# SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR THE SQUARE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach $C^*$-algebra of bounded linear operators on $H$. For $A\in\mathcal{B}(H)$ we define the modulus of $A$ by $|A|:=(A^*A)^{1/2}$. In this paper we obtain among others the following Hermite-Hadamard type inequalities for operator convex (concave) functions $f:[0,\infty)\to\mathbb{R}$ and $A,\ B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$,

$$f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\leq(\geq)\int_0^1 f\left(|(1-t)A+tB|^2\right)dt$$
$$\leq(\geq)\frac{1}{3}\left[f\left(|A|^2\right)+f\left[\operatorname{Re}(B^*A)\right]+f\left(|B|^2\right)\right].$$

Some examples for power functions and logarithm are also provided.

## 1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f(tx+(1-t)t)\leq tf(x)+(1-t)f(y)$$

for all $x,\ y\in I$ and $t\in[0,1]$.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if $f$ is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [13]). The inequalities in (1.1) hold in reversed direction if $f$ is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [10] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [3], Information Theory [2], Operator Theory [6], [7] and others.

Let $X$ be a vector space over the real or complex number field $\mathbb{K}$ and $x,y\in X,\ x\neq y$. Define the segment

$$[x,y]:=\{(1-t)x+ty,\ t\in[0,1]\}.$$

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We consider the function $f:[x,y]\to\mathbb{R}$ and the associated function

$$g(x,y):[0,1]\to\mathbb{R},\ g(x,y)(t):=f[(1-t)x+ty],\ t\in[0,1].$$

Note that $f$ is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x,y]\subset X$, we have the *Hermite-Hadamard integral inequality* (see [4, p. 2], [5, p. 2])

$$f\left(\frac{x+y}{2}\right)\leq\int_0^1 f[(1-t)x+ty]dt\leq\frac{f(x)+f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0,1]\to\mathbb{R}$.

Since $f(x)=\|x\|^p$ ($x\in X$ and $1\leq p<\infty$) is a convex function, then for any $x,y\in X$ we have the following norm inequality from (1.2) (see [12, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p\leq\int_0^1\|(1-t)x+ty\|^p dt\leq\frac{\|x\|^p+\|y\|^p}{2}. \tag{1.3}$$

A real valued continuous function $f$ on an interval $I$ is said to be *operator convex (operator concave)* on $I$ if

$$f\left((1-\lambda)A+\lambda B\right)\leq(\geq)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right) \tag{1.4}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

A real valued continuous function $f$ on an interval $I$ is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subset I$ imply $f(A)\leq f(B)$.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

In the recent paper [9] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions:

**Theorem 1.** *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval $I$. Then for any selfadjoint operators $A$ and $B$ with spectra in $I$ and for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned} & f\left(\frac{A+B}{2}\right) \\ & \leq(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \\ & \leq\int_0^1 f\left((1-s)A+sB\right)ds \\ & \leq\frac{1}{2}\left[f\left((1-\lambda)A+\lambda B\right)+(1-\lambda)f(B)+\lambda f(A)\right] \\ & \leq\frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.5}$$

A similar result and with a different proof was obtained by B. Li in [11]. For $\lambda = \frac{1}{2}$ in (1.5) we recapture the result obtained in the earlier paper [8] by the author. For other similar inequalities for operator convex functions see [1] and [14]-[18].

Denote by $\mathcal{B}(H)$ the Banach $C^*$-algebra of bounded linear operators on Hilbert space $H$. For $A \in \mathcal{B}(H)$ we define the modulus of $A$ by $|A| := (A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B| \le |A| + |B|$, so the classical arguments using this inequality can not be used.

In this paper we obtain among others the following Hermite-Hadamard type inequalities for operator convex (concave) functions $f : [0,\infty) \to \mathbb{R}$ and $A$, $B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \ge 0$,

$$f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \le (\ge) \int_0^1 f\left(|(1-t)A + tB|^2\right) dt$$
$$\le (\ge) \frac{1}{3}\left[f\left(|A|^2\right) + f\left[\operatorname{Re}(B^*A)\right] + f\left(|B|^2\right)\right].$$

Some examples for power functions and logarithm are also provided.

