NEW APPLICATIONS OF YOUNG'S TYPE INEQUALITIES FOR FUNCTIONALS AND TRACE

LOREDANA TIRTIRAU

ABSTRACT. New applications of Young's type inequalities will be presented in this work for functionals, norm and for trace of positive operators on Hilbert spaces.

1. Introduction

The inequality of Young is

$$a^{\nu} b^{1-\nu} < \nu a + (1-\nu) b,$$

where a and b are distinct positive real numbers and $0 < \nu < 1$. This inequality is also an inequality between arithmetic and geometric mean.

There are many generalizations and refinements of Young's inequality, see for example [1], [2], [14], [13], [16], [4], [3], [5], [8], [17] and references therein.

We have to recall the following three inequalities presented in [17] in Theorem 2.1, Theorem 2.3 and Theorem 3.3 which will be used below.

Theorem 2.1*Let ν, τ and a, b are real positive numbers with $0 < \tau, \nu < 1$. Then we have*

$$\frac{a \nabla_{\nu} b - a \#_{\nu} b}{a \nabla_{\tau} b - a \#_{\tau} b} \leq \frac{\nu(1-\nu)}{\tau(1-\tau)},$$

for all $(b-a)(\tau-\nu) \geq 0$ and

$$\frac{a \nabla_{\nu} b - a \#_{\nu} b}{a \nabla_{\tau} b - a \#_{\tau} b} \geq \frac{\nu(1-\nu)}{\tau(1-\tau)},$$

for all $(b-a)(\tau-\nu) \leq 0$.

Theorem 2.3*Let ν, τ and a, b are real positive numbers with $0 < \tau, \nu < 1$. Then we have*

$$\frac{(a \nabla_{\nu} b)^2 - (a \#_{\nu} b)^2}{(a \nabla_{\tau} b(^2 - (a \#_{\tau} b)^2} \leq \frac{\nu(1-\nu)}{\tau(1-\tau)},$$

for all $(b-a)(\tau-\nu) \geq 0$ and

$$\frac{(a \nabla_{\nu})^2 b - (a \#_{\nu} b)^2}{(a \nabla_{\tau} b)^2 - (a \#_{\tau} b)^2} \geq \frac{\nu(1-\nu)}{\tau(1-\tau)},$$

for all $(b-a)(\tau-\nu) \leq 0$.

Date: July 07, 2021.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Young type inequalities, Isotonic linear functionals, Positive operators.

RGMIA Res. Rep. Coll. 24 (2021), Art. 95, 10 pp. Received 30/07/21

Theorem 3.3 *Let $N \in \mathbb{R}$, $0 \le v \le 1, a, b > 0$. Then we have*

$$(1 - v^{N+1} + v^{N+2})a + 1 - v^2()b \le v^{vN-(N+1)}a^v b^{1-v} + (\sqrt{a} - \sqrt{b})^2.$$

We recall some basic things about the functional calculus with continuous functions on spectrum. As in [9], we recall that for selfadjoint operators $A, B \in B(\mathcal{H})$ we write $A \le B$ (or $B \ge A$) if $< Ax, x > \le < Bx, x >$ for every vector $x \in \mathcal{H}$, or $B - A$ is a positive operator. Firstly, we will consider A as a selfadjoint linear operator on a complex Hilbert space $(\mathcal{H}; < ., . >)$. A fundamental role will be played by *Gelfand map* $\Phi : C(Sp(A)) \to C^*(A)$, which is a $*$- isometrically isomorphism between the set of all *continuous functions* defined on the *spectrum* of A and the C^*- algebra generated by A and the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$. Φ has the following properties: for any $f, f \in C(Sp(A))$ and for any $\alpha, \beta \in \mathbf{C}$ we have
(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g) \quad \text{and} \quad \Phi(f) = \Phi(f^*)$;
(iii) $||\Phi(f)|| = ||f|| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H \quad \text{and} \quad \Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A.)$
The *continuous functional calculus* for a selfadjoint operator A is based on the notation

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)),$$

and on the following remark:
If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \ge 0$ for any $t \in Sp(A)$ implies that $f(A) \ge 0$, i.e. $f(A)$ is a *positive operator* on $\mathcal{H}$. In addition, if and f and g are real valued functions on $Sp(A)$ then the following property holds:

$$f(t) \ge g(t) \quad \text{for any} \quad t \in Sp(A) \quad \text{implies that} \quad f(A) \ge g(A) \tag{1.1}$$

in the operator order of $B(\mathcal{H})$.

