SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS AND THE SQUARE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$ and $\operatorname{Re}A:=\frac{1}{2}(A^*+A)$. In this paper we obtain among others the following Hermite-Hadamard type inequalities for convex functions $f:[0,\infty)\to\mathbb{R}$, A, $B\in\mathcal{B}(H)$ and $0\le m<M$ with $0\le m\le|(1-t)A+tB|^2\le M$ for all $t\in[0,1]$,

$$\begin{aligned}0&\le f(m)\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\&+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\&\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

Some examples for power functions, logarithm and exponential are also provided.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f(tx+(1-t)t)\le tf(x)+(1-t)f(y)$$

for all x, $y\in I$ and $t\in[0,1]$.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}\tag{1.1}$$

holds if f is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [17]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [13] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [3], Information Theory [2], Operator Theory [9], [10] and others.

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Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [4, p. 2], [5, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x) + f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [16, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)\,(1-\lambda) f(A) + \lambda f(B) \tag{1.4}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In the recent paper [12] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions:

Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we* *have the inequalities*

$$
\begin{aligned}
(1.5)\qquad & f\left(\frac{A+B}{2}\right)\\
&\leq (1-\lambda) f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\leq \int_0^1 f((1-s)A+sB)\,ds\\
&\leq \frac{1}{2}\left[f((1-\lambda)A+\lambda B)+(1-\lambda)f(B)+\lambda f(A)\right]\\
&\leq \frac{f(A)+f(B)}{2}.
\end{aligned}
$$

A similar result and with a different proof was obtained by B. Li in [15]. For $\lambda=\frac{1}{2}$ in (1.5) we recapture the result obtained in the earlier paper [11] by the author. For other similar inequalities for operator convex functions see [1] and [18]-[22].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

In the recent paper [14] we obtained among others the following Hermite-Hadamard type inequalities for operator convex (concave) functions $f:[0,\infty)\to\mathbb{R}$ and A, $B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$,

$$
\begin{aligned}
f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right) &\leq (\geq)\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\
&\leq (\geq)\frac{1}{3}\left[f\left(|A|^2\right)+f\left[\operatorname{Re}(B^*A)\right]+f\left(|B|^2\right)\right].
\end{aligned}
$$

Some examples for power functions and logarithm were also provided.

When we relax the condition of operator convexity and assume only that f is convex on $[0,\infty)$ then we can provide lower and upper bounds for the quantity

$$
\begin{aligned}
& f(m)\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\
&+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt
\end{aligned}
$$

provided that A, $B\in\mathcal{B}(H)$ with

$$0\leq m\leq|(1-t)A+tB|^2\leq M$$

for all $t\in[0,1]$.

Some examples for power functions, logarithm and exponential are also provided.

2. Main Results

The first main result is as follows:

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$. If $A,\ B\in\mathcal{B}(H)$ with $0\leq m\leq|(1-t)A+tB|^2\leq M$ for all $t\in[0,1]$, then*

$$\begin{aligned}(2.1)\quad 0&\leq\frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)1_H-\int_0^1\left||(1-t)A+tB|^2-\frac{m+M}{2}\right|dt\right\}\\&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq f(m)\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\&+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\&\leq\frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)1_H+\int_0^1\left||(1-t)A+tB|^2-\frac{m+M}{2}\right|dt\right\}\\&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

Proof. We use the double inequality, see [6]

$$\begin{aligned}(2.2)\quad &2\min\{\alpha,1-\alpha\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq(1-\alpha)f(m)+\alpha f(M)-f((1-\alpha)m+\alpha M)\\&\leq 2\max\{\alpha,1-\alpha\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],\end{aligned}$$

for $\alpha\in[0,1]$.

