|^2 ds \right) dt \int_a^b |B'(s)|^2 ds \\
&= (b-a) \int_a^b \left[\frac{1}{12} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right] dt \int_a^b |B'(s)|^2 ds \\
&= (b-a) \left[\frac{1}{12} (b-a)^3 + \int_a^b \left(t - \frac{a+b}{2} \right)^2 dt \right] \int_a^b |B'(s)|^2 ds \\
&= (b-a) \left[\frac{1}{12} (b-a)^3 + \frac{1}{12} (b-a)^3 \right] \int_a^b |B'(s)|^2 ds \\
&= \frac{1}{6} (b-a)^4 \int_a^b |B'(s)|^2 ds.
\end{aligned}$$

Therefore

$$K \leq \frac{1}{6} (b-a)^4 \int_a^b |\alpha(t) - \beta|^2 dt \int_a^b |B'(s)|^2 ds$$

and by (2.8) we derive the desired result. □

Corollary 1. *Let $B : [a,b] \to \mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ an integrable function, then*

$$\begin{aligned}
&\left| \int_a^b \alpha(t) B(t)\, dt - \frac{1}{b-a} \int_a^b \alpha(t)\, dt \int_a^b B(t)\, dt \right|^2 \\
&\leq \frac{1}{6} (b-a)^4 \left[\frac{1}{b-a} \int_a^b |\alpha(t)|^2 dt - \left| \frac{1}{b-a} \int_a^b \alpha(s)\, ds \right|^2 \right] \\
&\times \int_a^b |B'(s)|^2 ds.
\end{aligned} \tag{2.10}$$

Proof. Observe that for

$$\beta = \frac{1}{b-a} \int_a^b \alpha(s)\, ds,$$

we have

$$
\begin{aligned}
&\int_a^b |\alpha(t)-\beta|^2\,dt \\
&= \int_a^b \left|\alpha(t)-\frac{1}{b-a}\int_a^b \alpha(s)\,ds\right|^2 dt \\
&= \int_a^b \left[|\alpha(t)|^2-\frac{2}{b-a}\operatorname{Re}\left(\alpha(t)\int_a^b \overline{\alpha(s)}ds\right)+\left|\frac{1}{b-a}\int_a^b \alpha(s)\,ds\right|^2\right] dt \\
&= \int_a^b |\alpha(t)|^2\,dt-\frac{2}{b-a}\operatorname{Re}\left(\int_a^b \alpha(t)\,dt\int_a^b \overline{\alpha(s)}ds\right)+\frac{1}{b-a}\left|\int_a^b \alpha(s)\,ds\right|^2 \\
&= \int_a^b |\alpha(t)|^2\,dt-2\frac{1}{b-a}\left|\int_a^b \alpha(s)\,ds\right|^2+\frac{1}{b-a}\left|\int_a^b \alpha(s)\,ds\right|^2 \\
&= \int_a^b |\alpha(t)|^2\,dt-\frac{1}{b-a}\left|\int_a^b \alpha(s)\,ds\right|^2 \\
&= (b-a)\left[\frac{1}{b-a}\int_a^b |\alpha(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b \alpha(s)\,ds\right|^2\right].
\end{aligned}
$$

By (2.6) we then obtain the desired result (2.10). □

Corollary 2. *Assume that $\alpha:[a,b]\to\mathbb{C}$ is a Lebesgue integrable function such that there exists $\gamma,\Gamma\in\mathbb{C}$ with*

$$\left|\alpha(t)-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}|\Gamma-\gamma| \tag{2.11}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-\alpha(t))\left(\overline{\alpha(t)}-\overline{\gamma}\right)\right]\ge 0 \tag{2.12}$$

for a.e. $t\in\Omega$.

If $B:[a,b]\to\mathcal{B}(H)$ is a strongly differentiable function on the interval (a,b), then

$$
\begin{aligned}
&\left|\int_a^b \alpha(t)B(t)\,dt-\frac{1}{b-a}\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right|^2 \\
&\le \frac{1}{24}(b-a)^4|\Gamma-\gamma|^2\int_a^b |B'(s)|^2\,ds.
\end{aligned} \tag{2.13}
$$

The proof follows by (2.6) on taking $\beta=\frac{\gamma+\Gamma}{2}$.