## 2. Main Results

For an operator $T$ we consider the selfadjoint operator

$$\operatorname{Re}(T) := \frac{1}{2}(T^* + T).$$

We have the following Hermite-Hadamard type inequalities:

**Theorem 2.** *Let $f : [0,\infty) \to \mathbb{R}$ be operator convex (concave) on $[0,\infty)$. If $A$, $B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \ge 0$, then*

$$f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \tag{2.1}$$
$$\le (\ge) \int_0^1 f\left(|(1-t)A + tB|^2\right) dt$$
$$\le (\ge) \frac{1}{3}\left[f\left(|A|^2\right) + f\left[\operatorname{Re}(B^*A)\right] + f\left(|B|^2\right)\right].$$

*Proof.* Observe that, by the properties of modulus, we have for $t \in [0,1]$

$$\begin{aligned}
&|(1-t)A + tB|^2 \\
&= ((1-t)A + tB)^* (1-t)A + tB \\
&= ((1-t)A^* + tB^*)(1-t)A + tB \\
&= (1-t)^2 A^*A + t(1-t)B^*A + (1-t)tA^*B + t^2B^*B \\
&= (1-t)^2 |A|^2 + 2t(1-t)\operatorname{Re}(B^*A) + t^2|B|^2.
\end{aligned}$$

Assume that $f : [0,\infty) \to \mathbb{R}$ be operator convex (concave) on $[0,\infty)$. By using Jensen's integral inequality, we have that

$$f\left(\int_0^1 |(1-t)A + tB|^2 dt\right) \le \int_0^1 f\left(|(1-t)A + tB|^2\right) dt. \tag{2.2}$$

Since

$$\begin{aligned}
&\int_0^1 |(1-t)A+tB|^2\,dt \\
&=\left(\int_0^1 (1-t)^2\,dt\right)|A|^2+2\left(\int_0^1 t(1-t)\,dt\right)\operatorname{Re}(B^*A)+\left(\int_0^1 t^2dt\right)|B|^2 \\
&=\frac{1}{3}|A|^2+\frac{1}{3}\operatorname{Re}(B^*A)+\frac{1}{3}|B|^2=\frac{1}{3}\left(|A|^2+\operatorname{Re}(B^*A)+|B|^2\right),
\end{aligned}$$

hence by (2.2) we get the first inequality in (2.1).

Consider $\alpha=(1-t)^2$, $\beta=2t(1-t)$, $\gamma=t^2\geq 0$ for $t\in[0,1]$. Then

$$\alpha+\beta+\gamma=(1-t)^2+2t(1-t)+t^2=(1-t+t)^2=1$$

and by operator convexity of $f$ we have

$$\begin{aligned}
&f\left((1-t)^2|A|^2+2t(1-t)\operatorname{Re}(B^*A)+t^2|B|^2\right) \\
&\leq(1-t)^2 f\left(|A|^2\right)+2t(1-t)f[\operatorname{Re}(B^*A)]+t^2 f\left(|B|^2\right)
\end{aligned}\tag{2.3}$$

for all $t\in[0,1]$.

If we take the integral in (2.3) we get

$$\begin{aligned}
&\int_0^1 f\left(|(1-t)A+tB|^2\right)dt \\
&\leq\int_0^1\left[(1-t)^2 f\left(|A|^2\right)+2t(1-t)f[\operatorname{Re}(B^*A)]+t^2 f\left(|B|^2\right)\right]dt \\
&=\frac{1}{3}\left[f\left(|A|^2\right)+f[\operatorname{Re}(B^*A)]+f\left(|B|^2\right)\right],
\end{aligned}$$

which proves the second part of (2.1). □

The first inequality in (2.1) can be improved as follows:

**Theorem 3.** *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex (concave) on $[0,\infty)$. If $A$, $B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$, then*

$$\begin{aligned}
&f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right) \\
&\leq(\geq)\int_0^1 f\left(\left[t^2+(1-t)^2\right]\frac{|A|^2+|B|^2}{2}+2t(1-t)\operatorname{Re}(B^*A)\right)dt \\
&\leq(\geq)\int_0^1 f\left(|(1-t)A+tB|^2\right)dt.
\end{aligned}\tag{2.4}$$