Let $x\in[m,M]$ and take $\alpha=\frac{x-m}{M-m}\in[0,1]$ to get

$$\begin{aligned}&2\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f\left(\frac{M-x}{M-m}m+\frac{x-m}{M-m}M\right)\\&\leq 2\max\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],\end{aligned}$$

namely

$$\begin{aligned}(2.3)\quad 0&\leq\frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)-\left|x-\frac{m+M}{2}\right|\right\}\\&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\end{aligned}$$

$$\le \frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f(x)$$
$$\le \frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)+\left|x-\frac{m+M}{2}\right|\right\}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

If we use the continuous functional calculus for selfadjoint operators and the inequality (2.3) we get

$$0\le \frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)1_H-\left|T-\frac{m+M}{2}\right|\right\} \tag{2.4}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\le f(m)\frac{M-T}{M-m}+f(M)\frac{T-m}{M-m}-f(T)$$
$$\le \frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)1_H+\left|T-\frac{m+M}{2}\right|\right\}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

Now, if we replace in (2.4) T with $|(1-t)A+tB|^2$, then we get

$$0\le \frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)1_H-\left||(1-t)A+tB|^2-\frac{m+M}{2}\right|\right\} \tag{2.5}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\le f(m)\frac{M-|(1-t)A+tB|^2}{M-m}+f(M)\frac{|(1-t)A+tB|^2-m}{M-m}$$
$$-f\left(|(1-t)A+tB|^2\right)$$
$$\le \frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)1_H+\left||(1-t)A+tB|^2-\frac{m+M}{2}\right|\right\}$$
$$\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],$$

for all $t\in[0,1]$.

Now, if we take the integral (2.5) over $t\in[0,1]$ and use the fact that

$$\int_0^1 |(1-t)A+tB|^2\,dt$$
$$=\int_0^1\left[(1-t)^2|A|^2+2t(1-t)\operatorname{Re}(B^*A)+t^2|B|^2\right]dt$$
$$=\left(\int_0^1(1-t)^2dt\right)|A|^2+2\left(\int_0^1 t(1-t)\,dt\right)\operatorname{Re}(B^*A)+\left(\int_0^1 t^2dt\right)|B|^2$$
$$=\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right],$$

then we get the desired result (2.1). □

Corollary 1. *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$ with $f(0)=0$. If A, $B\in\mathcal{B}(H)$ with $|A|^2$, $|B|^2\le M$, where $M>0$, then*

$$
\begin{aligned}
(2.6)\qquad 0 &\le \frac{2}{M}\min\left\{\frac{1}{2}M1_H-\int_0^1\left||(1-t)A+tB|^2-\frac{1}{2}M\right|dt\right\}\\
&\times\left[\frac{1}{2}f(M)-f\left(\frac{1}{2}M\right)\right]\\
&\le \frac{f(M)}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\
&\le \frac{2}{M}\min\left\{\frac{1}{2}M1_H+\int_0^1\left||(1-t)A+tB|^2-\frac{1}{2}M\right|dt\right\}\\
&\times\left[\frac{1}{2}f(M)-f\left(\frac{1}{2}M\right)\right]\le 2\left[\frac{1}{2}f(M)-f\left(\frac{1}{2}M\right)\right].
\end{aligned}
$$

Proof. The following Cauchy-Bunyakowsky-Schwarz inequality holds

$$
(2.7)\qquad \sum_{k=1}^n w_k|z_k|^2\sum_{k=1}^n w_k|A_k|^2\ge\left|\sum_{k=1}^n w_kz_kA_k\right|^2,
$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\ge 0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n w_k=1$.

Let A, $B\in\mathcal{B}(H)$ and $t\in[0,1]$. Then by (2.7), we get

$$
\begin{aligned}
|(1-t)A+tB|^2 &= \left|(1-t)^{1/2}(1-t)^{1/2}A+t^{1/2}t^{1/2}B\right|^2\\
&\le\left[\left((1-t)^{1/2}\right)^2+\left(t^{1/2}\right)^2\right]\left[\left|(1-t)^{1/2}A\right|^2+\left|t^{1/2}B\right|^2\right]\\
&=(1-t+t)\left[(1-t)|A|^2+t|B|^2\right]\\
&=(1-t)|A|^2+t|B|^2.
\end{aligned}
$$

