DOUBLE DIFFERENCE HERMITE-HADAMARD TYPE INEQUALITIES FOR THE SQUARE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$ and $\operatorname{Re}A:=\frac{1}{2}(A^*+A)$. In this paper we obtain among others the following Hermite-Hadamard type inequalities for operator convex functions $f:[0,\infty)\to\mathbb{R}$ and A, $B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$,

$$\begin{aligned}0&\leq\frac{1}{3}\left[\frac{f\left(|A|^2\right)+2f\left(\operatorname{Re}(B^*A)\right)+f\left(|B|^2\right)}{4}-f\left(\left|\frac{A+B}{2}\right|^2\right)\right]\\&\leq\frac{f\left(|A|^2\right)+f\left(\operatorname{Re}(B^*A)\right)+f\left(|B|^2\right)}{3}-\int_0^1 f\left(\left|(1-t)A+tB\right|^2\right)dt\\&\leq\frac{7}{3}\left[\frac{f\left(|A|^2\right)+2f\left(\operatorname{Re}(B^*A)\right)+f\left(|B|^2\right)}{4}-f\left(\left|\frac{A+B}{2}\right|^2\right)\right].\end{aligned}$$

Some examples for power functions and logarithm are also provided.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\ldots,p_n)$ where $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n:=\sum_{j=1}^n p_j>0$ and $\mathbf{x}=(x_1,\ldots,x_n)\in C^n$, then

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\leq\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

In order to extend this inequality for operator convex functions of selfdjoint bounded linear operators on complex Hilbert spaces we need the following preliminary facts.

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A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A\leq B$ with $Sp(A), Sp(B)\subset I$ imply $f(A)\leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [10] and the references therein.

As examples of such functions, we note that $f(t)=t^r$ is operator monotone on $[0,\infty)$ if and only if $0\leq r\leq 1$. The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. The logarithmic function $f(t)=\ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t)=-t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t)=e^t$ is neither operator convex nor operator monotone.

We also have the following Jensen type inequality for operator convex functions $f:I\to\mathbb{R}$.

Let A_j be selfadjoint operators with $Sp(A_j)\subseteq I$, $j\in\{1,...,n\}$. If $p_j\geq 0$, $j\in\{1,...,n\}$ with $P_n>0$ and f is an operator convex function on I then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_iA_j\right)\leq\frac{1}{P_n}\sum_{i=1}^{n}p_if(A_j), \tag{1.2}$$

in the operator order.

This is a well known result and can be proved easily by mathematical induction over $n\geq 2$. The details are left to the reader.

For recent results related to the Jensen inequality for selfadjoint operators in Hilbert spaces see the papers [1]-[5], [11]-[18], [19] and the monograph [6].

In the recent paper [9] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions:

Theorem 1. *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda\in[0,1]$ we have the inequalities*

$$\begin{aligned}
&f\left(\frac{A+B}{2}\right) \\
&\leq(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right] \\
&\leq\int_0^1 f((1-s)A+sB)\,ds \\
&\leq\frac{1}{2}\left[f((1-\lambda)A+\lambda B)+(1-\lambda)f(B)+\lambda f(A)\right] \\
&\leq\frac{f(A)+f(B)}{2}.
\end{aligned} \tag{1.3}$$

A similar result and with a different proof was obtained by B. Li in [12]. For $\lambda = \frac{1}{2}$ in (1.3) we recapture the result obtained in the earlier paper [8] by the author. For other similar inequalities for operator convex functions see [1] and [20]-[24].

In this paper we obtain among others the following Hermite-Hadamard type inequalities for operator convex functions $f : [0, \infty) \to \mathbb{R}$ and $A, B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$,

$$\begin{aligned}
0 &\leq \frac{1}{3}\left[\frac{f\left(|A|^2\right) + 2f\left(\operatorname{Re}\left(B^*A\right)\right) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right] \\
&\leq \frac{f\left(|A|^2\right) + f\left(\operatorname{Re}\left(B^*A\right)\right) + f\left(|B|^2\right)}{3} - \int_0^1 f\left(|(1-t)A + tB|^2\right) dt \\
&\leq \frac{7}{3}\left[\frac{f\left(|A|^2\right) + 2f\left(\operatorname{Re}\left(B^*A\right)\right) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right].
\end{aligned}$$

Some examples for power functions and logarithm are also provided.