We also need basic properties for trace of operators. The main properties of the trace can be found in [5] and the references therein, but we mention just what we need. For any orthonormal basis $\{e_i\}_{i \in I}$ of a separable Hilbert space $\mathcal{H}$, the operator $A \in \mathcal{B}(\mathcal{H})$ is *trace class* if

$$||A||_1 = \sum_{i \in I} < |A|e_i, e_i > < \infty.$$

The definition of $||A||_1$ is independent of the choice of the orthogonal basis $\{e_i\}_{i \in I}$ and we denote the set of trace class operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{B}_1(\mathcal{H})$.

We need some of the well-known properties of trace:

(a) $||A||_1 = ||A^*||_1$, for any $A \in \mathcal{B}_1(\mathcal{H})$
(b) $(\mathcal{B}_1(\mathcal{H}), ||.||_1)$ is a Banach space.
(c) $\mathcal{B}(\mathcal{H})\mathcal{B}_1(\mathcal{H})\mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}_1(\mathcal{H})$ that is $\mathcal{B}_1(\mathcal{H})$ is a bilateral operator ideal in $\mathcal{B}(\mathcal{H})$.

We consider, following [5], the *trace* of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ to be

$$tr(A) = \sum_{i \in I} < Ae_i, e_i >,$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of $\mathcal{H}$. The previous series absolutely converges and it is basis independent. The definition is a an extension of the usual definition of the trace if $\mathcal{H}$ is finite dimensional.

$\mathcal{B}_1(\mathcal{H})$ is closed for to the $*$-operation, and $tr(A^*) = \overline{tr(A)}$. If $A \in \mathcal{B}_1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$ then $AT,\ TA \in \mathcal{B}_1(\mathcal{H})$ and $tr(AT) = tr(TA)$ and $|tr(AT)| \leq ||A||_1||T||$. The application $tr(.)$ is a bounded linear functional on $\mathcal{B}_1(\mathcal{H})$ with $||tr|| = 1$.

Many trace inequalities for matrices and operators can be found for example in [19], [21], [22], [5],[7], [16], [23], [18] and references therein.

2. New applications of Young's type inequalities for functionals

We start by recalling the definition of the isotonic linear functionals.

Let L be a linear class of real-valued functions $g : E \to \mathbb{R}$ having the following properties:

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;

(L2) $1 \in L$, *i.e.* if $f_0(t) = 1, t \in E$ then $f_0 \in L$.

An isotonic linear functional $A : L \to \mathbb{R}$ is a functional satisfying:

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.

(A2) If $f \in L$ and $f \geq 0$ then $A(f) \geq 0$.

The mapping A is said to be normalised if

(A3) $A(1) = 1$.

Using below Theorem 2.1, Theorem 2.3 and Theorem 3.1 from [17] we will get the following inequalities for isotonic linear functionals:

Proposition 1. *Let $A : L \to \mathbb{R}$ be a normalised isotonic linear functional.*

(a) If $f, g > 0$, $f^{1-\tau}g^{\tau} \in L$, $f^{1-v}g^{v} \in L$, $A(f), A(g) > 0$, $0 < \tau, v < 1$ and $(\tau - v)(g - f) \geq 0$ then the following inequality holds:

$$\frac{(1-v)A(f) + vA(g) - A(f^{1-v}g^{v})}{(1-\tau)A(f) + \tau A(g) - A(f^{1-\tau}g^{\tau})} \leq \frac{v(1-v)}{\tau(1-\tau)},$$

or

$$\frac{A(f)\nabla_v A(g) - A(f\#_v g)}{A(f)\nabla_\tau A(g) - A(f\#_\tau g)} \leq \frac{v(1-v)}{\tau(1-\tau)}.$$

(b) If $f, g > 0$, $f^2, g^2, f^{2(1-\tau)}g^{2\tau} \in L$, $f^{2(1-v)}g^{2v} \in L$, $A(f), A(g) > 0$, $0 < \tau, v < 1$ and $(\tau - v)(g - f) \geq 0$ then the following inequality holds:

$$\frac{(1-v)^2A(f^2) + v^2A(g^2) + 2v(1-v)A(fg) - A(f^{2(1-v)}g^{2v})}{(1-\tau)^2A(f^2) + \tau^2A(g^2) + 2\tau(1-\tau)A(fg) - A(f^{2(1-\tau)}g^{2\tau})} \leq \frac{v(1-v)}{\tau(1-\tau)}.$$

