SOME REVERSES OF HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR CONVEX FUNCTIONS IN HILBERT SPACES

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ABSTRACT. Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. In this paper we show among others that, if A, B are selfadjoint with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{R}$ is operator convex, then

$$\begin{aligned} 0 &\le \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\, dt \\ &\le \frac{f'_-(M) - f'_+(m)}{M - m} \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{A+B}{2} - \frac{m+M}{2} 1_H \right)^2 \right] \\ &\le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m) \right] 1_H \end{aligned}$$

and

$$\begin{aligned} 0 &\le \int_0^1 f((1-t)A + tB)\, dt - f\left(\frac{A+B}{2} \right) \\ &\le \frac{\left[f'_-(M) - f'_+(m) \right]}{M - m} \left(\frac{A+B}{2} - m 1_H \right) \left(M 1_H - \frac{A+B}{2} \right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m) \right] 1_H. \end{aligned}$$

Some examples for power functions and logarithm are also provided.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ where $p_j \ge 0$, $j \in \{1, ..., n\}$ with $P_n := \sum_{j=1}^n p_j > 0$ and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i \right) \le \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

In order to extend this inequality for operator convex functions of selfadjoint bounded linear operators on complex Hilbert spaces we need the following preliminary facts.

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A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left((1-\lambda)A+\lambda B\right)\leq(\geq)\,(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\mathrm{Sp}(A),\mathrm{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [11] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

We also have the following Jensen type inequality for operator convex functions $f:I\to\mathbb{R}$.

Let A_j be selfadjoint operators with $\mathrm{Sp}(A_j)\subseteq I$, $j\in\{1,...,n\}$. If $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n>0$ and f is an operator convex function on I then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_iA_j\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_if(A_j), \tag{1.2}$$

in the operator order.

This is a well known result and can be proved easily by mathematical induction over $n\geq 2$. The details are left to the reader.

For recent results related to the Jensen inequality for selfadjoint operators in Hilbert spaces see the papers [1]-[5], [12]-[19], [20] and the monograph [6].

In the recent paper [10] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions:

Theorem 1. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned} f\left(\frac{A+B}{2}\right) &\leq(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \\ &\leq\int_0^1 f\left((1-s)A+sB\right)ds \\ &\leq\frac{1}{2}\left[f\left((1-\lambda)A+\lambda B\right)+(1-\lambda)f(B)+\lambda f(A)\right] \\ &\leq\frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.3}$$

A similar result and with a different proof was obtained by B. Li in [13]. For $\lambda = \frac{1}{2}$ in (1.3) we recapture the result obtained in the earlier paper [8] by the author. For other similar inequalities for operator convex functions see [1] and [21]-[25].

In this paper we show among others that, if A, B are selfadjoint with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{R}$ is operator convex, then

$$\begin{aligned} 0 &\leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t)A + tB)\,dt \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m}\left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{A+B}{2} - \frac{m+M}{2}1_H\right)^2\right] \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]1_H \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t)A + tB)\,dt - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m}\left(\frac{A+B}{2} - m1_H\right)\left(M1_H - \frac{A+B}{2}\right) \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]1_H. \end{aligned}$$

Some examples for power functions and logarithm are also provided.

2. Some Discrete Inequality

We consider the functional

$$J_n(\mathbf{p}; \mathbf{A}, f, I) := \sum_{j=1}^{n} p_j f(A_j) - P_n f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{2.1}$$

where $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$, $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq I$ for $j \in \{1, ..., n\}$ and $f : I \to \mathbb{R}$ is a operator convex function defined on the interval I.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$. For $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1, ..., n\}$.

Lemma 1. *Assume that $f : I \to \mathbb{R}$ is an operator convex function and $\mathbf{A} = (A_1, ..., A_n)$ an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq I$, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, I) \geq J_n(\mathbf{p}; \mathbf{A}, f, I) + J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{2.2}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p}; \mathbf{A}, f, I) \geq J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{2.3}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a monotonic functional in the operator order.