Corollary 3. *Assume that* $\alpha : [a,b] \to \mathbb{C}$ *is of bounded variations and* $B : [a,b] \to \mathcal{B}(H)$ *is a strongly differentiable function on the interval* (a,b), *then*

$$\left| \int_a^b \alpha(t) B(t)\, dt - \frac{1}{b-a} \int_a^b \alpha(t)\, dt \int_a^b B(t)\, dt \right|^2 \le \frac{1}{24} (b-a)^4 \left(\bigvee_a^b (\alpha) \right)^2 \int_a^b |B'(s)|^2\, ds, \tag{2.14}$$

where $\bigvee_a^b (\alpha)$ *is the total variation of* α *on* $[a,b]$.

Proof. Since α is of bounded variation, then

$$\begin{aligned} \left| \alpha(t) - \frac{\alpha(a) + \alpha(b)}{2} \right| &= \frac{1}{2} |\alpha(t) - \alpha(a) + \alpha(t) - \alpha(b)| \\ &\le \frac{1}{2} [|\alpha(t) - \alpha(a)| + |\alpha(b) - \alpha(t)|] \\ &\le \frac{1}{2} \bigvee_a^b (\alpha) \end{aligned} \tag{2.15}$$

for all $t \in [a,b]$.

By utilising (2.6) for $\beta = \frac{\alpha(a)+\alpha(b)}{2}$ we get by (2.15) the desired result (2.14) □

Corollary 4. *Assume that* $\alpha : [a,b] \to \mathbb{C}$ *satisfies the condition in the point* $\frac{a+b}{2}$,

$$\left| \alpha(t) - \alpha\left(\frac{a+b}{2} \right) \right| \le L_{\frac{a+b}{2}} \left| t - \frac{a+b}{2} \right|^r \text{ for all } t \in [a,b], \tag{2.16}$$

where $r > 0$ *and* $L_{\frac{a+b}{2}} > 0$ *are given. Then*

$$\left| \int_a^b \alpha(t) B(t)\, dt - \frac{1}{b-a} \int_a^b \alpha(t)\, dt \int_a^b B(t)\, dt \right|^2 \le \frac{1}{6(2r+1)4^r} L_{\frac{a+b}{2}}^2 (b-a)^{2r+4} \int_a^b |B'(s)|^2\, ds. \tag{2.17}$$

Proof. We observe that, by (2.16),

$$\begin{aligned} \int_a^b \left| \alpha(t) - \alpha\left(\frac{a+b}{2} \right) \right|^2 dt &\le L_{\frac{a+b}{2}}^2 \int_a^b \left| t - \frac{a+b}{2} \right|^{2r} dt \\ &= L_{\frac{a+b}{2}}^2 \frac{1}{(2r+1)4^r} (b-a)^{2r+1}. \end{aligned}$$

Utilising (2.6), we derive (2.17). □

Remark 2. *If the function* $\alpha : [a,b] \to \mathbb{C}$ *is* L*-Lipschitzian with constant* L, *then from (2.17) we derive*

$$\left| \int_a^b \alpha(t) B(t)\, dt - \frac{1}{b-a} \int_a^b \alpha(t)\, dt \int_a^b B(t)\, dt \right|^2 \le \frac{1}{72} L^2 (b-a)^6 \int_a^b |B'(s)|^2\, ds. \tag{2.18}$$

We can introduce the following concept:

Definition 1. *We say that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$ if*

$$|B((1-t)u+tv)|^2\leq(\geq)(1-t)|B(u)|^2+t|B(v)|^2 \tag{2.19}$$

in the operator order of $\mathcal{B}(H)$, for all $u,v\in[a,b]$ and $t\in[0,1]$.

Let $A, B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (2.2) we get

$$\begin{aligned}|(1-\alpha)A+\alpha B|^2&=\left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^2\\&\leq\left[\left((1-\alpha)^{1/2}\right)^2+\left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2+\left|\alpha^{1/2}B\right|^2\right]\\&=(1-\alpha+\alpha)\left[(1-\alpha)|A|^2+\alpha|B|^2\right]\\&=(1-\alpha)|A|^2+\alpha|B|^2.\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let $t_1, t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}|C((1-\alpha)t_1+\alpha t_2)|^2&=|(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B|^2\\&=|(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)|^2\\&\leq(1-\alpha)|(1-t_1)A+t_1B|^2+\alpha|(1-t_2)A+t_2B|^2\\&=(1-\alpha)|C(t_1)|^2+\alpha|C(t_2)|^2,\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

We also observe that the function $D:[0,1]\to\mathcal{B}(H)$, $D(t)=(1-t)A+tB$ is also *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\leq(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t):=[f((1-t)A+tB)]^{1/2}, t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. Therefore for A, $B>0$, the function

$$B(t):=((1-t)A+tB)^{r/2},\ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\leq r\leq 2$ or $-1\leq r\leq 0$.