*Proof.* By the operator convexity, we also have

$$
\begin{aligned}
(2.5)\qquad & \frac{1}{2}\left[f\left(\left|(1-t)A+tB\right|^{2}\right)+f\left(\left|(1-t)B+tA\right|^{2}\right)\right] \\
& \geq f\left(\frac{\left|(1-t)A+tB\right|^{2}+\left|(1-t)B+tA\right|^{2}}{2}\right) \\
& = f\left(\frac{1}{2}\left[(1-t)^{2}\left|A\right|^{2}+2t(1-t)\operatorname{Re}(B^{*}A)+t^{2}\left|B\right|^{2}\right.\right. \\
& +\left.\left. (1-t)^{2}\left|B\right|^{2}+2t(1-t)\operatorname{Re}(A^{*}B)+t^{2}\left|A\right|^{2}\right]\right) \\
& = f\left(\left[t^{2}+(1-t)^{2}\right]\left(\frac{\left|A\right|^{2}+\left|B\right|^{2}}{2}\right)+2t(1-t)\operatorname{Re}(B^{*}A)\right)
\end{aligned}
$$

for all $t\in[0,1]$.

By taking the integral in (2.5) we get

$$
\begin{aligned}
(2.6)\qquad & \frac{1}{2}\int_{0}^{1}\left[f\left(\left|(1-t)A+tB\right|^{2}\right)+f\left(\left|(1-t)B+tA\right|^{2}\right)\right]dt \\
& \geq \int_{0}^{1} f\left(\left[t^{2}+(1-t)^{2}\right]\left(\frac{\left|A\right|^{2}+\left|B\right|^{2}}{2}\right)+2t(1-t)\operatorname{Re}(B^{*}A)\right)dt \\
& \geq f\left(\int_{0}^{1}\left\{\left[t^{2}+(1-t)^{2}\right]\left(\frac{\left|A\right|^{2}+\left|B\right|^{2}}{2}\right)+2t(1-t)\operatorname{Re}(B^{*}A)\right\}\right)dt \\
& = f\left(\int_{0}^{1}\left[t^{2}+(1-t)^{2}\right]dt\left(\frac{\left|A\right|^{2}+\left|B\right|^{2}}{2}\right)\right. \\
& \left. +2\left(\int_{0}^{1}t(1-t)\,dt\right)\operatorname{Re}(B^{*}A)\right)dt \\
& = f\left(\frac{\left|A\right|^{2}+\left|B\right|^{2}+\operatorname{Re}(B^{*}A)}{3}\right),
\end{aligned}
$$

where for the last inequality we used Jensen's inequality.

Since

$$
\int_{0}^{1} f\left(\left|(1-t)A+tB\right|^{2}\right)dt=\int_{0}^{1} f\left(\left|(1-t)B+tA\right|^{2}\right)dt,
$$

hence by (2.6) we deduce (2.4). □

**Lemma 1.** *Let* $f$ *be continuous on* $[0,\infty)$ *Then for any* $\lambda\in[0,1]$ *and* $A$, $B\in\mathcal{B}(H)$ *we have the representation*

$$
\begin{aligned}
(2.7)\qquad & \int_{0}^{1} f\left[\left|(1-t)A+tB\right|^{2}\right]dt \\
& =(1-\lambda)\int_{0}^{1} f\left[\left|(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right|^{2}\right]dt \\
& +\lambda\int_{0}^{1} f\left[\left|(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right|^{2}\right]dt.
\end{aligned}
$$

*In particular, for* $\lambda = \frac{1}{2}$,

$$\int_0^1 f\left[|(1-t)A + tB|^2\right] dt = \frac{1}{2}\int_0^1 f\left[\left|(1-t)\left(\frac{A+B}{2}\right) + tB\right|^2\right] dt + \frac{1}{2}\int_0^1 f\left[\left|(1-t)A + t\left(\frac{A+B}{2}\right)\right|^2\right] dt. \tag{2.8}$$