So, if $|A|^2\le M$ and $|B|^2\le M$, then $|(1-t)A+tB|^2\le(1-t)M+tM=M$ and by writing the inequality (2.1) for $m=0$ we derive (2.6). □

Theorem 3. *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$. If A, $B\in\mathcal{B}(H)$ with $0\le m\le|(1-t)A+tB|^2\le M$ for all $t\in[0,1]$, then*

$$
\begin{aligned}
(2.8)\qquad 0 &\le f(m)\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\
&+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt
\end{aligned}
$$

$$\leq \frac{f'_-(M)-f'_+(m)}{M-m}$$
$$\times\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left(|(1-t)A+tB|^2-\frac{m+M}{2}1_H\right)^2 dt\right]$$
$$\leq \frac{f'_-(M)-f'_+(m)}{M-m}$$
$$\times\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right]$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right],$$

provided that $f'_+(m)$ is finite.

Proof. We use the inequality, see [8]

$$(1-\alpha)f(m)+\alpha f(M)-f((1-\alpha)m+\alpha M)$$
$$\leq \alpha(1-\alpha)(M-m)\left[f'_-(M)-f'_+(m)\right]$$

for $\alpha\in[0,1]$.

Let $x\in[m,M]$ and take $\alpha=\frac{x-m}{M-m}\in[0,1]$ to get

$$\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f(x)\leq\frac{f'_-(M)-f'_+(m)}{M-m}(x-m)(M-x). \tag{2.9}$$

If we use the continuous functional calculus for selfadjoint operators and the inequality (2.9) we get

$$\begin{aligned} &f(m)\frac{M1_H-T}{M-m}+f(M)\frac{T-m1_H}{M-m}-f(T)\\ &\leq\frac{f'_-(M)-f'_+(m)}{M-m}(T-m1_H)(M1_H-T). \end{aligned} \tag{2.10}$$

Now, if we replace in (2.10) T with $|(1-t)A+tB|^2$, then we get

$$\begin{aligned} &f(m)\frac{M1_H-|(1-t)A+tB|^2}{M-m}+f(M)\frac{|(1-t)A+tB|^2-m1_H}{M-m}\\ &-f\left(|(1-t)A+tB|^2\right)\\ &\leq\frac{f'_-(M)-f'_+(m)}{M-m}\\ &\times\left(|(1-t)A+tB|^2-m1_H\right)\left(M1_H-|(1-t)A+tB|^2\right). \end{aligned} \tag{2.11}$$

By taking the integral over $t \in [0,1]$ in (2.11) we get

$$
\begin{aligned}
(2.12)\quad 0 &\le f(m)\,\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m} \\
&+ f(M)\,\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m} - \int_0^1 f\left(|(1-t)A+tB|^2\right)dt \\
&\le \frac{f'_-(M)-f'_+(m)}{M-m} \\
&\times \int_0^1 \left(|(1-t)A+tB|^2 - m1_H\right)\left(M1_H - |(1-t)A+tB|^2\right)dt.
\end{aligned}
$$

Now, observe that, by using the elementary identity

$$
(X-m1_H)(M1_H-X) = \frac{1}{4}(M-m)^2 1_H - \left(X-\frac{m+M}{2}1_H\right)^2
$$

we get

$$
\begin{aligned}
&\int_0^1 \left(|(1-t)A+tB|^2 - m1_H\right)\left(M1_H - |(1-t)A+tB|^2\right)dt \\
&= \frac{1}{4}(M-m)^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2 - \frac{m+M}{2}1_H\right)^2 dt.
\end{aligned}
$$