Proof. We have

$$
\begin{aligned}
(2.4)\qquad & J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I) \\
&= \sum_{j=1}^{n}(p_j+q_j)f(A_j) - (P_n+Q_n)f\left(\frac{1}{P_n+Q_n}\sum_{j=1}^{n}(p_j+q_j)A_j\right) \\
&= \sum_{j=1}^{n}(p_j+q_j)f(A_j) \\
&- (P_n+Q_n)f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)}{P_n+Q_n}\right).
\end{aligned}
$$

Now, consider the operators

$$
A := \frac{1}{P_n}\sum_{j=1}^{n}p_jA_j \text{ and } B := \frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j.
$$

Then $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Applying the inequality (OC) for A and B given above and $\lambda = \frac{Q_n}{P_n+Q_n}$ we have

$$
\begin{aligned}
(2.5)\qquad & f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)}{P_n+Q_n}\right) \\
&\le \frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) + \frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)
\end{aligned}
$$

in the operator order.

Making use of (2.4) and (2.5) we have

$$
\begin{aligned}
(2.6)\qquad & J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I) \\
&\ge \sum_{j=1}^{n}(p_j+q_j)f(A_j) - (P_n+Q_n) \\
&\times\left[\frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) + \frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)\right] \\
&= \sum_{j=1}^{n}p_jf(A_j) - P_nf\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) \\
&+ \sum_{j=1}^{n}q_jf(A_j) - Q_nf\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right) \\
&= J_n(\mathbf{p};\mathbf{A},f,I) + J_n(\mathbf{q};\mathbf{A},f,I)
\end{aligned}
$$

in the operator order, and the inequality (2.2) is proved.

Now, let $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$. Then by the super-additivity property (2.2) we have

$$J_n(\mathbf{p};\mathbf{A},f,I)=J_n((\mathbf{p}-\mathbf{q})+\mathbf{q};\mathbf{A},f,I)\geq J_n((\mathbf{p}-\mathbf{q});\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)\geq J_n(\mathbf{q};\mathbf{A},f,I) \tag{2.7}$$

in the operator order, and the monotonicity property (2.3) is proved. □

Corollary 1. *Assume that the function $f:I\to\mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1,...,A_n)$ satisfies the condition $\operatorname{Sp}(A_j)\subseteq I$ for any $j\in\{1,...,n\}$. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m,M such that*

$$m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q} \tag{2.8}$$

then

$$mJ_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq MJ_n(\mathbf{q};\mathbf{A},f,I) \tag{2.9}$$

in the operator order.

Proof. Observe that for $\alpha>0$ we have $J_n(\alpha\mathbf{p};\mathbf{A},f,I)=\alpha J_n(\mathbf{p};\mathbf{A},f,I)$.

Utilising the monotonicity property (2.3) we have

$$J_n(m\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq J_n(M\mathbf{q};\mathbf{A},f,I)$$

which imply the desired result (2.9). □

Remark 1. *We observe that if all $q_j>0$ then we have the inequality*

$$\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I) \tag{2.10}$$

in the operator order.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}, j\in\{1,...,n\}$, then we have the inequalities

$$n\min_{j\in\{1,\dots,n\}}\{p_j\}J_n(\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq n\max_{j\in\{1,\dots,n\}}\{p_j\}J_n(\mathbf{A},f,I) \tag{2.11}$$

where

$$J_n(\mathbf{A},f,I):=\frac{1}{n}\sum_{j=1}^{n}f(A_j)-f\left(\frac{1}{n}\sum_{j=1}^{n}A_j\right). \tag{2.12}$$

For $n=2$ and by choosing $p_1=\alpha, p_2=1-\alpha$ with $\alpha\in[0,1]$, we get from (2.11) the inequality

$$\begin{aligned}2\min\{\alpha,1-\alpha\}&\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right]\\&\leq(1-\alpha)f(A)+\alpha f(B)-f((1-\alpha)A+\alpha B)\\&\leq 2\max\{\alpha,1-\alpha\}\left[\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\right],\end{aligned} \tag{2.13}$$

in the operator order, where $f:I\to\mathbb{R}$ is an operator convex function and A and B are two bounded selfadjoint operators on the complex Hilbert space H with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subseteq I$.