*Proof.* For $\lambda = 0$ and $\lambda = 1$ the equality (2.7) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 f\left[|(1-t)(\lambda B + (1-\lambda)A) + tB|^2\right] dt = \int_0^1 f\left[|((1-t)\lambda + t)B + (1-t)(1-\lambda)A|^2\right] dt$$

and

$$\int_0^1 f\left[|t(\lambda B + (1-\lambda)A) + (1-t)A|^2\right] dt = \int_0^1 f\left[|t\lambda B + (1-\lambda t)A|^2\right] dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)du$. Then

$$\int_0^1 f\left[|((1-t)\lambda + t)B + (1-t)(1-\lambda)A|^2\right] dt = \frac{1}{1-\lambda}\int_\lambda^1 f\left[|uB + (1-u)A|^2\right] du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f\left[|t\lambda B + (1-\lambda t)A|^2\right] dt = \frac{1}{\lambda}\int_0^\lambda f\left[|uB + (1-u)A|^2\right] du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 f\left[|(1-t)(\lambda B + (1-\lambda)A) + tB|^2\right] dt \\
&+ \lambda\int_0^1 f\left[|t(\lambda B + (1-\lambda)A) + (1-t)A|^2\right] dt \\
&= \int_\lambda^1 f\left[|uB + (1-u)A|^2\right] du + \int_0^\lambda f\left[|uB + (1-u)A|^2\right] du \\
&= \int_0^1 f\left[|uB + (1-u)A|^2\right] du
\end{aligned}$$

and the identity (2.7) is proved. □

**Theorem 4.** *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$. If $A,\ B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$, then for $\lambda\in[0,1]$*

$$\begin{aligned}
(2.9)\quad & f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\
&\leq (1-\lambda) f\left(\frac{|(1-\lambda)A+\lambda B|^2+(1-\lambda)\operatorname{Re}(B^*A)+(\lambda+1)|B|^2}{3}\right)\\
&+\lambda f\left(\frac{(2-\lambda)|A|^2+\lambda\operatorname{Re}(B^*A)+|(1-\lambda)A+\lambda B|^2}{3}\right)\\
&\leq \int_0^1 f\left[|(1-t)A+tB|^2\right]dt\\
&\leq \frac{1}{3}f\left(|(1-\lambda)A+\lambda B|^2\right)+\frac{1}{3}\lambda f\left(|A|^2\right)+\frac{1}{3}(1-\lambda)f\left(|B|^2\right)\\
&+\frac{1}{3}(1-\lambda)f\left[(1-\lambda)\operatorname{Re}(B^*A)+\lambda|B|^2\right]+\frac{1}{3}\lambda f\left[(1-\lambda)A+\lambda\operatorname{Re}(B^*A)\right]\\
&\leq \frac{1}{3}\left[f\left(|A|^2\right)+f\left(\operatorname{Re}(B^*A)\right)+f\left(|B|^2\right)\right].
\end{aligned}$$

*In particular,*

$$\begin{aligned}
(2.10)\quad & f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\
&\leq \frac{1}{2}f\left(\frac{1}{3}\left[\left|\frac{A+B}{2}\right|^2+\frac{1}{2}\operatorname{Re}(B^*A)+\frac{3}{2}|B|^2\right]\right)\\
&+\frac{1}{2}f\left(\frac{1}{3}\left[\frac{3}{2}|A|^2+\frac{1}{2}\operatorname{Re}(B^*A)+\left|\frac{A+B}{2}\right|^2\right]\right)\\
&\leq \int_0^1 f\left[|(1-t)A+tB|^2\right]dt\\
&\leq \frac{1}{3}f\left(\left|\frac{A+B}{2}\right|^2\right)+\frac{1}{6}f\left(|A|^2\right)+\frac{1}{6}f\left(|B|^2\right)\\
&+\frac{1}{6}f\left[\frac{\operatorname{Re}(B^*A)+|B|^2}{2}\right]+\frac{1}{6}f\left[\frac{A+\operatorname{Re}(B^*A)}{2}\right]\\
&\leq \frac{1}{3}\left[f\left(|A|^2\right)+f\left(\operatorname{Re}(B^*A)\right)+f\left(|B|^2\right)\right].
\end{aligned}$$

*If $f$ is operator concave on $[0,\infty)$, then the sign of inequality reverses in (2.9) and (2.10).*