By the operator convexity of the square function and Jensen's inequality we have

$$
\begin{aligned}
&\int_0^1 \left(|(1-t)A+tB|^2 - \frac{m+M}{2}1_H\right)^2 \\
&\ge \left(\int_0^1 |(1-t)A+tB|^2 - \frac{m+M}{2}1_H\right)^2 \\
&= \left(\int_0^1 \left[(1-t)^2|A|^2 + 2t(1-t)\operatorname{Re}(B^*A) + t^2|B|^2\right]dt - \frac{m+M}{2}1_H\right)^2 \\
&= \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3} - \frac{m+M}{2}1_H\right)^2,
\end{aligned}
$$

which gives that

$$
\begin{aligned}
&\frac{1}{4}(M-m)^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2 - \frac{m+M}{2}1_H\right)^2 \\
&\le \frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3} - \frac{m+M}{2}1_H\right)^2 \\
&\le \frac{1}{4}(M-m)^2 1_H.
\end{aligned}
$$

By utilising (2.12) we derive (2.8). □

Corollary 2. *Let $f:[0,\infty)\to\mathbb{R}$ be convex on $[0,\infty)$ with $f(0)=0$. If A, $B\in\mathcal{B}(H)$ with $|A|^2$, $|B|^2\leq M$, where $M>0$, then*

$$\begin{aligned}(2.13)\qquad 0 &\leq \frac{f(M)}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\ &\leq \frac{f'_-(M)-f'_+(0)}{M}\\ &\quad\times\left[\frac{1}{4}M^2 1_H-\int_0^1\left(|(1-t)A+tB|^2-\frac{1}{2}M1_H\right)^2 dt\right]\\ &\leq \frac{f'_-(M)-f'_+(0)}{M}\\ &\quad\times\left[\frac{1}{4}M^2 1_H-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{1}{2}M1_H\right)^2\right]\\ &\leq \frac{1}{4}M\left[f'_-(M)-f'_+(0)\right],\end{aligned}$$

provided that $f'_+(0)$ is finite.

We also have:

Theorem 4. *Let $f:[0,\infty)\to\mathbb{R}$ be twice differentiable and so that there exists the constants $0\leq d$, D such that $d\leq f''(t)\leq D$ for any $t\in(m,M)\subset[0,\infty)$. If A, $B\in\mathcal{B}(H)$ with $0\leq m\leq|(1-t)A+tB|^2\leq M$ for all $t\in[0,1]$, then*

$$\begin{aligned}(2.14)\quad 0 &\leq \frac{1}{2}d\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left(|(1-t)A+tB|^2-\frac{m+M}{2}1_H\right)^2 dt\right]\\ &\leq f(m)\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\ &\quad+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\ &\leq \frac{1}{2}D\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left(|(1-t)A+tB|^2-\frac{m+M}{2}1_H\right)^2 dt\right]\\ &\leq \frac{1}{2}D\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right]\\ &\leq \frac{1}{8}D(M-m)^2 1_H.\end{aligned}$$

Proof. If there exists the constants $0\leq d$, D such that

$$d\leq f''(t)\leq D \text{ for any } t\in(m,M),$$

then, see for instance [7],

$$\begin{aligned}(2.15)\qquad \frac{1}{2}\nu(1-\nu)d(M-m)^2 &\leq (1-\nu)f(m)+\nu f(M)-f((1-\nu)m+\nu M)\\ &\leq \frac{1}{2}\nu(1-\nu)D(M-m)^2\end{aligned}$$

for any $\nu \in [0,1]$.

Let $x \in [m,M]$ and take $\alpha = \frac{x-m}{M-m} \in [0,1]$ in (2.15) to get

$$\frac{1}{2}d(x-m)(M-x) \leq \frac{M-x}{M-m}f(m) + \frac{x-m}{M-m}f(M) - f(x) \leq \frac{1}{2}D(x-m)(M-x). \tag{2.16}$$