The following result also holds:

Theorem 2. *If the function $f:[m,M]\to\mathbb{R}$ is operator convex and if the n-tuple of selfadjoint operators $(A_1,...,A_n)$ has the property that $\operatorname{Sp}(A_j)\subseteq[m,M]$ for any $j\in\{1,...,n\}$, then for any $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n:=\sum_{j=1}^n p_j>0$ we have*

$$0\leq\frac{1}{P_n}\sum_{j=1}^n p_j f(A_j)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\times\left(\frac{1}{2}(M-m)1_H+\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j-\frac{m+M}{2}1_H\right|\right)\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H \tag{2.14}$$

in the operator order.

Proof. Since the function $f:[m,M]\to\mathbb{R}$ is operator convex, then it is convex and we have the inequality

$$f(t)=f\left(\frac{(M-t)m+(t-m)M}{M-m}\right)\leq\frac{(M-t)f(m)+(t-m)f(M)}{M-m}$$

for any $t\in[m,M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator A with spectrum $\operatorname{Sp}(A)\subseteq[m,M]$, we have in the operator order

$$f(A_j)\leq\frac{f(m)(M1_H-A_j)+f(M)(A_j-m1_H)}{M-m} \tag{2.15}$$

for any $j\in\{1,...,n\}$.

If we multiply the inequality (2.15) by p_j and sum over j from 1 to n we get

$$\frac{1}{P_n}\sum_{j=1}^n p_j f(A_j)\leq\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m} \tag{2.16}$$

in the operator order.

Therefore we have

$$0\leq\frac{1}{P_n}\sum_{j=1}^n p_j f(A_j)-f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)\leq\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m}-f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right) \tag{2.17}$$

in the operator order, which is a reverse of Jensen's inequality that is of interest in itself.

Now, from the scalar version of (2.13) we have

$$\begin{aligned}0 &\leq (1-t)\,f(m)+tf(M)-f((1-t)\,m+tM) \\ &\leq 2\max\{t,1-t\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\ &= 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\end{aligned} \tag{2.18}$$

for any $t\in[m,M]$, where $f:[m,M]\to\mathbb{R}$ is a continuous convex function on $[m,M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator T with $0\leq T\leq 1_H$ we have from (2.18) that

$$\begin{aligned}0 &\leq f(m)\,(1_H-T)+f(M)\,T-f((1_H-T)\,m+TM) \\ &\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}+\left|T-\frac{1}{2}1_H\right|\right)\end{aligned} \tag{2.19}$$

in the operator order.

Writing the inequality (2.19) for the operator

$$0\leq T=\frac{\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-m1_H}{M-m}\leq 1_H$$

we have

$$\begin{aligned}&\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-m1_H\right)}{M-m} \\ &-f\left[\frac{m\left(M1_H-\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+M\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-m1_H\right)}{M-m}\right] \\ &=\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-m1_H\right)}{M-m} \\ &-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) \\ &\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\ &\times\left(\frac{1}{2}(M-m)\,1_H+\left|\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-\frac{m+M}{2}1_H\right|\right)\end{aligned} \tag{2.20}$$

in the operator order.

The last part is obvious since

$$\left|\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-\frac{m+M}{2}1_H\right|\leq\frac{1}{2}(M-m)\,1_H.$$

□

Corollary 2. *Assume that A, B are selfadjoint with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{R}$ is operator convex, then*

$$
\begin{aligned}
0 &\leq (1-t) f(A) + t f(B) - f((1-t) A + tB) \\
&\leq \frac{2}{M-m} \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] \\
&\quad \times \left(\frac{1}{2} (M-m) 1_H + \left| (1-t) A + tB - \frac{m+M}{2} 1_H \right| \right) \\
&\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] 1_H,
\end{aligned} \tag{2.21}
$$

for all $t \in [0, 1]$.