*Proof.* By using (2.1) we get by replacing $A$ with $(1-\lambda)A+\lambda B$ that

$$\begin{aligned}
&f\left(\frac{\left|(1-\lambda)A+\lambda B\right|^{2}+\operatorname{Re}\left(B^{*}\left((1-\lambda)A+\lambda B\right)\right)+\left|B\right|^{2}}{3}\right)\\
&\leq\int_{0}^{1}f\left[\left|(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right|^{2}\right]dt\\
&\leq\frac{1}{3}\left[f\left(\left|(1-\lambda)A+\lambda B\right|^{2}\right)+f\left[\operatorname{Re}\left(B^{*}\left((1-\lambda)A+\lambda B\right)\right)\right]+f\left(\left|B\right|^{2}\right)\right],
\end{aligned}$$

which, by multiplication with $(1-\lambda)$ gives

$$\begin{aligned}
&(1-\lambda)f\left(\frac{\left|(1-\lambda)A+\lambda B\right|^{2}+(1-\lambda)\operatorname{Re}\left(B^{*}A\right)+(\lambda+1)\left|B\right|^{2}}{3}\right)\\
&\leq(1-\lambda)\int_{0}^{1}f\left[\left|(1-t)\left((1-\lambda)A+\lambda B\right)+tB\right|^{2}\right]dt\\
&\leq\frac{1}{3}\left[(1-\lambda)f\left(\left|(1-\lambda)A+\lambda B\right|^{2}\right)+f\left[(1-\lambda)\operatorname{Re}\left(B^{*}A\right)+\lambda\left|B\right|^{2}\right]\right.\\
&\left.+f\left(\left|B\right|^{2}\right)\right].
\end{aligned}\tag{2.11}$$

By using (2.1) we get by replacing $B$ with $(1-\lambda)A+\lambda B$ that

$$\begin{aligned}
&f\left(\frac{\left|A\right|^{2}+\operatorname{Re}\left(\left((1-\lambda)A+\lambda B\right)^{*}A\right)+\left|(1-\lambda)A+\lambda B\right|^{2}}{3}\right)\\
&\leq\int_{0}^{1}f\left[\left|(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right|^{2}\right]dt\\
&\leq\frac{1}{3}\left[f\left(\left|A\right|^{2}\right)+f\left[\operatorname{Re}\left(\left((1-\lambda)A+\lambda B\right)^{*}A\right)\right]+f\left(\left|(1-\lambda)A+\lambda B\right|^{2}\right)\right],
\end{aligned}$$

which, by multiplication with $\lambda$ gives

$$\begin{aligned}
&\lambda f\left(\frac{(2-\lambda)\left|A\right|^{2}+\lambda\operatorname{Re}\left(B^{*}A\right)+\left|(1-\lambda)A+\lambda B\right|^{2}}{3}\right)\\
&\leq\lambda\int_{0}^{1}f\left[\left|(1-t)A+t\left((1-\lambda)A+\lambda B\right)\right|^{2}\right]dt\\
&\leq\frac{1}{3}\lambda\left[f\left(\left|A\right|^{2}\right)+f\left[(1-\lambda)\left|A\right|^{2}+\lambda\operatorname{Re}\left(B^{*}A\right)\right]\right.\\
&\left.+f\left(\left|(1-\lambda)A+\lambda B\right|^{2}\right)\right].
\end{aligned}\tag{2.12}$$

If we add (2.11) and (2.12) and use (2.7), then we get

$$
\begin{aligned}
&(1-\lambda) f\left(\frac{|(1-\lambda) A+\lambda B|^2+(1-\lambda) \operatorname{Re}(B^* A)+(\lambda+1)|B|^2}{3}\right) \\
&+\lambda f\left(\frac{(2-\lambda)|A|^2+\lambda \operatorname{Re}(B^* A)+|(1-\lambda) A+\lambda B|^2}{3}\right) \\
&\leq \int_0^1 f\left[|(1-t) A+t B|^2\right] dt \\
&\leq \frac{1}{3}(1-\lambda)\left[f\left(|(1-\lambda) A+\lambda B|^2\right)+f\left[(1-\lambda) \operatorname{Re}(B^* A)+\lambda|B|^2\right]+f\left(|B|^2\right)\right]. \\
&+\frac{1}{3} \lambda\left[f\left(|A|^2\right)+f\left[(1-\lambda)|A|^2+\lambda \operatorname{Re}(B^* A)\right]+f\left(|(1-\lambda) A+\lambda B|^2\right)\right] \\
&=\frac{1}{3} f\left(|(1-\lambda) A+\lambda B|^2\right)+\frac{1}{3} \lambda f\left(|A|^2\right)+\frac{1}{3}(1-\lambda) f\left(|B|^2\right) \\
&+\frac{1}{3}(1-\lambda) f\left[(1-\lambda) \operatorname{Re}(B^* A)+\lambda|B|^2\right]+\frac{1}{3} \lambda f\left[(1-\lambda)|A|^2+\lambda \operatorname{Re}(B^* A)\right] \\
&=\frac{1}{3} f\left((1-\lambda)^2|A|^2+2 \lambda(1-\lambda) \operatorname{Re}(B^* A)+\lambda|B|^2\right) \\
&+\frac{1}{3} \lambda f\left(|A|^2\right)+\frac{1}{3}(1-\lambda) f\left(|B|^2\right) \\
&+\frac{1}{3}(1-\lambda) f\left[(1-\lambda) \operatorname{Re}(B^* A)+\lambda|B|^2\right]+\frac{1}{3} \lambda f\left[(1-\lambda)|A|^2+\lambda \operatorname{Re}(B^* A)\right],
\end{aligned}
$$