(c) If $N \in \mathbb{N}$, $0 \leq v \leq 1$ and $f, g > 0$ and $f^{\frac{1}{2}}g^{\frac{1}{2}} \in L$, $f^v g^{1-v} \in L$, $A(f), A(g) > 0$ then we have,

$$2A(f^{\frac{1}{2}}g^{\frac{1}{2}}) \leq v^{vN-(N+1)}A(f^v g^{1-v}) + v^{N+1}(1-v)A(f) + v^2A(g).$$

Proof. We will take into account the inequalities (9) from Theorem 2.1, (11) from Theorem 2.3 and (20) from Theorem 3.1, see [17] respectively, and we consider here $a = f$ and $b = g$, obtaining respectively:

$$\frac{a\nabla_v b - a\#_v b}{a\nabla_\tau b - a\#_\tau b} \leq \frac{v(1-v)}{\tau(1-\tau)},$$

$$\frac{(a\nabla_v b)^2 - (a\#_v b)^2}{(a\nabla_\tau b)^2 - (a\#_\tau b)^2} \leq \frac{v(1-v)}{\tau(1-\tau)}$$

and

$$(1-v^{N+1}+v^{N+2})a+(1-v^2)b\leq v^{vN-(N+1)}a^v b^{1-v}+(\sqrt{a}-\sqrt{b})^2.$$

Now we apply the functional A before and we get,

$$\tau(1-\tau)[(1-v)A(f)+vA(g)-A(f^{1-v}g^v)]\leq v(1-v)[(1-\tau)A(f)+\tau A(g)-A(f^{1-\tau}g^\tau)],$$

$$\tau(1-\tau)[A(((1-v)f+vg)^2)-A((f^{1-v}g^v)^2)]\leq v(1-v)[A(((1-\tau)f+\tau g)^2)-A((f^{1-\tau}g^\tau)^2)]$$

and

$$(1-v^{N+1}+v^{N+2})A(f)+(1-v^2)A(g)\leq v^{vN-(N+1)}A(f^v g^{1-v})+A(f)+A(g)-2A(f^{\frac{1}{2}}g^{\frac{1}{2}}).$$

or the required inequalities by calculus, taking into account the properties ofl A.

□

Theorem 1. *Let $A:L\to\mathbb{R}$ be a normalised isotonic linear functional. If $f,g>0$, $f^{\frac{1}{2}}g^{\frac{1}{2}}\in L$, $f^v g^{1-v}\in L$, $A(f),A(g)>0$ and $N\in\mathbf{N}, 0\leq v\leq 1$ then the following inequality holds:*

$$2\frac{A(f^{\frac{1}{2}}g^{\frac{1}{2}})}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}\leq v^2+v^{N+1}(1-v)+v^{vN-(N+1)}\frac{A(f^v g^{1-v})}{A^v(f)A^{1-v}(g)}$$

Proof. In inequality

$$(1-v^{N+1}+v^{N+2})a+(1-v^2)b\leq v^{vN-(N+1)}a^v b^{1-v}+(\sqrt{a}-\sqrt{b})^2$$

from Theorem 3.1, we take $a=\frac{f}{A(f)}, b=\frac{g}{A(g)}$ and we have,

$$(1-v^{N+1}+v^{N+2})\frac{f}{A(f)}+(1-v^2)\frac{g}{A(g)}\leq v^{vN-(N+1)}\frac{f^v}{A^v(f)}\frac{g^{1-v}}{A^{1-v}(g)}+\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2.$$

We apply the functional A in previous inequality and by using $(A1)$ and $(A2)$, after calculus, we find that,

$$(1-v^{N+1}+v^{N+2})+(1-v^2)\leq v^{vN-(N+1)}\frac{A(f^v g^{1-v})}{A^v(f)A^{1-v}(g)}+1+1-2\frac{A(f^{\frac{1}{2}}g^{\frac{1}{2}})}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)},$$

i.e. the desired inequality.

