PERTURBED GRÜSS' TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. In this paper we show among others that, if $B : [a, b] \to \mathcal{B}(H)$ is a strongly differentiable function on the interval (a, b) and $\alpha : [a, b] \to \mathbb{C}$ an integrable function, then for all $\beta \in \mathbb{C}$ and $D \in \mathcal{B}(H)$,

$$\begin{aligned}&\Big|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \\ &- (b-a)\left(\int_a^b \left(t - \frac{a+b}{2}\right)\alpha(t)\,dt\right) D\Big|^2 \\ &\le \frac{1}{6}(b-a)^4 \int_a^b |\alpha(t) - \beta|^2\,dt \int_a^b \left|B'(s) - D\right|^2 ds.\end{aligned}$$

Applications for the exponential functions are also given.

1. Introduction

For two Lebesgue integrable functions f, $g : [a, b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f, g) := \frac{1}{b-a}\int_a^b f(t) g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt \int_a^b g(t)\,dt.$$

In 1934, G. Grüss [12] showed that

$$|D(f, g)| \le \frac{1}{4}(M - m)(N - n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on} \quad [a, b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by the author in [3]:

Theorem 1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e \in H$, $\|e\| = 1$. If φ, ϕ, γ, $\Gamma \in \mathbb{K}$ and x, $y \in H$ are such that*

$$\operatorname{Re}\langle \phi e - x, x - \varphi e \rangle \ge 0 \quad \textit{and} \quad \operatorname{Re}\langle \Gamma e - y, y - \gamma e \rangle \ge 0 \tag{1.3}$$

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or, equivalently (see [5]*)*

$$\left\| x-\frac{\varphi+\phi}{2}e\right\| \leq \frac{1}{2}\left|\phi-\varphi\right| \quad \text{and} \quad \left\| y-\frac{\gamma+\Gamma}{2}e\right\| \leq \frac{1}{2}\left|\Gamma-\gamma\right|, \tag{1.4}$$

then

$$\left|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle\right| \leq \frac{1}{4}\left|\phi-\varphi\right|\left|\Gamma-\gamma\right|. \tag{1.5}$$

The constant $\frac{1}{4}$ *is best possible in (1.5).*

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by the author in 2001, [4].

Theorem 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex Hilbert space,* $\Omega\subset\mathbb{R}^n$ *be a Lebesgue measurable set and* $\rho:\Omega\to[0,\infty)$ *a Lebesgue measurable function with* $\int_\Omega \rho(s)\,ds=1$*. We denote by* $L_{2,\rho}(\Omega,H)$ *the set of all Bochner measurable functions* f *on* Ω *such that* $\|f\|_{2,\rho}^2:=\int_\Omega \rho(s)\|f(s)\|^2 ds<\infty$*. If* f*,* g *belong to* $L_{2,\rho}(\Omega,H)$ *and there exist the vectors* x*,* X*,* y*,* $Y\in H$ *such that*

$$\begin{aligned} &\int_\Omega \rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt \geq 0, \\ &\int_\Omega \rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt \geq 0, \end{aligned} \tag{1.6}$$

then we have the inequality

$$\begin{aligned} &\left|\int_\Omega \rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega \rho(t)f(t)\,dt,\int_\Omega \rho(t)g(t)\,dt\right\rangle\right| \\ &\leq \frac{1}{4}\|X-x\|\|Y-y\|. \end{aligned} \tag{1.7}$$

The constant $\frac{1}{4}$ *is sharp in the sense mentioned above.*

Remark 1. *A practical sufficient condition for (1.6) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle\geq 0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle\geq 0$$

or, equivalently

$$\left\|f(t)-\frac{X+x}{2}\right\|\leq\frac{1}{2}\|X-x\| \ \text{ and } \ \left\|g(t)-\frac{Y+y}{2}\right\|\leq\frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$*.*

For related results, see [2], [5]-[11] and [13]-[14].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

2. Some Preparation

In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that $p : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b p(s)\,ds = 1$. We have for $\alpha : [a,b] \to \mathbb{C}$ and $A : [a,b] \to \mathcal{B}(H)$,

$$\begin{aligned}0 &\le \left|\overline{\alpha(t)}A(s) - \overline{\alpha(s)}A(t)\right|^2 = |\alpha(t)|\,|A(s)|^2 - \alpha(s)\overline{\alpha(t)}A^*(t)A(s)\\ &- \alpha(t)\overline{\alpha(s)}A^*(s)A(t) + |\alpha(s)|^2|A(t)|^2,\end{aligned}$$

which gives that

$$|\alpha(t)|^2|A(s)|^2 + |\alpha(s)|^2|A(t)|^2 \ge \alpha(s)\overline{\alpha(t)}A^*(t)A(s) + \alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s,\,t \in [a,b]$.