We have the following result for general convex functions, see [9]:

Lemma 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $a, b \in \mathring{I}$, the interior of I, with $a < b$ and $\nu \in [0, 1]$. Then*

$$
\begin{aligned}
&\nu (1-\nu)(b-a) \left[f'_+ ((1-\nu) a + \nu b) - f'_- ((1-\nu) a + \nu b) \right] \\
&\leq (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \\
&\leq \nu (1-\nu)(b-a) \left[f'_- (b) - f'_+ (a) \right].
\end{aligned} \tag{2.22}
$$

In particular, we have

$$
\begin{aligned}
\frac{1}{4} (b-a) \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] &\leq \frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2} \right) \\
&\leq \frac{1}{4} (b-a) \left[f'_- (b) - f'_+ (a) \right].
\end{aligned} \tag{2.23}
$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (2.23).

Corollary 3. *If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a, b \in \mathring{I}$ and $\nu \in [0, 1]$ we have*

$$
\begin{aligned}
0 &\leq (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \\
&\leq \nu (1-\nu)(b-a) \left[f'(b) - f'(a) \right].
\end{aligned} \tag{2.24}
$$

We also have:

Theorem 3. *If the function $f : [m, M] \to \mathbb{R}$ is operator convex and if the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ has the property that $\operatorname{Sp}(A_j) \subseteq [m, M]$ for any $j \in \{1, ..., n\}$, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^n p_j > 0$ we have*

$$
\begin{aligned}
0 &\leq \frac{1}{P_n} \sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n} \sum_{j=1}^n p_j A_j \right) \\
&\leq \frac{\left[f'_- (M) - f'_+ (m) \right]}{M-m} \frac{1}{P_n} \sum_{j=1}^n p_j (A_j - m 1_H) \frac{1}{P_n} \sum_{j=1}^n p_j (M 1_H - A_j) \\
&\leq \frac{1}{4} (M-m) \left[f'_- (M) - f'_+ (m) \right] 1_H.
\end{aligned} \tag{2.25}
$$

Proof. Using the *continuous functional calculus* for a selfadjoint operator T with $0 \leq T \leq 1_H$ we have from (2.22) that

$$
\begin{aligned}
0 &\leq f(m)(1-T) + f(M)T - f(m(1-T)+MT) \\
&\leq (M-m)\left[f'_{-}(M) - f'_{+}(m)\right] T(1-T).
\end{aligned} \tag{2.26}
$$

Writing the inequality (2.26) for the operator

$$
0 \leq T = \frac{\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H}{M-m} \leq 1_H
$$

we have

$$
\begin{aligned}
&\frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m} \\
&- f\left[\frac{m\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + M\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m}\right] \\
&= \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m} \\
&- f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \\
&\leq (M-m)\left[f'_{-}(M) - f'_{+}(m)\right] \\
&\times \frac{\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H}{M-m}\left(1 - \frac{\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H}{M-m}\right) \\
&= \frac{\left[f'_{-}(M) - f'_{+}(m)\right]}{M-m} \frac{1}{P_n}\sum_{j=1}^{n} p_j (A_j - m1_H) \frac{1}{P_n}\sum_{j=1}^{n} p_j (M1_H - A_j).
\end{aligned} \tag{2.27}
$$

By making use of (2.17) and (2.27) we derive the first inequality in (2.25).

We also have

$$
\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^{n} p_j (A_j - m1_H) \frac{1}{P_n}\sum_{j=1}^{n} p_j (M1_H - A_j) \\
&\leq \frac{1}{4}\left[\frac{1}{P_n}\sum_{j=1}^{n} p_j (A_j - m1_H) + \frac{1}{P_n}\sum_{j=1}^{n} p_j (M1_H - A_j)\right]^2 \\
&= \frac{1}{4}(M-m)^2 1_H,
\end{aligned}
$$

which proves the last part of (2.25). □

Corollary 4. *Assume that A, B are selfadjoint with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$ and $f : [m, M] \to \mathbb{R}$ is operator convex, then*

$$0 \leq (1-t) f(A) + t f(B) - f((1-t) A + tB) \tag{2.28}$$
$$\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m} ((1-t) A + tB - m1_H)(M1_H - (1-t) A - tB)$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right] 1_H$$

for all $t \in [0,1]$.