□

Theorem 2. *Let $A,B:L\to\mathbb{R}$ be two normalised isotonic linear functionals. If $f,g:E\to\mathbb{R}$ $f\geq 0, g>0$, $f^{\frac{1}{2}}g^{\frac{1}{2}}\in L$, $f^v g^{1-v}\in L$, $f^{1-v}g^v\in L$, and $N\in\mathbf{N}, 0\leq v\leq 1$ then the following inequality holds:*

$$2A(f^{\frac{1}{2}}g^{\frac{1}{2}})B(f^{\frac{1}{2}}g^{\frac{1}{2}})\leq$$
$$\leq v^{N+1}(1-v)A(f)B(g)+v^2A(g)B(f)+v^{vN-(N+1)}A(f^v g^{1-v})B(f^{1-v}g^v),$$

or

$$2A(f\#g)B(f\#g)\leq v^{N+1}(1-v)A(f)B(g)+v^2A(g)B(f)+v^{vN-(N+1)}A(g\#_v f)B(f\#_v g)$$

Proof. We use inequality from Theorem 3.1, see [17] for $a = \frac{f(z)}{g(z)}$ and $b = \frac{f(t)}{g(t)}$ and we have,

$$(1-v^{N+1}+v^{N+2})\frac{f(z)}{g(z)}+(1-v^2)\frac{f(t)}{g(t)} \leq v^{vN-(N+1)}\left(\frac{f(z)}{g(z)}\right)^v\left(\frac{f(t)}{g(t)}\right)^{1-v}+\left(\frac{f^{\frac{1}{2}}(z)}{g^{\frac{1}{2}}(z)}-\frac{f^{\frac{1}{2}}(t)}{g^{\frac{1}{2}}(t)}\right)^2.$$

Multiplying by $g(z)g(t) > 0$ we obtain after calculus,

$$2f^{\frac{1}{2}}(z)g^{\frac{1}{2}}(z)f^{\frac{1}{2}}(t)g^{\frac{1}{2}}(t) \leq v^{N+1}(1-v)f(z)g(t)+v^2 f(t)g(z)+v^{vN-(N+1)}f^v(z)g^v(t)f^{1-v}(t)g^{1-v}(z)$$

for any $z, t \in E$.

Fix $t \in E$ and then by previous inequality we have in the order of L that

$$2f^{\frac{1}{2}}g^{\frac{1}{2}}f^{\frac{1}{2}}(t)g^{\frac{1}{2}}(t) \leq v^{N+1}(1-v)fg(t) + v^2 f(t)g + v^{vN-(N+1)}f^v g^v(t)f^{1-v}(t)g^{1-v}$$

for any $t \in E$.

If we take now the functional A in previous inequality, then we have,

$$2A(f^{\frac{1}{2}}g^{\frac{1}{2}})f^{\frac{1}{2}}(t)g^{\frac{1}{2}}(t) \leq v^{N+1}(1-v)A(f)g(t)+v^2 f(t)A(g)+v^{vN-(N+1)}A(f^v g^{1-v})g^v(t)f^{1-v}(t)$$

for any $t \in E$.

This inequality can be written in the order of L as

$$2A(f^{\frac{1}{2}}g^{\frac{1}{2}})f^{\frac{1}{2}}g^{\frac{1}{2}} \leq v^{N+1}(1-v)A(f)g + v^2 fA(g) + v^{vN-(N+1)}A(f^v g^{1-v})g^v f^{1-v}.$$

Now we take into account the functional B in last inequality and we obtain the desired result.

□

3. New applications of Young's type inequalities for norms

Proposition 2. *Let A, B be two positive operators on a complex Hilbert space $\mathcal{H}$ and $N \in \mathbb{N}$, $0 \leq v \leq 1$. Then the following inequality takes place:*

$$2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y > \leq v^{N+1}(1-v)||A^{\frac{1}{2}}x||^2||y||^2 + v^2||x||^2||B^{\frac{1}{2}}y||^2 + \\ + v^{vN-(N+1)} < A^v x, x >< B^{1-v}y, y >,$$

for any $x, y \in \mathcal{H}$.

Proof. Using inequality from Theorem 3.1 and the functional calculus for continuous functions we have

$$2b^{\frac{1}{2}}A^{\frac{1}{2}} - v^{N+1}(1-v)A - v^2 bI \leq v^{vN-(N+1)}b^{1-v}A^v,$$

for any $b > 0$ and $N \in \mathbb{N}$, $0 \leq v \leq 1$. Here A is positive operator and I is the identity operator. Moreover, using the inner product, we also can have,

$$2b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x > -v^{N+1}(1-v) < Ax, x > -v^2 b < x, x > \leq v^{vN-(N+1)}b^{1-v} < A^v x, x >,$$

for any $x, y \in \mathcal{H}$ using the technique given in [5].