Now, multiply this with $p(s)p(t) \ge 0$ to get

$$\begin{aligned}&p(t)|\alpha(t)|^2 p(s)|A(s)|^2 + p(s)|\alpha(s)|^2 p(t)|A(t)|^2\\ &\ge p(t)\overline{\alpha(t)}A^*(t)p(s)\alpha(s)A(s) + p(s)\overline{\alpha(s)}A^*(s)p(t)\alpha(t)A(t)\end{aligned}$$

for all $s,\,t \in [a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned}&\int_a^b p(t)|\alpha(t)|^2 dt \int_a^b |A(s)|^2 ds + \int_a^b |\alpha(s)|^2 ds \int_a^b p(t)|A(t)|^2 dt\\ &\ge \int_a^b p(t)\overline{\alpha(t)}A^*(t)\,dt \int_a^b \alpha(s)A(s)\,ds\\ &+ \int_a^b p(s)\overline{\alpha(s)}A^*(s)\,ds \int_a^b \alpha(t)A(t)\,dt\\ &= 2\left|\int_a^b p(s)\alpha(s)A(s)\,ds\right|^2,\end{aligned}$$

which proves that

$$\int_a^b p(t)|\alpha(t)|^2 dt \int_a^b p(t)|A(t)|^2 dt \ge \left|\int_a^b p(t)\alpha(t)A(t)\,dt\right|^2, \tag{2.1}$$

provided that $\alpha \in L_{2,p}([a,b],\mathbb{C})$ and

$$A \in L_{2,p}([a,b],\mathcal{B}(H)) := \left\{A : [a,b] \to B(H),\ \int_a^b p(t)\|A(t)\|^2 dt < \infty\right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^n p_k|z_k|^2 \sum_{k=1}^n p_k|A_k|^2 \ge \left|\sum_{k=1}^n p_k z_k A_k\right|^2, \tag{2.2}$$

where $z_k \in \mathbb{C}$, $A_k \in \mathcal{B}(H)$, $p_k \ge 0$ for $k \in \{1,...,n\}$ and $\sum_{k=1}^n p_k = 1$.

We have the following perturbed identity:

Lemma 1. *Let $B:[a,b]\rightarrow\mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\rightarrow\mathbb{C}$ an integrable function, then for all $\beta\in\mathbb{C}$ and $D\in\mathcal{B}(H)$,*

$$
\begin{aligned}
&(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt \\
&-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)D \\
&=\int_a^b(\alpha(t)-\beta) \\
&\times\left(\int_a^t(s-a)\left[B'(s)-D\right]ds+\int_t^b(s-b)\left[B'(s)-D\right]ds\right)dt.
\end{aligned}\tag{2.3}
$$

Proof. We start to the Montgomery identity for a strongly differentiable function $B:[a,b]\rightarrow\mathcal{B}(H)$

$$
B(t)(b-a)-\int_a^b B(s)\,ds=\int_a^t(s-a)B'(s)\,ds+\int_t^b(s-b)B'(s)\,ds \tag{2.4}
$$

that holds for all $t\in[a,b]$.

Indeed, integrating by parts, we have

$$
\int_a^t(s-a)B'(s)\,ds=(t-a)B(t)-\int_a^t B(s)\,ds
$$

and

$$
\int_t^b(s-b)B'(s)\,ds=(b-t)B(t)-\int_t^b B(s)\,ds
$$

which by addition gives (2.4).

If we multiply this identity by $\alpha(t)$ and integrate over t in $[a,b]$, then we get

$$
\begin{aligned}
&(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(s)\,ds \\
&=\int_a^b\alpha(t)\left(\int_a^t(s-a)B'(s)\,ds+\int_t^b(s-b)B'(s)\,ds\right)dt.
\end{aligned}\tag{2.5}
$$

Now, if we replace $\alpha(t)$ by $\alpha(t)-\beta$, then we get

$$
\begin{aligned}
&(b-a)\int_a^b[\alpha(t)-\beta]B(t)\,dt-\int_a^b[\alpha(t)-\beta]\,dt\int_a^b B(t)\,dt \\
&=(b-a)\int_a^b\alpha(t)B(t)\,dt-(b-a)\beta\int_a^b B(t)\,dt \\
&-\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt+(b-a)\beta\int_a^b B(t)\,dt \\
&=(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(s)\,ds
\end{aligned}
$$

and by (2.5) we derive

$$\begin{aligned}(2.6)\qquad &(b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\\ &=\int_a^b(\alpha(t)-\beta)\left(\int_a^t(s-a)\,B'(s)\,ds+\int_t^b(s-b)\,B'(s)\,ds\right)dt.\end{aligned}$$