We also have the following result:

Lemma 3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{I}, \tag{2.29}$$

then

$$\frac{1}{2}\nu(1-\nu) d (b-a)^2 \leq (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \tag{2.30}$$
$$\leq \frac{1}{2}\nu(1-\nu) D (b-a)^2$$

for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)^2 D, \tag{2.31}$$

for any a, $b \in \mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (2.31).

We also have:

Theorem 4. *If the function $f : [m, M] \to \mathbb{R}$ is operator convex with $f''(x) \leq D$ for all $x \in (m, M)$ and if the n-tuple of selfadjoint operators $(A_1, ..., A_n)$ has the property that $\operatorname{Sp}(A_j) \subseteq [m, M]$ for any $j \in \{1, ..., n\}$, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^n p_j > 0$ we have*

$$0 \leq \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) \tag{2.32}$$
$$\leq \frac{1}{2} D \frac{1}{P_n}\sum_{j=1}^n p_j (A_j - m1_H) \frac{1}{P_n}\sum_{j=1}^n p_j (M1_H - A_j) \leq \frac{1}{8}(M-m)^2 D.$$

Proof. Using the *continuous functional calculus* for a selfadjoint operator T with $0 \leq T \leq 1_H$ we have from (2.30) that

$$0 \leq f(m)(1-T) + f(M) T - f(m(1-T) + MT) \tag{2.33}$$
$$\leq \frac{1}{2} T (1-T)(M-m)^2 D$$

Writing the inequality (2.33) for the operator

$$0 \leq T = \frac{\frac{1}{P_n}\sum_{j=1}^n p_j A_j - m1_H}{M-m} \leq 1_H,$$

we derive (2.32). □

Corollary 5. *Assume that* A, B *are selfadjoint with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$ *and* $f : [m, M] \to \mathbb{R}$ *is operator convex with* $f''(x) \leq D$ *for all* $x \in (m, M)$*, then*

$$\begin{aligned} 0 &\leq (1-t) f(A) + t f(B) - f((1-t) A + tB) \\ &\leq \frac{1}{2} D ((1-t) A + tB - m 1_H)(M 1_H - (1-t) A - tB) \\ &\leq \frac{1}{8} (M-m)^2 D. \end{aligned} \tag{2.34}$$

for all $t \in [0, 1]$.

3. Hermite-Hadamard Type Inequalities

We have the following reverses of Hermite-Hadamard type inequalities:

Theorem 5. *Assume that* A, B *are selfadjoint with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$ *and* $f : [m, M] \to \mathbb{R}$ *is operator convex, then*

$$\begin{aligned} 0 &\leq \frac{f(A) + f(B)}{2} - \int_0^1 f((1-t) A + tB)\, dt \\ &\leq \frac{2}{M-m} \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] \\ &\times \left(\frac{1}{2} (M-m) 1_H + \int_0^1 \left| (1-t) A + tB - \frac{m+M}{2} 1_H \right| dt \right) \\ &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] 1_H \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} 0 &\leq \int_0^1 f((1-t) A + tB)\, dt - f\left(\frac{A+B}{2} \right) \\ &\leq \frac{2}{M-m} \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] \\ &\times \left(\frac{1}{2} (M-m) 1_H + \left| \frac{A+B}{2} - \frac{m+M}{2} 1_H \right| \right) \\ &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] 1_H. \end{aligned} \tag{3.2}$$

Proof. Take the integral over $t \in [0, 1]$ in (2.21) to get

$$\begin{aligned} 0 &\leq \int_0^1 [(1-t) f(A) + t f(B) - f((1-t) A + tB)]\, dt \\ &\leq \frac{2}{M-m} \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] \\ &\times \int_0^1 \left(\frac{1}{2} (M-m) 1_H + \left| (1-t) A + tB - \frac{m+M}{2} 1_H \right| \right) dt \\ &\leq 2 \left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2} \right) \right] 1_H, \end{aligned} \tag{3.3}$$

which gives (3.1).