Now we use the same method for $b > 0$ and we obtain

$$2 < A^{\frac{1}{2}}x, x > B^{\frac{1}{2}} - v^{N+1}(1-v) < Ax, x > I - v^2 < x, x > B \leq \\ \leq v^{vN-(N+1)} < A^v x, x > B^{1-v},$$

for any $x \in \mathcal{H}$, where B is positive operator, and, from here also,

$$2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y > -v^{N+1}(1-v) < Ax, x >< y, y > -v^2 < x, x >< By, y > \leq \\ \leq v^{vN-(N+1)} < A^v x, x >< B^{1-v}y, y >,$$

for any $x, y \in \mathcal{H}$.

$\square$

Remark 1. *(a) Under previous conditions, if we take $x = y$ then we have,*

$$2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}x, x > \leq v^{N+1}(1-v)||A^{\frac{1}{2}}x||^2||x||^2 + v^2||x||^2||B^{\frac{1}{2}}x||^2 + \\ + v^{vN-(N+1)} < A^v x, x >< B^{1-v}x, x >,$$

for any $x \in \mathcal{H}$.

(b) Under previous conditions,

$$2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y > \leq v^{N+1}(1-v)||A^{\frac{1}{2}}x||^2 + v^2||B^{\frac{1}{2}}y||^2 + \\ + v^{vN-(N+1)} < A^v x, x >< B^{1-v}y, y >,$$

when $||x|| = ||y|| = 1$.

(c) Under conditions from point (a), we have,

$$2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}x, x > \leq v^{N+1}(1-v)||A^{\frac{1}{2}}x||^2 + v^2||B^{\frac{1}{2}}x||^2 + \\ + v^{vN-(N+1)} < A^v x, x >< B^{1-v}x, x >,$$

for any $x \in \mathcal{H}$ with $||x|| = 1$.

4. New applications of Young's type inequalities for trace

Proposition 3. *Let τ, v be real positive numbers with $0 < v < \tau < 1$, C be a positive operator with the property that $C < I$ and $P \in \mathcal{B}_1(\mathcal{H})$, $P \geq 0$ with $trP > 0$. Then we have,*

$$\frac{tr(PC)\nabla_v trP - tr(PC^{1-v})}{tr(PC)\nabla_\tau trP - tr(PC^{1-\tau})} \leq \frac{v(1-v)}{\tau(1-\tau)}.$$

Proof. Now C is an operator with $0 < C < I$ and using the functional calculus we get from

$$\tau(1-\tau)[(1-v)c + v - c^{1-v}] \leq v(1-v)[(1-\tau)c + \tau - c^{1-\tau}],$$

which is inequality (9) from Theorem 2.1, see [17] applied for $0 < c = \frac{a}{b} < 1$, we get the vector inequality,

$$\tau(1-\tau)[(1-v)C + vI - C^{1-v}] \leq v(1-v)[(1-\tau)C + \tau I - C^{1-\tau}].$$

Applying the inner product here, we obtain,

$$\tau(1-\tau)[(1-v) < Cx, x > + v < x, x > - < C^{1-v}x, x >] \leq \\ \leq v(1-v)[(1-\tau) < Cx, x > + \tau < x, x > - < C^{1-\tau}x, x >].$$

for any $x \in \mathcal{H}$. Now we take $x = P^{\frac{1}{2}}e, e \in \mathcal{H}$ in previous inequality and we get,

$$\tau(1-\tau)[(1-v) < P^{\frac{1}{2}}CP^{\frac{1}{2}}e, e > + v < Pe, e > - < P^{\frac{1}{2}}C^{1-v}P^{\frac{1}{2}}e, e >] \leq \\ \leq v(1-v)[(1-\tau)P^{\frac{1}{2}}CP^{\frac{1}{2}}e, e > + \tau < Pe, e > - < P^{\frac{1}{2}}C^{1-\tau}P^{\frac{1}{2}}e, e >].$$

for any $e \in \mathcal{H}$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. We take above $e = e_i, i \in I$ and summing over $i \in I$ and we find