If we replace $B(t)$ with $B(t)-tD$ in (2.6), then we get

$$\begin{aligned}(2.7)\qquad &(b-a)\int_a^b \alpha(t)\,[B(t)-tD]\,dt-\int_a^b \alpha(t)\,dt\int_a^b (B(t)-tD)\,dt\\ &=\int_a^b(\alpha(t)-\beta)\\ &\times\left(\int_a^t(s-a)\,[B'(s)-D]\,ds+\int_t^b(s-b)\,[B'(s)-D]\,ds\right)dt.\end{aligned}$$

Observe that

$$\begin{aligned}(2.8)\qquad &(b-a)\int_a^b \alpha(t)\,[B(t)-tD]\,dt-\int_a^b \alpha(t)\,dt\int_a^b (B(t)-tD)\,dt\\ &=(b-a)\left[\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b t\alpha(t)\,dtD\right]\\ &-\int_a^b \alpha(t)\,dt\left[\int_a^b B(t)\,dt-\frac{1}{2}\left(b^2-a^2\right)D\right]\\ &=(b-a)\int_a^b \alpha(t)\,B(t)\,dt-(b-a)\int_a^b t\alpha(t)\,dtD\\ &-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt+\frac{1}{2}\left(b^2-a^2\right)\int_a^b \alpha(t)\,dtD\\ &=(b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\\ &-(b-a)\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dtD\end{aligned}$$

and by (2.7), we get (2.3). □

Remark 2. *If α is symmetric on $[a,b]$, namely $\alpha(a+b-t)=\alpha(t)$ for all $t\in[a,b]$, then $h(t):=\left(t-\frac{a+b}{2}\right)\alpha(t)$ is antisymmetric, which gives that*

$$\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt=0,$$

and by (2.3) we derive

$$(2.9) \quad (b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt$$
$$= \int_a^b (\alpha(t) - \beta)$$
$$\times \left(\int_a^t (s-a)\left[B'(s) - D\right] ds + \int_t^b (s-b)\left[B'(s) - D\right] ds \right) dt$$

for all $\beta \in \mathbb{C}$, $D \in \mathcal{B}(H)$.

3. Main Results

Our main result is as follows:

Theorem 3. *Let $B : [a,b] \to \mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ an integrable function, then for all $\beta \in \mathbb{C}$ and $D \in \mathcal{B}(H)$,*

$$(3.1) \quad \left| (b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right.$$
$$\left. - (b-a)\left(\int_a^b \left(t - \frac{a+b}{2}\right) \alpha(t)\,dt \right) D \right|^2$$
$$\le \frac{1}{6}(b-a)^4 \int_a^b |\alpha(t) - \beta|^2\,dt \int_a^b |B'(s) - D|^2\,ds.$$

In particular, for $D = 0$, we have the Grüss' type inequality

$$(3.2) \quad (b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt$$
$$\le \frac{1}{6}(b-a)^4 \int_a^b |\alpha(t) - \beta|^2\,dt \int_a^b |B'(s)|^2\,ds.$$

Proof. By taking the modulus and using the CBS integral inequality, we get

$$(3.3) \quad \left| (b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right.$$
$$\left. - (b-a)\left(\int_a^b \left(t - \frac{a+b}{2}\right) \alpha(t)\,dt \right) D \right|^2$$
$$=: U.$$

$$= \left| \int_a^b (\alpha(t) - \beta) \right.$$
$$\left. \times \left(\int_a^t (s-a) [B'(s) - D] ds + \int_t^b (s-b) [B'(s) - D] ds \right) dt \right|^2$$
$$\leq \int_a^b |\alpha(t) - \beta|^2 dt$$
$$\times \int_a^b \left| \int_a^t (s-a) [B'(s) - D] ds + \int_t^b (s-b) [B'(s) - D] ds \right|^2 dt$$

If we consider the kernel for $t \in [a, b]$,

$$L(t, s) := \begin{cases} s - a, & a \leq s \leq t, \\ s - b, & t < s \leq b, \end{cases}$$

then we have

$$\int_a^t (s-a) [B'(s) - D] ds + \int_t^b (s-b) [B'(s) - D] ds = \int_a^b L(t, s) [B'(s) - D] ds,$$

for $t \in [a, b]$.

By the CBS integral inequality we get

$$\left| \int_a^t (s-a) [B'(s) - D] ds + \int_t^b (s-b) [B'(s) - D] ds \right|^2 \tag{3.4}$$
$$= \left| \int_a^b L(t, s) [B'(s) - D] ds \right|^2 \leq \int_a^b |L(t, s)|^2 ds \int_a^b |B'(s) - D|^2 ds.$$

Now, observe that

$$\begin{aligned} \int_a^b |L(t, s)|^2 ds &= \int_a^t |L(t, s)|^2 ds + \int_t^b |L(t, s)|^2 ds \\ &= \int_a^t (s-a)^2 ds + \int_t^b (s-b)^2 ds \\ &= \frac{1}{3} \left[(t-a)^3 + (b-t)^3 \right] \\ &= (b-a) \left[\frac{1}{12} (b-a)^2 + \left(t - \frac{a+b}{2} \right)^2 \right], \end{aligned}$$

for $t \in [a, b]$.