which proves the second, third and fourth inequalities in (2.9).

By the operator convexity of $f$ we have

$$
\begin{aligned}
&(1-\lambda) f\left(\frac{|(1-\lambda) A+\lambda B|^2+(1-\lambda) \operatorname{Re}(B^* A)+(\lambda+1)|B|^2}{3}\right) \\
&+\lambda f\left(\frac{(2-\lambda)|A|^2+\lambda \operatorname{Re}(B^* A)+|(1-\lambda) A+\lambda B|^2}{3}\right) \\
&\geq f\left[(1-\lambda) \frac{|(1-\lambda) A+\lambda B|^2+(1-\lambda) \operatorname{Re}(B^* A)+(\lambda+1)|B|^2}{3}\right. \\
&\left.+\lambda \frac{(2-\lambda)|A|^2+\lambda \operatorname{Re}(B^* A)+|(1-\lambda) A+\lambda B|^2}{3}\right] \\
&=f\left[\frac{1}{3}\left(|(1-\lambda) A+\lambda B|^2+\left[(1-\lambda)^2+\lambda^2\right] \operatorname{Re}(B^* A)\right.\right. \\
&\left.\left.\quad+\left(1-\lambda^2\right)|B|^2+(2-\lambda) \lambda|A|^2\right)\right] \\
&=f\left[\frac{1}{3}\left((1-\lambda)^2|A|^2+2(1-\lambda) \lambda \operatorname{Re}(B^* A)+\lambda^2|B|^2\right.\right. \\
&\left.\left.\quad+\left[(1-\lambda)^2+\lambda^2\right] \operatorname{Re}(B^* A)+\left(1-\lambda^2\right)|B|^2+(2-\lambda) \lambda|A|^2\right)\right] \\
&=f\left(\frac{|A|^2+\operatorname{Re}(B^* A)+|B|^2}{3}\right),
\end{aligned}
$$

which proves the first inequality in (2.9).

By the operator convexity, we also have

$$\begin{aligned}
&\frac{1}{3}f\left((1-\lambda)^2|A|^2+2\lambda(1-\lambda)\operatorname{Re}(B^*A)+\lambda^2|B|^2\right)\\
&+\frac{1}{3}\lambda f\left(|A|^2\right)+\frac{1}{3}(1-\lambda)f\left(|B|^2\right)\\
&+\frac{1}{3}(1-\lambda)f\left[(1-\lambda)\operatorname{Re}(B^*A)+\lambda|B|^2\right]+\frac{1}{3}\lambda f\left[(1-\lambda)|A|^2+\lambda\operatorname{Re}(B^*A)\right]\\
&\le\frac{1}{3}(1-\lambda)^2f\left(|A|^2\right)+\frac{2}{3}\lambda(1-\lambda)f(\operatorname{Re}(B^*A))+\frac{1}{3}\lambda^2f\left(|B|^2\right)\\
&+\frac{1}{3}\lambda f\left(|A|^2\right)+\frac{1}{3}(1-\lambda)f\left(|B|^2\right)\\
&+\frac{1}{3}(1-\lambda)^2f(\operatorname{Re}(B^*A))+\frac{1}{3}(1-\lambda)\lambda f\left(|B|^2\right)\\
&+\frac{1}{3}\lambda(1-\lambda)f\left(|A|^2\right)+\lambda^2f(\operatorname{Re}(B^*A))\\
&=\frac{1}{3}\left[(1-\lambda)^2+\lambda+\lambda(1-\lambda)\right]f\left(|A|^2\right)\\
&+\frac{1}{3}\left[2\lambda(1-\lambda)+(1-\lambda)^2+\lambda^2\right]f(\operatorname{Re}(B^*A))\\
&+\frac{1}{3}\left[\lambda^2+(1-\lambda)+(1-\lambda)\lambda\right]f\left(|B|^2\right)\\
&=\frac{1}{3}\left[f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)\right],
\end{aligned}$$