Corollary 5. *With the assumptions of Theorem 3 and if $B':[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$, then*

$$\begin{aligned}&\left|\int_a^b\alpha(t)B(t)\,dt-\frac{1}{b-a}\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt\right|^2\\&\leq\frac{1}{6}(b-a)^3\int_a^b|\alpha(t)-\beta|^2\,dt\frac{|B'(a)|^2+|B'(b)|^2}{2}.\end{aligned} \tag{2.20}$$

In particular,

$$\left|\int_a^b \alpha(t) B(t) dt - \frac{1}{b-a}\int_a^b \alpha(t) dt \int_a^b B(t) dt\right|^2 \leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t)|^2 dt \frac{|B'(a)|^2 + |B'(b)|^2}{2}. \tag{2.21}$$

Proof. It follows by (2.6) on observing that

$$\begin{aligned}\int_a^b |B'(t)|^2 dt &= (b-a)\int_0^1 |B'((1-s)a + sb)|^2 ds \\ &\leq (b-a)\int_0^1 \left[(1-s)|B'(a)|^2 + s|B'(b)|^2\right] ds \\ &= (b-a)\frac{|B'(a)|^2 + |B'(b)|^2}{2}.\end{aligned}$$

□

Remark 3. *Consider the function* $B(t) = \frac{1}{2}t[(2-t)A + tB]$, $t \in [0,1]$ *with* A, $B \in \mathcal{B}(H)$, $B \neq A$. *Then* $B'(t) = (1-t)A + tB$ *is square modulus convex on* $[0,1]$. *Then by (2.20) we get for an integrable function* $\alpha : [0,1] \to \mathbb{C}$ *and* $\beta \in \mathbb{C}$ *that*

$$\left|\frac{1}{2}\int_0^1 \alpha(t) t[(2-t)A + tB] dt - \frac{1}{2}\int_0^1 \alpha(t) dt \int_0^1 t[(2-t)A + tB]\right|^2 \leq \frac{1}{6}\int_0^1 |\alpha(t) - \beta|^2 dt \frac{|B'(0)|^2 + |B'(1)|^2}{2},$$

namely

$$\left|\int_0^1 t\alpha(t)[(2-t)A + tB] dt - \int_0^1 \alpha(t) dt \left(\frac{2A+B}{3}\right)\right|^2 \leq \frac{1}{3}\int_0^1 |\alpha(t) - \beta|^2 dt \frac{|A|^2 + |B|^2}{2}. \tag{2.22}$$

In particular,

$$\left|\int_0^1 t\alpha(t)[(2-t)A + tB] dt - \int_0^1 \alpha(t) dt \left(\frac{2A+B}{3}\right)\right|^2 \leq \frac{1}{3}\int_0^1 |\alpha(t)|^2 dt \frac{|A|^2 + |B|^2}{2}. \tag{2.23}$$

We also have:

Corollary 6. *With the assumptions of Theorem 3 and if* $B' : [a,b] \to \mathcal{B}(H)$ *is square modulus concave on* $[a,b]$, *then*

$$\left|\int_a^b \alpha(t) B(t) dt - \frac{1}{b-a}\int_a^b \alpha(t) dt \int_a^b B(t) dt\right|^2 \leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t) - \beta|^2 dt \left|B'\left(\frac{a+b}{2}\right)\right|^2. \tag{2.24}$$

In particular,

$$\left|\int_a^b \alpha(t) B(t)\,dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t)|^2\,dt \left|B'\left(\frac{a+b}{2}\right)\right|^2. \tag{2.25}$$

Proof. Since $B' : [a,b] \to \mathcal{B}(H)$ is square modulus concave, then

$$\left|B'\left(\frac{u+v}{2}\right)\right|^2 \geq \frac{|B'(u)|^2 + |B'(v)|^2}{2}$$

for all $u, v \in [a,b]$.

By taking $u = (1-s)a + sb$ and $v = sa + (1-s)b$, $s \in [0,1]$ we get

$$\left|B'\left(\frac{a+b}{2}\right)\right|^2 \geq \frac{|B'((1-s)a+sb)|^2 + |B'(sa+(1-s)b)|^2}{2}$$

$s \in [0,1]$.