This implies that

$$\begin{aligned}
&\int_a^b \left| \int_a^t (s-a)\left[B'(s)-D\right] ds + \int_t^b (s-b)\left[B'(s)-D\right] ds \right|^2 dt \\
&\le \int_a^b \left(\int_a^b |L(t,s)|^2 ds \right) dt \int_a^b |B'(s)-D|^2 ds \\
&= (b-a) \int_a^b \left[\frac{1}{12}(b-a)^2 + \left(t-\frac{a+b}{2}\right)^2 \right] dt \int_a^b |B'(s)-D|^2 ds \\
&= (b-a) \left[\frac{1}{12}(b-a)^3 + \int_a^b \left(t-\frac{a+b}{2}\right)^2 dt \right] \int_a^b |B'(s)-D|^2 ds \\
&= (b-a) \left[\frac{1}{12}(b-a)^3 + \frac{1}{12}(b-a)^3 \right] \int_a^b |B'(s)-D|^2 ds \\
&= \frac{1}{6}(b-a)^4 \int_a^b |B'(s)-D|^2 ds.
\end{aligned}$$

Therefore

$$U \le \frac{1}{6}(b-a)^4 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b |B'(s)-D|^2 ds$$

and by (3.3) we derive the desired result. □

Corollary 1. *With the assumptions of Theorem 3 and if α is symmetric on $[a,b]$, then*

$$\begin{aligned}
&\left| (b-a) \int_a^b \alpha(t) B(t)\, dt - \int_a^b \alpha(t)\, dt \int_a^b B(t)\, dt \right| \\
&\le \frac{1}{6}(b-a)^4 \int_a^b |\alpha(t)-\beta|^2 dt \int_a^b |B'(s)-D|^2 ds
\end{aligned} \tag{3.5}$$

for all $\beta \in \mathbb{C}$ and $D \in \mathcal{B}(H)$.

The proof follows by (3.1) observing that

$$\int_a^b \left(t-\frac{a+b}{2}\right) \alpha(t)\, dt = 0.$$

We have the following identity of interest:

Lemma 2. *For any A, X, $Y \in \mathcal{B}(H)$, we have*

$$\left| A - \frac{X+Y}{2} \right|^2 - \frac{1}{4}|X-Y|^2 = \operatorname{Re}\left[(A^*-X^*)(A-Y)\right]. \tag{3.6}$$

Proof. We have

$$\begin{aligned}
&\left|A-\frac{X+Y}{2}\right|^2-\frac{1}{4}\left|X-Y\right|^2\\
&=\left|A\right|^2-\frac{X^*+Y^*}{2}A-A^*\frac{X+Y}{2}+\frac{1}{4}\left(\left|X\right|^2+X^*Y+Y^*X+\left|Y\right|^2\right)\\
&-\frac{1}{4}\left(\left|X\right|^2-X^*Y-Y^*X+\left|Y\right|^2\right)\\
&=\left|A\right|^2-\frac{X^*+Y^*}{2}A-A^*\frac{X+Y}{2}+\frac{1}{2}\left(X^*Y+Y^*X\right)
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Re}\left[\left(A^*-X^*\right)\left(A-Y\right)\right]\\
&=\operatorname{Re}\left[\left|A\right|^2-X^*A-A^*Y+X^*Y\right]\\
&=\left|A\right|^2-\operatorname{Re}\left(X^*A\right)-\operatorname{Re}\left(A^*Y\right)+\operatorname{Re}\left(X^*Y\right)\\
&=\left|A\right|^2-\frac{1}{2}\left(X^*A+A^*X\right)-\frac{1}{2}\left(A^*Y+Y^*A\right)+\frac{1}{2}\left(X^*Y+Y^*X\right)\\
&=\left|A\right|^2-\frac{1}{2}\left(X^*+Y^*\right)A-\frac{1}{2}A^*\left(X+Y\right)+\frac{1}{2}\left(X^*Y+Y^*X\right),
\end{aligned}$$

which proves the desired identity (3.6). □

Corollary 2. *Let $A,\ X,\ Y\in\mathcal{B}(H)$. The following statements are equivalent*

$$\left|A-\frac{X+Y}{2}\right|^2\leq\frac{1}{4}\left|X-Y\right|^2$$

and

$$\operatorname{Re}\left[\left(X^*-A^*\right)\left(A-Y\right)\right]\geq 0.$$

We also have:

Corollary 3. *Let $B:[a,b]\to\mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\to\mathbb{C}$ an integrable function. If there exist the operators $U,\ V\in\mathcal{B}(H)$ such that either,*

$$\left|B'(s)-\frac{U+V}{2}\right|^2\leq\frac{1}{4}\left|U-V\right|^2$$

or

$$\operatorname{Re}\left[\left(X^*-\left(B'(s)\right)^*\right)\left(B'(s)-Y\right)\right]\geq 0$$

for a.e. $s\in[a,b]$, then

$$\begin{aligned}
&\left|(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt\right.\\
&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{U+V}{2}\right|^2\\
&\leq\frac{1}{24}(b-a)^5\left|U-V\right|^2\int_a^b\left|\alpha(t)-\beta\right|^2dt
\end{aligned}\tag{3.7}$$

for all $\beta\in\mathbb{C}$.

If α is symmetric on $[a,b]$, then we have the Grüss' type inequality

$$\left|(b-a)\int_a^b \alpha(t)B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right| \leq \frac{1}{24}(b-a)^5|U-V|^2\int_a^b|\alpha(t)-\beta|^2\,dt \tag{3.8}$$

for all $\beta\in\mathbb{C}$.

Proof. We have by (3.1) for $D=\frac{U+V}{2}$ that

$$\begin{aligned}&\left|(b-a)\int_a^b \alpha(t)B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right.\\&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{U+V}{2}\right|^2\\&\leq\frac{1}{6}(b-a)^4\int_a^b|\alpha(t)-\beta|^2\,dt\int_a^b\left|B'(s)-\frac{U+V}{2}\right|^2ds\\&\leq\frac{1}{24}(b-a)^5|U-V|^2\int_a^b|\alpha(t)-\beta|^2\,dt,\end{aligned}$$

which gives (3.7). □

We have the following particular inequalities of interest:

Proposition 1. *With the assumptions of Corollary 3 we have*

$$\begin{aligned}&\left|(b-a)\int_a^b \alpha(t)B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right.\\&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{U+V}{2}\right|^2\\&\leq\frac{1}{24}(b-a)^6|U-V|^2\left[\frac{1}{b-a}\int_a^b|\alpha(t)|^2-\left|\frac{1}{b-a}\int_a^b\alpha(t)\,dt\right|^2\right]\end{aligned} \tag{3.9}$$

and, if α is symmetric on $[a,b]$, then

$$\begin{aligned}&\left|(b-a)\int_a^b \alpha(t)B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right|^2\\&\leq\frac{1}{24}(b-a)^6|U-V|^2\left[\frac{1}{b-a}\int_a^b|\alpha(t)|^2-\left|\frac{1}{b-a}\int_a^b\alpha(t)\,dt\right|^2\right].\end{aligned} \tag{3.10}$$

If there exists γ, $\Gamma\in\mathbb{C}$ with

$$\left|\alpha(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma| \tag{3.11}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-\alpha(t))\left(\overline{\alpha(t)}-\overline{\gamma}\right)\right]\geq 0 \tag{3.12}$$

for a.e. $t \in [a,b]$, *then*

$$\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt - (b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{U+V}{2}\right|^2 \leq \frac{1}{96}(b-a)^6 |\Gamma-\gamma|^2 |U-V|^2 \tag{3.13}$$

and, if α *is symmetric on* $[a,b]$, *then*

$$\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \leq \frac{1}{96}(b-a)^6 |\Gamma-\gamma|^2 |U-V|^2. \tag{3.14}$$

If α *is of bounded variation, then*

$$\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt - (b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{U+V}{2}\right|^2 \leq \frac{1}{96}(b-a)^6 |U-V|^2 \left(\bigvee_a^b(\alpha)\right)^2 \tag{3.15}$$

and, if α *is symmetric on* $[a,b]$, *then*

$$\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \leq \frac{1}{96}(b-a)^6 |U-V|^2 \left(\bigvee_a^b(\alpha)\right)^2. \tag{3.16}$$

If $\alpha : [a,b] \to \mathbb{C}$ *satisfies the condition in the point* $\frac{a+b}{2}$,

$$\left|\alpha(t) - \alpha\left(\frac{a+b}{2}\right)\right| \leq L_{\frac{a+b}{2}} \left|t-\frac{a+b}{2}\right|^r \text{ for all } t \in [a,b], \tag{3.17}$$

where $r > 0$ *and* $L_{\frac{a+b}{2}} > 0$ *are given, then*

$$\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt - (b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{U+V}{2}\right|^2 \leq \frac{1}{24}\frac{1}{(2r+1)4^r} L^2_{\frac{a+b}{2}} (b-a)^{2r+6} |U-V|^2 \tag{3.18}$$

and, if α *is symmetric on* $[a,b]$ *, then*

$$\left|(b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right|^2 \le \frac{1}{24}\frac{1}{(2r+1)\,4^r}L^2_{\frac{a+b}{2}}(b-a)^{2r+6}|U-V|^2. \tag{3.19}$$