which proves the last part of (2.9). □

## 3. Some Examples

Assume that $A, B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\ge 0$, then by writing inequality (2.1) for the operator convex function $f(t)=t^r$ for $r\in[1,2]$ we get

$$\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^r\le\int_0^1|(1-t)A+tB|^{2r}dt \le\frac{1}{3}\left[|A|^{2r}+[\operatorname{Re}(B^*A)]^r+|B|^{2r}\right]. \tag{3.1}$$

If we take the power $\frac{1}{2r}\in(0,1)$ then by Heinz-Löwner theorem we also derive

$$\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^{1/2}\le\left(\int_0^1|(1-t)A+tB|^{2r}dt\right)^{1/2r} \le\frac{1}{3^{1/2r}}\left[|A|^{2r}+[\operatorname{Re}(B^*A)]^r+|B|^{2r}\right]^{1/2r}. \tag{3.2}$$

If $p\in(0,1)$ then by (2.1) for the operator concave function $f(t)=t^p$ we get

$$\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^p\ge\int_0^1|(1-t)A+tB|^{2p}dt \ge\frac{1}{3}\left[|A|^{2p}+[\operatorname{Re}(B^*A)]^p+|B|^{2p}\right]. \tag{3.3}$$

In particular, for $p = 1/2$, we get

$$\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right)^{1/2} \geq \int_0^1 |(1-t)A + tB|\,dt \geq \frac{1}{3}\left[|A| + [\operatorname{Re}(B^*A)]^{1/2} + |B|\right] \tag{3.4}$$

for $A,\ B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$.

If $|A|^2,\ |B|^2,\ \operatorname{Re}(B^*A) > 0$ then we have for the operator concave function $f(t) = \ln t$ on $(0,\infty)$ that

$$\ln\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \geq \int_0^1 \ln\left(|(1-t)A + tB|^2\right)dt \geq \frac{1}{3}\left[\ln\left(|A|^2\right) + \ln[\operatorname{Re}(B^*A)] + \ln\left(|B|^2\right)\right]. \tag{3.5}$$

If $|A|^2,\ |B|^2,\ \operatorname{Re}(B^*A) > 0$ then we have for the operator convex function $f(t) = t^{-p}$ on $(0,\infty)$ with $p \in (0,1]$ that

$$\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right)^{-p} \leq \int_0^1 |(1-t)A + tB|^{-2p}\,dt \leq \frac{1}{3}\left[|A|^{-2p} + [\operatorname{Re}(B^*A)]^{-p} + |B|^{-2p}\right] \tag{3.6}$$

and, in particular

$$\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right)^{-1} \leq \int_0^1 |(1-t)A + tB|^{-2}\,dt \leq \frac{1}{3}\left[|A|^{-2} + [\operatorname{Re}(B^*A)]^{-1} + |B|^{-2}\right]. \tag{3.7}$$

Also, if we take $p = 1/2$ in (3.6) we get

$$\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right)^{-1/2} \leq \int_0^1 |(1-t)A + tB|^{-1}\,dt \leq \frac{1}{3}\left[|A|^{-1} + [\operatorname{Re}(B^*A)]^{-1/2} + |B|^{-1}\right]. \tag{3.8}$$

If $|A|^2$, $|B|^2$, $\operatorname{Re}(B^*A) > 0$ then we have for the operator convex function $f(t) = t\ln t$ on $(0,\infty)$ that

$$\begin{aligned} &\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} \ln\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \\ &\leq \int_0^1 |(1-t)A + tB|^2 \ln\left(|(1-t)A + tB|^2\right) dt \\ &\leq \frac{1}{3}\left[|A|^2 \ln\left(|A|^2\right) + \operatorname{Re}(B^*A)\ln\left[\operatorname{Re}(B^*A)\right] + |B|^2 \ln\left(|B|^2\right)\right]. \end{aligned} \tag{3.9}$$

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