If we take the integral over we $s \in [0,1]$ get

$$\left|B'\left(\frac{a+b}{2}\right)\right|^2 \geq \frac{1}{2}\int_0^1 \left[|B'((1-s)a+sb)|^2 + |B'(sa+(1-s)b)|^2\right] ds = \int_0^1 |B'((1-s)a+sb)|^2\,ds.$$

The results follow now by Theorem 3. □

3. Some Examples

Further, let $A, B \in \mathcal{B}(H)$ such that $(1-t)A + tB$ is invertible for all $t \in [0,1]$. For this to happen, it is enough to assume that $A, B > 0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t) := ((1-t)A + tB)^{-1}$, $t \in [0,1]$ and observe that

$$B'(t) = -((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1},\ t \in [0,1].$$

By (2.6) we get

$$\begin{aligned}&\left|\int_a^b \alpha(t)((1-t)A+tB)^{-1}\,dt\right.\\ &\left.-\frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b ((1-t)A+tB)^{-1}\,dt\right|^2\\ &\leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t)-\beta|^2\,dt\\ &\times \int_a^b \left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2 ds\end{aligned} \tag{3.1}$$

for all $\beta \in \mathbb{C}$.

Since for any operator $V \in \mathcal{B}(H)$ we have $|V|^2 \leq \|V\|^2$, then

$$\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2$$
$$\leq \left\|((1-t)A+tB)^{-1}\right\|^4 \|B-A\|^2$$

for all $t \in [0,1]$, which implies that

$$\left|\int_a^b \alpha(t)((1-t)A+tB)^{-1}dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b ((1-t)A+tB)^{-1}dt\right|^2 \leq \frac{1}{6}(b-a)^3 \|B-A\|^2 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b \left\|((1-t)A+tB)^{-1}\right\|^4 ds. \tag{3.2}$$

Now, if $A \geq m > 0$ and $B \geq m > 0$, then $((1-t)A+tB)^{-1} \leq m^{-1}$ for $t \in [0,1]$, which implies $\left\|((1-t)A+tB)^{-1}\right\|^4 \leq m^{-4}$ and by (3.2) we get

$$\left|\int_a^b \alpha(t)((1-t)A+tB)^{-1}dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b ((1-t)A+tB)^{-1}dt\right|^2 \leq \frac{1}{6}(b-a)^3 \frac{\|B-A\|^2}{m^4}\int_a^b |\alpha(t)-\beta|^2 dt \tag{3.3}$$

for all $\beta \in \mathbb{C}$.

Consider the function $B(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $B'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. By making use of (2.6) we get

$$\left|\int_a^b \alpha(t)\exp(tT)\,dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b \exp(tT)\,dt\right|^2 \leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b |T\exp(sT)|^2 ds \tag{3.4}$$

for all $\beta \in \mathbb{C}$.

Since $\|\exp(tT)\| \leq \exp(|t|\,\|T\|)$, $t \in \mathbb{R}$, $T \in \mathcal{B}(H)$, then by (3.4)

$$\begin{aligned}
&\left|\int_a^b \alpha(t)\exp(tT)\,dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b \exp(tT)\,dt\right|^2 \\
&\leq \frac{1}{6}(b-a)^3 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b |T\exp(sT)|^2 ds \\
&\leq \frac{1}{6}(b-a)^3 \|T\|^2 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b \|\exp(tT)\|^2 ds \\
&\leq \frac{1}{6}(b-a)^3 \|T\|^2 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b \exp(2\|T\|\,|t|)\,dt
\end{aligned} \tag{3.5}$$

for all $\beta \in \mathbb{C}$.

Observe that, if $0 \leq a \leq b$, then

$$\int_a^b \exp(2\|T\||t|)\,dt = \int_a^b \exp(2\|T\|t)\,dt = \frac{\exp(2\|T\|b) - \exp(2\|T\|a)}{2\|T\|}$$

and by (3.5) we get

$$\left|\int_a^b \alpha(t)\exp(tT)\,dt - \frac{1}{b-a}\int_a^b \alpha(t)\,dt \int_a^b \exp(tT)\,dt\right|^2 \leq \frac{1}{12}(b-a)^3\|T\|\int_a^b |\alpha(t)-\beta|^2\,dt\,[\exp(2\|T\|b) - \exp(2\|T\|a)] \tag{3.6}$$

for all $\beta \in \mathbb{C}$.

If T is invertible, then [1]

$$\int_a^b \exp(tT)\,dt = T^{-1}[\exp(bT) - \exp(aT)] \tag{3.7}$$

and by (3.6) we get

$$\left|\int_a^b \alpha(t)\exp(tT)\,dt - \left(\frac{1}{b-a}\int_a^b \alpha(t)\,dt\right)T^{-1}[\exp(bT) - \exp(aT)]\right|^2 \leq \frac{1}{12}(b-a)^3\|T\|\int_a^b |\alpha(t)-\beta|^2\,dt\,[\exp(2\|T\|b) - \exp(2\|T\|a)] \tag{3.8}$$

for all $\beta \in \mathbb{C}$.

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