Proof. The inequality (3.9) follows by (3.7) by taking

$$\beta=\frac{1}{b-a}\int_a^b \alpha(t)\,dt$$

and observing that

$$\frac{1}{b-a}\int_a^b\left|\alpha(t)-\frac{1}{b-a}\int_a^b \alpha(t)\,dt\right|^2 dt=\frac{1}{b-a}\int_a^b|\alpha(t)|^2-\left|\frac{1}{b-a}\int_a^b \alpha(t)\,dt\right|^2.$$

The inequality (3.13) follows by (3.7) by taking

$$\beta=\frac{\gamma+\Gamma}{2}$$

and observing that

$$\int_a^b\left|\alpha(t)-\frac{\gamma+\Gamma}{2}\right|^2 dt\le\frac{1}{4}(b-a)|\Gamma-\gamma|^2$$

for a.e. $t\in[a,b]$.

If we take

$$\beta=\frac{\alpha(a)+\alpha(b)}{2}$$

and take into account that α is of bounded variation, then we have

$$\begin{aligned}\left|\alpha(t)-\frac{\alpha(a)+\alpha(b)}{2}\right| &= \frac{1}{2}|\alpha(t)-\alpha(a)+\alpha(t)-\alpha(b)|\\ &\le\frac{1}{2}[|\alpha(t)-\alpha(a)|+|\alpha(b)-\alpha(t)|]\\ &\le\frac{1}{2}\bigvee_a^b(\alpha)\end{aligned}$$

for all $t\in[a.b]$.

By utilising (3.7) we derive (3.15).

We observe that, by (3.17),

$$\begin{aligned}\int_a^b\left|\alpha(t)-\alpha\left(\frac{a+b}{2}\right)\right|^2 dt &\le L^2_{\frac{a+b}{2}}\int_a^b\left|t-\frac{a+b}{2}\right|^{2r}dt\\ &= L^2_{\frac{a+b}{2}}\frac{1}{(2r+1)\,4^r}(b-a)^{2r+1}.\end{aligned}$$

Utilising (3.1), we derive (3.18). □

Corollary 4. *Let $B : [a,b] \to \mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ an integrable function. Then for all $\beta \in \mathbb{C}$*

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right. \\ &\left. - \int_a^b \left(t - \frac{a+b}{2}\right)\alpha(t)\,dt\,[B(b)-B(a)]\right|^2 \\ &\leq \frac{1}{6}(b-a)^5 \int_a^b |\alpha(t)-\beta|^2\,dt \\ &\times \left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right). \end{aligned} \tag{3.20}$$

If α is symmetric on $[a,b]$, then

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \\ &\leq \frac{1}{6}(b-a)^5 \int_a^b |\alpha(t)-\beta|^2\,dt \\ &\times \left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right). \end{aligned} \tag{3.21}$$

Proof. If we take in (3.1)

$$D = \frac{1}{b-a}\int_a^b B'(t)\,dt = \frac{B(b)-B(a)}{b-a}$$

then we get

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t) B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right. \\ &\left. -(b-a)\left(\int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{B(b)-B(a)}{b-a}\right|^2 \\ &\leq \frac{1}{6}(b-a)^4 \int_a^b |\alpha(t)-\beta|^2\,dt \int_a^b \left|B'(s) - \frac{1}{b-a}\int_a^b B'(t)\,dt\right|^2 ds. \end{aligned} \tag{3.22}$$

Since

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \left|B'(s) - \frac{1}{b-a}\int_a^b B'(t)\,dt\right|^2 ds \\ &= \frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{1}{b-a}\int_a^b B'(t)\,dt\right|^2 \\ &= \frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2, \end{aligned}$$

hence by (3.22) we derive (3.20). □

Proposition 2. *With the assumptions of Corollary 4 we have*

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t)B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right. \\ &\left. -\int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\,[B(b)-B(a)]\right|^2 \\ &\leq \frac{1}{6}(b-a)^6\left[\frac{1}{b-a}\int_a^b |\alpha(t)|^2 - \left|\frac{1}{b-a}\int_a^b \alpha(t)\,dt\right|^2\right] \\ &\times\left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right). \end{aligned} \tag{3.23}$$

If α is symmetric on $[a,b]$, then

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t)B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \\ &\leq \frac{1}{6}(b-a)^6\left[\frac{1}{b-a}\int_a^b |\alpha(t)|^2 - \left|\frac{1}{b-a}\int_a^b \alpha(t)\,dt\right|^2\right] \\ &\times\left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right). \end{aligned} \tag{3.24}$$

If there exists γ, $\Gamma \in \mathbb{C}$ such that either (3.11) or (3.12) is valid, then