$$\tau(1-\tau)[(1-v)\sum_{i \in I} < P^{\frac{1}{2}}CP^{\frac{1}{2}}e_i, e_i > + v\sum_{i \in I} < Pe_i, e_i > - \sum_{i \in I} < P^{\frac{1}{2}}C^{1-v}P^{\frac{1}{2}}e_i, e_i >] \leq$$

$$\leq v(1-v)[(1-\tau)\sum_{i\in I} < P^{\frac{1}{2}}CP^{\frac{1}{2}}e_i, e_i > +\tau\sum_{i\in I} < Pe_i, e_i > -\sum_{i\in I} < P^{\frac{1}{2}}C^{1-\tau}P^{\frac{1}{2}}e_i, e_i >].$$

and by the properties of trace

$$\tau(1-\tau)[(1-v)tr(PC)+v.trP-tr(PC^{1-v})] \leq v(1-v)[(1-\tau)tr(Pc)+\tau trP-tr(PC^{1-\tau})].$$

□

Theorem 3. *Let τ, v be real positive numbers with $0 < v < \tau < 1$ and A and B be two positive invertible operators with $0 < A \leq B$ and $B \in \mathcal{B}_1(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$. Then we have,*

$$\frac{trA\nabla_v trB - tr(B\#_{1-v}A)}{trA\nabla_\tau trB - tr(B\#_{1-\tau}A)} \leq \frac{v(1-v)}{\tau(1-\tau)}.$$

Proof. In inequality

$$\frac{[(1-v)tr(PC) + v.trP - tr(PC^{1-v})]}{[(1-\tau)tr(PC) + \tau.trP - tr(PC^{1-\tau})]} \leq \frac{v(1-v)}{\tau(1-\tau)}$$

we take $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ because from our hypothesis, $C \leq I$ and $P = B \in \mathcal{B}_1(\mathcal{H})$. Taking into account the property that $tr(AT) = tr(TA)$ if $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{H})$, see [9], we obtain $tr(PC) = tr(B(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})) = trA$ and we will use the notation, for weighted geometric mean, $A\#_\nu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}$ obtaining $tr(PC^{1-v}) = tr(B\#_{1-v}A)$ and the desired inequality.

□

Theorem 4. *Let τ, v be real positive numbers with $0 < v < \tau < 1$ and A and B be two positive invertible operators with $0 < A \leq B$ and $B \in \mathcal{B}_1(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$. Then the following inequality takes place,*

$$(\tau - v)[(1-\tau)(1-v)trA - \tau v.trB] \leq \tau(1-\tau)tr(B\#_{1-v}A) - v(1-v)tr(B\#_{1-\tau}A)$$

Proof. We use inequality (11) from Theorem 2.3, see [17], which can be written as

$$(\tau - v)[(1-\tau)(1-v)a^2 - \tau v b^2] \leq \tau(1-\tau)a^{2(1-v)}b^{2v} - v(1-v)a^{1(1-\tau)}b^{2\tau}$$

when $0 < a < b$ and here we consider $c = \left(\frac{a}{b}\right)^2 < 1$ so this inequality becomes

$$(\tau - v)[(1-\tau)(1-v)c - \tau v] \leq \tau(1-\tau)c^{1-v} - v(1-v)c^{1-\tau}.$$

By the same method as in previous theorem, by functional calculus for the operator C and then replacing $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $P = B$ we get the desired inequality.

□

Proposition 4. *Let $N \in \mathbf{N}, 0 \leq v \leq 1$. If $P \in \mathcal{B}_1(\mathcal{H})$, $P \geq 0$ with $trP > 0$ and C is a positive operator then the following inequality holds:*

$$2tr(PC^{\frac{1}{2}}) \leq v^{vN-(N+1)}tr(PC^v) + v^{N+1}(1-v)tr(PC) + v^2 trP.$$

Proof. We use the same method as in the case of Proposition 3. We divide the inequality

$$(1 - v^{N+1} + v^{N+2})a + (1-v^2)b \leq v^{vN-(N+1)}a^v b^{1-v} + (\sqrt{a} - \sqrt{b})^2$$

from Theorem 3.1, by b and we denote $c = \frac{a}{b}$ in order to obtain after calculus,

$$2c^{\frac{1}{2}} \leq v^{vN-(N+1)}c^v + v^{N+1}(1-v)c + v^2.$$

By functional calculus for the operator C and using then the inner product, we get,

$$2 < C^{\frac{1}{2}}x, x >\leq v^{vN-(N+1)} < C^{v}x, x > +v^{N+1}(1-v) < Cx, x > +v^{2} < x.x >,$$

for any $x \in \mathcal{H}$. Then taking $x = P^{\frac{1}{2}}e$, $e \in \mathcal{H}$ in previous inequality and then for an orthonormal basis of $\mathcal{H}$, $\{e_i\}_{i\in I}$ we put in last inequality $e = e_i, i \in I$ and summ over $i \in I$. By the properties of the trace we get the desired inequality.