From (2.21) we also have for $t=1/2$ that

$$0\leq\frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right)\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\times\left(\frac{1}{2}(M-m)1_H+\left|\frac{A+B}{2}-\frac{m+M}{2}1_H\right|\right)\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H, \tag{3.4}$$

which is an inequality of interest in itself.

If in this inequality we replace A with $(1-t)A+tB$, B with $(1-t)B+tA$, then we get

$$0\leq\frac{f((1-t)A+tB)+f((1-t)B+tA)}{2}-f\left(\frac{A+B}{2}\right)\leq\frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\times\left(\frac{1}{2}(M-m)1_H+\left|\frac{A+B}{2}-\frac{m+M}{2}1_H\right|\right)\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H$$

for all $t\in[0,1]$.

If we take the integral over $t\in[0,1]$ in this inequality and observe that

$$\int_0^1 f((1-t)A+tB)\,dt=\int_0^1 f((1-t)B+tA)\,dt,$$

then we obtain the desired result (3.2). □

We also have the second pair of reverses for the Hermite-Hadamard inequalities

Theorem 6. *Assume that A, B are selfadjoint with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subseteq[m,M]$ and $f:[m,M]\to\mathbb{R}$ is operator convex, then*

$$0\leq\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\leq\frac{f'_-(M)-f'_+(m)}{M-m}\times\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left((1-t)A+tB-\frac{m+M}{2}1_H\right)^2\right]\leq\frac{f'_-(M)-f'_+(m)}{M-m}\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{A+B}{2}-\frac{m+M}{2}1_H\right)^2\right]\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H \tag{3.5}$$

and

$$0 \le \int_0^1 f\left((1-t)A + tB\right)dt - f\left(\frac{A+B}{2}\right) \le \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m}\left(\frac{A+B}{2} - m1_H\right)\left(M1_H - \frac{A+B}{2}\right) \le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]1_H. \tag{3.6}$$

Proof. By taking the integral in (2.28) we get

$$0 \le \frac{f(A)+f(B)}{2} - \int_0^1 f\left((1-t)A+tB\right)dt \le \frac{f'_-(M) - f'_+(m)}{M-m} \times \int_0^1 \left((1-t)A + tB - m1_H\right)\left(M1_H - (1-t)A - tB\right)dt \tag{3.7}$$

Now, observe that, by using the elementary equality

$$(X - m1_H)(M1_H - X) = \frac{1}{4}(M-m)^2 1_H - \left(X - \frac{m+M}{2}1_H\right)^2$$

we get

$$\int_0^1 \left((1-t)A + tB - m1_H\right)\left(M1_H - (1-t)A - tB\right)dt = \frac{1}{4}(M-m)^2 1_H - \int_0^1 \left((1-t)A + tB - \frac{m+M}{2}1_H\right)^2.$$

By the operator convexity of the square function and Jensen's inequality we have

$$\int_0^1 \left((1-t)A + tB - \frac{m+M}{2}1_H\right)^2 \ge \left(\int_0^1 (1-t)A + tBdt - \frac{m+M}{2}1_H\right)^2 = \left(\frac{A+B}{2} - \frac{m+M}{2}1_H\right)^2,$$

which gives that

$$\frac{1}{4}(M-m)^2 1_H - \int_0^1 \left((1-t)A + tB - \frac{m+M}{2}1_H\right)^2 \le \frac{1}{4}(M-m)^2 1_H - \left(\frac{A+B}{2} - \frac{m+M}{2}1_H\right)^2 \le \frac{1}{4}(M-m)^2 1_H$$

and by (3.7) we obtain the desired result (3.5).

From (2.28) we also have

$$0 \le \frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right) \le \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m}\left(\frac{A+B}{2} - m1_H\right)\left(M1_H - \frac{A+B}{2}\right) \le \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]1_H, \tag{3.8}$$

which is an inequality of interest in itself.