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t)B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right. \\ &\left. -\int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\,[B(b)-B(a)]\right|^2 \\ &\leq \frac{1}{24}(b-a)^6|\Gamma-\gamma|^2 \\ &\times\left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right). \end{aligned} \tag{3.25}$$

If α is symmetric on $[a,b]$, then

$$\begin{aligned} &\left|(b-a)\int_a^b \alpha(t)B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \\ &\leq \frac{1}{24}(b-a)^6|\Gamma-\gamma|^2 \\ &\times\left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right). \end{aligned} \tag{3.26}$$

If α is of bounded variation, then

$$\begin{aligned}(3.27)\quad & \left|(b-a)\int_a^b \alpha(t)\,B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right.\\ & \left. - \int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\,[B(b)-B(a)]\right|^2 \\ & \le \frac{1}{24}(b-a)^6 \left(\bigvee_a^b(\alpha)\right)^2 \\ & \times \left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right).\end{aligned}$$

If α is symmetric on $[a,b]$, then

$$\begin{aligned}(3.28)\quad & \left|(b-a)\int_a^b \alpha(t)\,B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \\ & \le \frac{1}{24}(b-a)^6 \left(\bigvee_a^b(\alpha)\right)^2 \\ & \times \left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right).\end{aligned}$$

If $\alpha:[a,b]\to\mathbb{C}$ satisfies the condition (3.17), then

$$\begin{aligned}(3.29)\quad & \left|(b-a)\int_a^b \alpha(t)\,B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt \right.\\ & \left. - \int_a^b \left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\,[B(b)-B(a)]\right|^2 \\ & \le \frac{1}{6}\frac{1}{(2r+1)\,4^r}(b-a)^{2r+6} L^2_{\frac{a+b}{2}} \\ & \times \left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right).\end{aligned}$$

If α is symmetric on $[a,b]$, then

$$\begin{aligned}(3.30)\quad & \left|(b-a)\int_a^b \alpha(t)\,B(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b B(t)\,dt\right|^2 \\ & \le \frac{1}{6}\frac{1}{(2r+1)\,4^r}(b-a)^{2r+6} L^2_{\frac{a+b}{2}} \\ & \times \left(\frac{1}{b-a}\int_a^b |B'(s)|^2\,ds - \left|\frac{B(b)-B(a)}{b-a}\right|^2\right).\end{aligned}$$

Corollary 5. *Let $B:[a,b]\to\mathcal{B}(H)$ be a strongly differentiable function on the interval (a,b) with the property that*

$$\left|B'(s)-B'\left(\frac{a+b}{2}\right)\right|^2\le U^2\left|s-\frac{a+b}{2}\right|,\ s\in(a,b) \tag{3.31}$$

for some selfadjoint operator U and $\alpha:[a,b]\to\mathbb{C}$ an integrable function, then for all $\beta\in\mathbb{C}$

$$\begin{aligned}&\left|(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt\right.\\&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)B'\left(\frac{a+b}{2}\right)\right|^2\\&\le\frac{1}{24}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)U^2.\end{aligned} \tag{3.32}$$

If α is symmetric on $[a,b]$, then

$$\begin{aligned}&\left|(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt\right|^2\\&\le\frac{1}{24}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)U^2.\end{aligned} \tag{3.33}$$

Remark 3. *If B is strongly twice differentiable, then*

$$B'(s)-B'\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^s B''(\tau)\,d\tau,\ s\in[a,b].$$

Taking the modulus and observing that

$$\begin{aligned}\left|B'(s)-B'\left(\frac{a+b}{2}\right)\right|^2&=\left|\int_{\frac{a+b}{2}}^s B''(\tau)\,d\tau\right|^2\\&\le\left|s-\frac{a+b}{2}\right|\left|\int_{\frac{a+b}{2}}^s|B''(\tau)|^2\,d\tau\right|\\&\le\left|s-\frac{a+b}{2}\right|\int_a^b|B''(\tau)|^2\,d\tau\end{aligned}$$

for all $s\in(a,b)$.

Therefore, if B is strongly twice differentiable with $B''\in L_2([a,b],\mathcal{B}(H))$, then for all $\beta\in\mathbb{C}$

$$\begin{aligned}&\left|(b-a)\int_a^b\alpha(t)B(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b B(t)\,dt\right.\\&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)B'\left(\frac{a+b}{2}\right)\right|^2\\&\le\frac{1}{24}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)\int_a^b|B''(\tau)|^2\,d\tau.\end{aligned} \tag{3.34}$$

If α is symmetric on $[a,b]$, then

$$\left|(b-a)\int_a^b \alpha(t)\,B(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b B(t)\,dt\right|^2 \le \frac{1}{24}(b-a)^6\left(\int_a^b |\alpha(t)-\beta|^2\,dt\right)\int_a^b |B''(\tau)|^2\,d\tau. \tag{3.35}$$

By taking various particular cases of β we can obtain similar results as above. We omit the details.