As above, we then get

$$\begin{aligned} 0 &\leq \frac{1}{2}d\int_0^1 \left(|(1-t)A+tB|^2 - m\right)\left(M - |(1-t)A+tB|^2\right)dt \\ &\leq f(m)\frac{M - \frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right]}{M-m} \\ &+ f(M)\frac{\frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right] - m}{M-m} - \int_0^1 f\left(|(1-t)A+tB|^2\right)dt \\ &\leq \frac{1}{2}D\int_0^1 \left(|(1-t)A+tB|^2 - m\right)\left(M - |(1-t)A+tB|^2\right)dt, \end{aligned}$$

which gives the desired inequality (2.14). □

Corollary 3. *Let $f:[0,\infty) \to \mathbb{R}$ be twice differentiable with $f(0)=0$ and so that there exists the constants $0 \leq d$, D such that $d \leq f''(t) \leq D$ for any $t \in (m,M) \subset [0,\infty)$. If $A, B \in \mathcal{B}(H)$ with $|A|^2, |B|^2 \leq M$, where $M>0$, then*

$$\begin{aligned} 0 &\leq \frac{1}{2}d\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2 - \frac{1}{2}M1_H\right)^2 dt\right] \\ &\leq \frac{f(M)}{M}\left[\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right] - \int_0^1 f\left(|(1-t)A+tB|^2\right)dt \\ &\leq \frac{1}{2}D\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2 - \frac{1}{2}M1_H\right)^2 dt\right] \\ &\leq \frac{1}{2}D\left[\frac{1}{4}M^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{1}{2}M1_H\right)^2\right] \\ &\leq \frac{1}{8}DM^2 1_H. \end{aligned} \tag{2.17}$$

3. Some Examples

The function $f(t) = t^p$, $p \geq 1$ is convex on $[0,\infty)$ with $f(0)=0$. From (2.6) we derive

$$\begin{aligned} 0 &\leq M^{p-1}\left[\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right] - \int_0^1 |(1-t)A+tB|^{2p}\,dt \\ &\leq M^p\left(\frac{2^{p-1}-1}{2^{p-1}}\right), \end{aligned} \tag{3.1}$$

provided that $A, B \in \mathcal{B}(H)$ with $|A|^2, |B|^2 \leq M$, $M>0$.

The function $f(t)=t^{r}$, $r\in(0,1)$ is concave on $[0,\infty)$ with $f(0)=0$. From (2.6) we derive

$$0\le\int_0^1 f\left|(1-t)A+tB\right|^{2r}dt-\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3M^{1-r}}\le M^{r}\left(2^{1-r}-1\right). \tag{3.2}$$

For $r=1/2$ we get

$$0\le\int_0^1 f\left|(1-t)A+tB\right|dt-\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3\sqrt{M}}\le\sqrt{M}\left(\sqrt{2}-1\right). \tag{3.3}$$

Consider the concave function $f(t)=\ln(t+1)$. By (2.6) we get

$$\begin{aligned}0&\le\int_0^1\ln\left(\left|(1-t)A+tB\right|^2+1\right)dt\\&\quad-\frac{\ln(M+1)}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]\\&\le2\left[\ln\left(\frac{1}{2}M+1\right)-\frac{1}{2}\ln(M+1)\right]\end{aligned} \tag{3.4}$$

provided that $A,B\in\mathcal{B}(H)$ with $|A|^2,|B|^2\le M$, with $M>0$.

Consider the convex function $f(t)=\exp t-1$, $t\ge0$. Then by (2.6) we get

$$\begin{aligned}0&\le\frac{\exp M-1}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]+1\\&\quad-\int_0^1\exp\left(\left|(1-t)A+tB\right|^2\right)dt\\&\le2\left[\frac{1}{2}(\exp M+1)-\exp\left(\frac{1}{2}M\right)\right],\end{aligned} \tag{3.5}$$

provided that $A,B\in\mathcal{B}(H)$ with $|A|^2,|B|^2\le M$, with $M>0$.

The function $f(t)=t^{p}$, $p\ge1$ is convex on $[0,\infty)$ with $f(0)=0$. From (2.13) we derive

$$\begin{aligned}0&\le M^{p-1}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]-\int_0^1\left|(1-t)A+tB\right|^{2p}dt\\&\le pM^{p-2}\left[\frac{1}{4}M^2 1_H-\int_0^1\left(\left|(1-t)A+tB\right|^2-\frac{1}{2}M1_H\right)^2dt\right]\\&\le pM^{p-2}\left[\frac{1}{4}M^2 1_H-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{1}{2}M1_H\right)^2\right]\\&\le\frac{1}{4}pM^{p}1_H,\end{aligned} \tag{3.6}$$

provided that $A,B\in\mathcal{B}(H)$ with $|A|^2,|B|^2\le M$, $M>0$.