Now, if in (3.8) we replace A with $(1-t)A+tB$, B with $(1-t)B+tA$, then we get

$$\begin{aligned}
0 &\le \frac{f((1-t)A+tB)+f((1-t)B+tA)}{2}-f\left(\frac{A+B}{2}\right)\\
&\le \frac{\left[f'_-(M)-f'_+(m)\right]}{M-m}\left(\frac{A+B}{2}-m1_H\right)\left(M1_H-\frac{A+B}{2}\right)\\
&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H,
\end{aligned}$$

which by integration over t on $[0,1]$ produces (3.6). □

By utilising (2.32) we can also obtain the following reverses of Hermite-Hadamard inequalities for operator convex functions.

Theorem 7. *Assume that A, B are selfadjoint with $\operatorname{Sp}(A), \operatorname{Sp}(B)\subseteq[m,M]$ and $f:[m,M]\to\mathbb{R}$ is operator convex with $f''(x)\le D$ for all $x\in(m,M)$, then*

$$\begin{aligned}
0 &\le \frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt \\
&\le \frac{1}{2}D\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left((1-t)A+tB-\frac{m+M}{2}1_H\right)^2\right]\\
&\le \frac{1}{2}D\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{A+B}{2}-\frac{m+M}{2}1_H\right)^2\right]\\
&\le \frac{1}{8}D(M-m)^2 1_H
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
0 &\le \int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\\
&\le \frac{1}{2}D\left(\frac{A+B}{2}-m1_H\right)\left(M1_H-\frac{A+B}{2}\right)\le\frac{1}{8}(M-m)^2 D1_H.
\end{aligned} \tag{3.10}$$

4. Some Examples

By writing the inequalities from Theorems 5 and 6 for the operator convex function $f(t)=t^r$ for $r\in[1,2]$ we get for $0\le m\le A$, $B\le M$ that

$$\begin{aligned}
0 &\le \frac{A^r+B^r}{2}-\int_0^1((1-t)A+tB)^r\,dt\\
&\le \frac{2}{M-m}\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]\\
&\times\left(\frac{1}{2}(M-m)1_H+\int_0^1\left|(1-t)A+tB-\frac{m+M}{2}1_H\right|dt\right)\\
&\le 2\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]1_H,
\end{aligned} \tag{4.1}$$

$$
\begin{aligned}
0 &\leq \int_0^1 ((1-t)A+tB)^r\,dt - \left(\frac{A+B}{2}\right)^r \\
&\leq \frac{2}{M-m}\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right] \\
&\quad\times\left(\frac{1}{2}(M-m)1_H+\left|\frac{A+B}{2}-\frac{m+M}{2}1_H\right|\right) \\
&\leq 2\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]1_H,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
0 &\leq \frac{A^r+B^r}{2}-\int_0^1 ((1-t)A+tB)^r\,dt \\
&\leq r\frac{M^{r-1}-m^{r-1}}{M-m} \\
&\quad\times\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left((1-t)A+tB-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{M^{r-1}-m^{r-1}}{M-m}\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{A+B}{2}-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{4}r(M-m)\left(M^{r-1}-m^{r-1}\right)1_H
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
0 &\leq \int_0^1 ((1-t)A+tB)^r\,dt - \left(\frac{A+B}{2}\right)^r \\
&\leq r\frac{M^{r-1}-m^{r-1}}{M-m}\left(\frac{A+B}{2}-m1_H\right)\left(M1_H-\frac{A+B}{2}\right) \\
&\leq \frac{1}{4}r(M-m)\left(M^{r-1}-m^{r-1}\right)1_H.
\end{aligned} \tag{4.4}
$$