□

Theorem 5. *Let $N \in \mathbf{N}, 0 \leq v \leq 1$ and A, B two positive invertible operators with $B \in \mathcal{B}_1(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$. Then we have,*

$$2tr(B\#A) \leq v^{vN-(N+1)}tr(B\#_{v}A) + v^{N+1}(1-v)trA + v^{2}trB.$$

Proof. We consider the inequality from Proposition 4, where instead of C we put $C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $P = B$. We see that $trPC = trA$ and in this way the inequality can be rewritten using the notations of the weighted geometric mean. □

Theorem 6. *Let A, B be two positive invertible operators and $N \in \mathbf{N}, 0 \leq v \leq 1$. If $P, Q \in \mathcal{B}_1(\mathcal{H})$, $P, Q \geq 0$ with $trP > 0, trQ > 0$. We have*

$$2.\frac{tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}})}{trP.trQ} \leq v^{N+1}(1-v)\frac{tr(PA)}{trP} + v^{2}\frac{tr(QB)}{trQ} + v^{vN-(N+1)}\frac{tr(PA^{v})tr(QB^{1-v})}{trP.trQ}$$

Proof. Using the functional calculus with continuous functions on spectrum for the operator A, we get

$$2b^{\frac{1}{2}}A^{\frac{1}{2}} \leq v^{N+1}(1-v)A + v^{2}bI + v^{vN-(N+1)}b^{1-v}A^{v}$$

which come from inequality of Theorem 3.1, see [17] for b fixed. By applying inner the product, we get

$$2b^{\frac{1}{2}} < A^{\frac{1}{2}}x, x >\leq v^{N+1}(1-v) < Ax, x > +v^{2}b < x, x > +v^{vN-(N+1)}b^{1-v} < A^{v}x, x >$$

, for any $x \in \mathcal{H}$ and by taking $x = P^{\frac{1}{2}}e, e \in \mathcal{H}$ we have

$$2b^{\frac{1}{2}} < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e, e >\leq v^{N+1}(1-v) < P^{\frac{1}{2}}AP^{\frac{1}{2}}e, e > +v^{2}b < Pe, e > +$$
$$+v^{vN-(N+1)}b^{1-v} < P^{\frac{1}{2}}A^{v}P^{\frac{1}{2}}e, e > .$$

Let $\{e_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{H}$ and we put in previous inequality $e = e_i$ and then summing over $i \in I$ we obtain the following

$$2b^{\frac{1}{2}}\sum_{i\in I} < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e_i, e_i >\leq v^{N+1}(1-v)\sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i > +v^{2}b\sum_{i\in I} < Pe_i, e_i > +$$
$$+v^{vN-(N+1)}b^{1-v}\sum_{i\in I} < P^{\frac{1}{2}}A^{v}P^{\frac{1}{2}}e_i, e_i >,$$

or by the trace property,

$$2b^{\frac{1}{2}}tr(PA^{\frac{1}{2}}) \leq v^{N+1}(1-v)tr(PA) + v^{2}btrP+$$
$$+v^{vN-(N+1)}b^{1-v}tr(PA^{v}.$$

Now we use the same method for $B > 0$ and $y = Q^{\frac{1}{2}}f, f \in \mathcal{H}$ and an orthononormal basis $\{f_i\}$ of $\mathcal{H}$ and we get the desired inequality.