Consider the concave function $f(t)=\ln(t+1)$. By (2.13) we get

$$\begin{aligned}
0 &\leq \int_0^1 \ln\left(|(1-t)A+tB|^2+1\right)dt \\
&\quad - \frac{\ln(M+1)}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right] \\
&\leq \frac{1}{M+1}\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2-\frac{1}{2}M1_H\right)^2 dt\right] \\
&\leq \frac{1}{M+1}\left[\frac{1}{4}M^2 1_H - \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{1}{2}M1_H\right)^2\right] \\
&\leq \frac{M^2}{4(M+1)}1_H
\end{aligned} \tag{3.7}$$

provided that $A, B\in\mathcal{B}(H)$ with $|A|^2, |B|^2\leq M$, with $M>0$.

Consider the convex function $f(t)=\exp t-1$, $t\geq 0$. Then by (2.13) we get

$$\begin{aligned}
0 &\leq \frac{\exp M-1}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]+1 \\
&\quad - \int_0^1 \exp\left(|(1-t)A+tB|^2\right)dt \\
&\leq \frac{\exp M-1}{M}\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2-\frac{1}{2}M1_H\right)^2 dt\right] \\
&\leq \frac{\exp M-1}{M} \\
&\quad \times \left[\frac{1}{4}M^2 1_H - \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{1}{2}M1_H\right)^2\right] \\
&\leq \frac{1}{4}M(\exp M-1),
\end{aligned} \tag{3.8}$$

provided that $A, B\in\mathcal{B}(H)$ with $|A|^2, |B|^2\leq M$, with $M>0$.

Consider the concave function $f(t)=\ln(t+1)$. By (2.17) we get

$$\begin{aligned}
0 &\leq \frac{1}{2(M+1)^2}\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2-\frac{1}{2}M1_H\right)^2 dt\right] \\
&\leq \int_0^1 \ln\left(|(1-t)A+tB|^2+1\right)dt \\
&\quad - \frac{\ln(M+1)}{M}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]
\end{aligned} \tag{3.9}$$

$$\leq \frac{1}{2}\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A + tB|^2 - \frac{1}{2}M 1_H\right)^2 dt\right]$$
$$\leq \frac{1}{2}\left[\frac{1}{4}M^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^* A) + |B|^2}{3} - \frac{1}{2}M 1_H\right)^2\right]$$
$$\leq \frac{1}{8}M^2 1_H$$

provided that $A,\, B \in \mathcal{B}(H)$ with $|A|^2,\, |B|^2 \leq M$, with $M > 0$.

Consider the convex function $f(t) = \exp t - 1$, $t \geq 0$. By (2.17) we get

$$0 \leq \frac{1}{2}\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A + tB|^2 - \frac{1}{2}M 1_H\right)^2 dt\right] \tag{3.10}$$
$$\leq \frac{\exp M - 1}{M}\left[\frac{|A|^2 + \operatorname{Re}(B^* A) + |B|^2}{3}\right] + 1$$
$$- \int_0^1 \exp\left(|(1-t)A + tB|^2\right) dt$$
$$\leq \frac{1}{2}\exp(M)\left[\frac{1}{4}M^2 1_H - \int_0^1 \left(|(1-t)A + tB|^2 - \frac{1}{2}M 1_H\right)^2 dt\right]$$
$$\leq \frac{1}{2}\exp(M)\left[\frac{1}{4}M^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^* A) + |B|^2}{3} - \frac{1}{2}M 1_H\right)^2\right]$$
$$\leq \frac{1}{8}\exp(M)\, M^2 1_H,$$

provided that $A,\, B \in \mathcal{B}(H)$ with $|A|^2,\, |B|^2 \leq M$, with $M > 0$.

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