By utilising Theorem 6 we derive

$$
\begin{aligned}
0 &\leq \frac{A^r+B^r}{2}-\int_0^1 ((1-t)A+tB)^r\,dt \\
&\leq \frac{1}{2}r(r-1)m^{r-2} \\
&\quad\times\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left((1-t)A+tB-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{2}r(r-1)m^{r-2}\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{A+B}{2}-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{8}r(r-1)m^{r-2}(M-m)^2 1_H
\end{aligned} \tag{4.5}
$$

and

$$\begin{aligned}
(4.6)\qquad 0 &\leq \int_0^1 ((1-t)A+tB)^r\,dt - \left(\frac{A+B}{2}\right)^r \\
&\leq \frac{1}{2}r(r-1)m^{r-2}\left(\frac{A+B}{2}-m1_H\right)\left(M1_H-\frac{A+B}{2}\right) \\
&\leq \frac{1}{8}(M-m)^2 r(r-1)m^{r-2}1_H.
\end{aligned}$$

If we write the inequalities from Theorems 5 and 6 for the operator convex function $f(t)=-\ln t$, we get for $0<m\leq A,\ B\leq M$ that

$$\begin{aligned}
(4.7)\qquad 0 &\leq \int_0^1 \ln((1-t)A+tB)\,dt - \frac{\ln A+\ln B}{2} \\
&\leq \frac{2}{M-m}\ln\left(\frac{m+M}{2\sqrt{mM}}\right) \\
&\times\left(\frac{1}{2}(M-m)1_H+\int_0^1\left|(1-t)A+tB-\frac{m+M}{2}1_H\right|dt\right) \\
&\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^2 1_H,
\end{aligned}$$

$$\begin{aligned}
(4.8)\qquad 0 &\leq \ln\left(\frac{A+B}{2}\right) - \int_0^1 \ln((1-t)A+tB)\,dt \\
&\leq \frac{2}{M-m}\ln\left(\frac{m+M}{2\sqrt{mM}}\right)\left(\frac{1}{2}(M-m)1_H+\left|\frac{A+B}{2}-\frac{m+M}{2}1_H\right|\right) \\
&\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^2 1_H,
\end{aligned}$$

$$\begin{aligned}
(4.9)\qquad 0 &\leq \int_0^1 \ln((1-t)A+tB)\,dt - \frac{\ln A+\ln B}{2} \\
&\leq \frac{1}{mM}\left[\frac{1}{4}(M-m)^2 1_H - \int_0^1\left((1-t)A+tB-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{mM}\left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{A+B}{2}-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{4mM}(M-m)^2 1_H
\end{aligned}$$

and

$$\begin{aligned}
(4.10)\qquad 0 &\leq \ln\left(\frac{A+B}{2}\right) - \int_0^1 \ln((1-t)A+tB)\,dt \\
&\leq \frac{1}{mM}\left(\frac{A+B}{2}-m1_H\right)\left(M1_H-\frac{A+B}{2}\right) \\
&\leq \frac{1}{4mM}(M-m)^2 1_H.
\end{aligned}$$

By utilising Theorem 6 we derive

$$
\begin{aligned}
(4.11)\qquad 0 &\leq \int_0^1 \ln\left(\left(1-t\right)A+tB\right)dt-\frac{\ln A+\ln B}{2}\\
&\leq \frac{1}{2m^2}\left[\frac{1}{4}\left(M-m\right)^2 1_H-\int_0^1\left(\left(1-t\right)A+tB-\frac{m+M}{2}1_H\right)^2\right]\\
&\leq \frac{1}{2m^2}\left[\frac{1}{4}\left(M-m\right)^2 1_H-\left(\frac{A+B}{2}-\frac{m+M}{2}1_H\right)^2\right]\\
&\leq \frac{1}{8}\frac{\left(M-m\right)^2}{m^2}1_H
\end{aligned}
$$

and

$$
\begin{aligned}
(4.12)\qquad 0 &\leq \ln\left(\frac{A+B}{2}\right)-\int_0^1 \ln\left(\left(1-t\right)A+tB\right)dt\\
&\leq \frac{1}{2m^2}\left(\frac{A+B}{2}-m1_H\right)\left(M1_H-\frac{A+B}{2}\right)\leq \frac{1}{8}\frac{\left(M-m\right)^2}{m^2}1_H.
\end{aligned}
$$

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