4. Some Examples

Consider the function $B(t)=\exp(tT)$, where $t\in\mathbb{R}$ and $T\in\mathcal{B}(H)$. Then $B'(t)=T\exp(tT)$ and $B''(t)=T^2\exp(tT)$ for $t\in\mathbb{R}$ and $T\in\mathcal{B}(H)$. If $\alpha:[a,b]\to\mathbb{C}$ an integrable function, then for all $\beta\in\mathbb{C}$ we get from (3.34)

$$\begin{aligned}&\Bigg|(b-a)\int_a^b \alpha(t)\exp(tT)\,dt-\int_a^b \alpha(t)\,dt\int_a^b \exp(tT)\,dt\\&-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)T\exp\left(\frac{a+b}{2}T\right)\Bigg|^2\\&\le \frac{1}{24}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)\int_a^b\left|T^2\exp(tT)\right|^2d\tau.\end{aligned} \tag{4.1}$$

If α is symmetric on $[a,b]$, then

$$\left|(b-a)\int_a^b \alpha(t)\exp(tT)\,dt-\int_a^b \alpha(t)\,dt\int_a^b \exp(tT)\,dt\right|^2 \le \frac{1}{24}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)\int_a^b\left|T^2\exp(tT)\right|^2d\tau, \tag{4.2}$$

for all $\beta\in\mathbb{C}$.

Since for any operator $V\in\mathcal{B}(H)$ we have $|V|^2\le\|V\|^2$ and $\|\exp(tT)\|\le\exp(|t|\,\|T\|)$, $t\in\mathbb{R}$, $T\in\mathcal{B}(H)$, then

$$\begin{aligned}\int_a^b\left|T^2\exp(tT)\right|^2d\tau&\le\int_a^b\left\|T^2\exp(tT)\right\|^2d\tau\le\|T\|^2\int_a^b\|\exp(tT)\|^2\,d\tau\\&\le\|T\|^2\int_a^b\exp(2|t|\,\|T\|)\,d\tau\end{aligned}$$

for $t\in\mathbb{R}$, $T\in\mathcal{B}(H)$.

Observe that, if $0\le a\le b$, then

$$\begin{aligned}\int_a^b\exp(2\|T\|\,|t|)\,dt&=\int_a^b\exp(2\|T\|\,t)\,dt\\&=\frac{\exp(2\|T\|\,b)-\exp(2\|T\|\,a)}{2\|T\|}.\end{aligned}$$

By utilising (4.1) we derive

$$\left|(b-a)\int_a^b \alpha(t)\exp(tT)\,dt-\int_a^b \alpha(t)\,dt\int_a^b \exp(tT)\,dt\right. \\ \left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)T\exp\left(\frac{a+b}{2}T\right)\right|^2 \\ \leq\frac{1}{48}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)\|T\|\left[\exp(2\|T\|b)-\exp(2\|T\|a)\right] \tag{4.3}$$

for $\alpha:[a,b]\to\mathbb{C}$ an integrable function and $\beta\in\mathbb{C}$.

If α is symmetric on $[a,b]$, with $0\leq a\leq b$, then

$$\left|(b-a)\int_a^b \alpha(t)\exp(tT)\,dt-\int_a^b \alpha(t)\,dt\int_a^b \exp(tT)\,dt\right| \\ \leq\frac{1}{48}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)\|T\|\left[\exp(2\|T\|b)-\exp(2\|T\|a)\right] \tag{4.4}$$

for $\beta\in\mathbb{C}$.

If T is invertible, then [1]

$$\int_a^b \exp(tT)\,dt=T^{-1}\left[\exp(bT)-\exp(aT)\right] \tag{4.5}$$

and by (4.4) we get

$$\left|(b-a)\int_a^b \alpha(t)\exp(tT)\,dt-\int_a^b \alpha(t)\,dtT^{-1}\left[\exp(bT)-\exp(aT)\right]\right| \\ \leq\frac{1}{48}(b-a)^6\left(\int_a^b|\alpha(t)-\beta|^2\,dt\right)\|T\|\left[\exp(2\|T\|b)-\exp(2\|T\|a)\right] \tag{4.6}$$

for $\beta\in\mathbb{C}$.

If we take $\alpha(t)=\left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$ in (4.6) we derive for $\beta=0$ that

$$\left|\int_a^b\left|t-\frac{a+b}{2}\right|\exp(tT)\,dt-\frac{1}{4}(b-a)T^{-1}\left[\exp(bT)-\exp(aT)\right]\right| \\ \leq\frac{1}{576}(b-a)^8\|T\|\left[\exp(2\|T\|b)-\exp(2\|T\|a)\right] \tag{4.7}$$

if T is invertible and $0\leq a\leq b$.

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