2. Main Results

We consider the functional

$$J_n(\mathbf{p}; \mathbf{A}, f, I) := \sum_{j=1}^{n} p_j f(A_j) - P_n f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{2.1}$$

where $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$, $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint operators with $Sp(A_j) \subseteq I$ for $j \in \{1, ..., n\}$ and $f : I \to \mathbb{R}$ is a operator convex function defined on the interval I.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p} = (p_1, ..., p_n)$, $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n > 0$. For $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1, ..., n\}$.

Lemma 1. *Assume that $f : I \to \mathbb{R}$ is an operator convex function and $\mathbf{A} = (A_1, ..., A_n)$ an n-tuple of selfadjoint operators with $Sp(A_j) \subseteq I$, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, f, I) \geq J_n(\mathbf{p}; \mathbf{A}, f, I) + J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{2.2}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p}; \mathbf{A}, f, I) \geq J_n(\mathbf{q}; \mathbf{A}, f, I) \geq 0, \tag{2.3}$$

i.e., $J_n(\cdot; \mathbf{A}, f, I)$ is a monotonic functional in the operator order.

Proof. We have

$$
\begin{aligned}
&J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I) \\
&=\sum_{j=1}^{n}(p_j+q_j)\,f(A_j)-(P_n+Q_n)\,f\left(\frac{1}{P_n+Q_n}\sum_{j=1}^{n}(p_j+q_j)\,A_j\right) \\
&=\sum_{j=1}^{n}(p_j+q_j)\,f(A_j) \\
&-(P_n+Q_n)\,f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)}{P_n+Q_n}\right).
\end{aligned}
\tag{2.4}
$$

Now, consider the operators

$$
A:=\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j \text{ and } B:=\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j.
$$

Then $Sp(A), Sp(B)\subseteq I$.

Applying the inequality (OC) for A and B given above and $\lambda=\frac{Q_n}{P_n+Q_n}$ we have

$$
\begin{aligned}
&f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)}{P_n+Q_n}\right) \\
&\leq\frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)
\end{aligned}
\tag{2.5}
$$

in the operator order.

Making use of (2.4) and (2.5) we have

$$
\begin{aligned}
&J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I) \\
&\geq\sum_{j=1}^{n}(p_j+q_j)\,f(A_j)-(P_n+Q_n) \\
&\times\left[\frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)\right] \\
&=\sum_{j=1}^{n}p_jf(A_j)-P_nf\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) \\
&+\sum_{j=1}^{n}q_jf(A_j)-Q_nf\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right) \\
&=J_n(\mathbf{p};\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)
\end{aligned}
\tag{2.6}
$$

in the operator order, and the inequality (2.2) is proved.

Now, let $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$. Then by the super-additivity property (2.2) we have

$$\begin{aligned} J_n(\mathbf{p};\mathbf{A},f,I) &= J_n((\mathbf{p}-\mathbf{q})+\mathbf{q};\mathbf{A},f,I) \\ &\geq J_n((\mathbf{p}-\mathbf{q});\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)\geq J_n(\mathbf{q};\mathbf{A},f,I) \end{aligned} \tag{2.7}$$

in the operator order, and the monotonicity property (2.3) is proved. □

Corollary 1. *Assume that the function $f:I\to\mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1,...,A_n)$ satisfies the condition $Sp(A_j)\subseteq I$ for any $j\in\{1,...,n\}$. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m,M such that*

$$m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q} \tag{2.8}$$

then

$$mJ_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq MJ_n(\mathbf{q};\mathbf{A},f,I) \tag{2.9}$$

in the operator order.

Proof. Observe that for $\alpha>0$ we have $J_n(\alpha\mathbf{p};\mathbf{A},f,I)=\alpha J_n(\mathbf{p};\mathbf{A},f,I)$.