□

Corollary 1.

$$2\frac{tr(P\#S).tr(Q\#V)}{trP.trQ} \leq v^{N+1}(1-v)\frac{trS}{trP} + v^2\frac{trV}{trQ} + v^{vN-(N+1)}\frac{tr(P\#_v S).tr(Q\#_{1-v}S)}{trP.trQ}$$

Proof. By the same procedure as in Theorem 5, we replace A by $P^{-\frac{1}{2}}SP^{-\frac{1}{2}}$ and B by $Q^{-\frac{1}{2}}VQ^{-\frac{1}{2}}$ in inequality from Theorem 6 and by calculus we obtain the desired inequality. □

Corollary 2. *Under conditions from Theorem 6, if in addition, we takeP* $= Q$*, we obtain*

$$2.\frac{tr(PA^{\frac{1}{2}})tr(PB^{\frac{1}{2}})}{tr^2P} \leq v^{N+1}(1-v)\frac{tr(PA)}{trP} + v^2\frac{tr(PB)}{trP} + v^{vN-(N+1)}\frac{tr(PA^v)tr(PB^{1-v})}{tr^2P}$$

References

[1] H. ALZER, C. M. FONSECA and A. KOVACEC, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra* **63**, 3, (2015), pp. 622-635.
[2] T. ANDO, Matrix Young inequalities, *Oper. Theory Adv. Appl.* **75**, (1995), pp. 33-38.
[3] S. S. DRAGOMIR, Some new reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.*, **18**, (2015), pp. 12.
[4] S. S. DRAGOMIR, Some asymmeetric reverses of Young's scalar and operator inequalities with applications, *RGMIA Res. Rep. Coll.*, **19**, (2016), Art. 44, pp. 16.
[5] S. S. DRAGOMIR, Trace inequalities for positive operators via recent refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **19**, (2015), Art. 157, pp. 16.
[6] S. S. DRAGOMIR, A note on new refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 131.
[7] S. S. DRAGOMIR, Trace inequalities of Jensen type seldadjoint operators in Hilbert spaces: A survey of recent results, *RGMIA Res. Rep. Coll.*, **20**, (2017), Art. 20, pp. 52.
[8] S. S. DRAGOMIR, A note on new refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **18**, (2015), Art. 131.
[9] S. S. DRAGOMIR, Cebysev's type inequalities for functions on selfadjoint operators in Hilbert spaces, *RGMIA Res. Rep. Coll.*, (2009).
[10] S. FURUICHI, N. MINCULETE, Alternative reverse inequalities for Young's inequality, *J. Math. Inequalities*, 2011;5:595-600.
[11] R. A. HORN, C. R. JOHNSON, *Matrix analysis*, 2nd ed. New York : Cambridge University Press; 2013.
[12] J. I. FUJII, E. KAMEI, Uhlmann's interpolational method for operators means, *Math. Japon,*. **34** (1989), no. 4, pp. 541-547.
[13] F. KITTANEH, Y. MANASRAH, Reverse Young and Heinz inequalities for matrices, *Linear and Multilinear Algebra*, **59**,9 (2011), pp. 1031–1037.
[14] F. KITTANEH, Y. MANASRAH, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.*, **361** (2010), 262–269.
[15] L. LIU, A trace class operator inequality, *J. Math. Anal. Appl.*, 328 (2007), pp. 1484-1486.
[16] S. MANJEGANI, Holder and Young inequalities for the trace of operators, *Positivity*, **11**, (2007), pp. 239-250.
[17] Y. REN, Some results of Young-type inequalities, *RACSAM*, (2020), 114:143.
[18] M. B. RUSKAI, Inequalities for trace on von Neumann algebras, *Commun. Math. Phis.*, **26** (1972), pp. 280-289.
[19] K. SHEBRAWI, H. ABADAWI, Trace inequalities for matrices, *Bull. Aust. Math. Soc.*, **87**, (2013), pp. 139-148.
[20] B. SIMON, *Trace Ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.
[21] Z. ULUKOK, R. TURKMEN, On some matrix trace inequalities, *J. Inequal. Appl. (2010)*, Art ID 201486, pp. 8.

[22] X. M. YANG, X. Q. YANG and K. L. TEO, A matrix trace inequality, *J. Math. Anal. Appl.*, **263** (2001), pp. 327-331.

[23] X. YANG, A matrix trace inequality, *J. Math. Anal. Appl.*, **250**, (2000), pp. 372-374.

[24] WH. YOUNG, On classes of summable functions and their Fourier series, *Proc. Royal Soc. London*, Ser. A., **87**, (1912), pp. 225-229.

Department of Mathematics, "Politehnica" University of Timisoara, P-ta. Victoriei, No.2, 300006-Timisoara