Utilising the monotonicity property (2.3) we have

$$J_n(m\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq J_n(M\mathbf{q};\mathbf{A},f,I)$$

which imply the desired result (2.9). □

Remark 1. *We observe that if all $q_j>0$ then we have the inequality*

$$\begin{aligned} \min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I) &\leq J_n(\mathbf{p};\mathbf{A},f,I) \\ &\leq \max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I) \end{aligned} \tag{2.10}$$

in the operator order.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}, j\in\{1,...,n\}$, then we have the inequalities

$$n\min_{j\in\{1,\dots,n\}}\{p_j\}J_n(\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq n\max_{j\in\{1,\dots,n\}}\{p_j\}J_n(\mathbf{A},f,I) \tag{2.11}$$

where

$$J_n(\mathbf{A},f,I):=\frac{1}{n}\sum_{j=1}^n f(A_j)-f\left(\frac{1}{n}\sum_{j=1}^n A_j\right). \tag{2.12}$$

Theorem 2. *Let* $f:[0,\infty)\to\mathbb{R}$ *be operator convex on* $[0,\infty)$. *If* $A,\ B\in\mathcal{B}(H)$ *with* $\operatorname{Re}(B^*A)\geq 0$, *then for all* $t\in[0,1]$

(2.13)
$$0\leq 3m(t)\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\}$$
$$\leq(1-t)^2f\left(|A|^2\right)+2t(1-t)f(\operatorname{Re}(B^*A))+t^2f\left(|B|^2\right)-f\left(|(1-t)A+tB|^2\right)$$
$$\leq 3M(t)\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\},$$

where

$$m(t):=\begin{cases}t^2,\ t\in[0,1/2],\\ (1-t)^2,\ (1/2,1]\end{cases}\quad\text{and } M(t):=\begin{cases}(1-t)^2,\ t\in[0,1/3]\\ 2t(1-t),\ t\in(1/3,2/3)\\ t^2,\ t\in(2/3,1].\end{cases}$$

We also have the integral inequality

(2.14)
$$0\leq\frac{1}{4}\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\}$$
$$\leq\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-\int_0^1f\left(|(1-t)A+tB|^2\right)dt$$
$$\leq\frac{17}{4}\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\}.$$

Proof. Observe that, by the properties of modulus, we have for $t\in[0,1]$

$$\begin{aligned}&|(1-t)A+tB|^2\\&=((1-t)A+tB)^*(1-t)A+tB\\&=((1-t)A^*+tB^*)(1-t)A+tB\\&=(1-t)^2A^*A+t(1-t)B^*A+(1-t)tA^*B+t^2B^*B\\&=(1-t)^2|A|^2+2t(1-t)\operatorname{Re}(B^*A)+t^2|B|^2.\end{aligned}$$

Put

$$p_1=(1-t)^2,\ p_2=2t(1-t)\ \text{and } p_3=t^2,\ t\in[0,1].$$

Then

$$p_1+p_2+p_3=(1-t)^2+2t(1-t)+t^2=(1-t+t)^2=1$$

We use the inequality (2.11) for $n=3,\ A_1=|A|^2,\ A_2=\operatorname{Re}(B^*A)$ and $A_3=|B|^2$ to get

(2.15)

$$\begin{aligned}
&3\min\left\{(1-t)^2, 2t(1-t), t^2\right\}\\
&\times\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\}\\
&\le (1-t)^2 f\left(|A|^2\right)+2t(1-t) f(\operatorname{Re}(B^*A))+t^2 f\left(|B|^2\right)\\
&-f\left(|(1-t)A+tB|^2\right)\\
&\le 3\max\left\{(1-t)^2, 2t(1-t), t^2\right\}\\
&\times\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\}
\end{aligned}$$

for all $t\in[0,1]$.

Observe that

$$\max\left\{(1-t)^2, 2t(1-t), t^2\right\}=\begin{cases}(1-t)^2,\ t\in[0,1/3]\\ 2t(1-t),\ t\in(1/3,2/3)\\ t^2,\ t\in(2/3,1]\end{cases}=M(t)$$

and

$$\min\left\{(1-t)^2, 2t(1-t), t^2\right\}=\begin{cases}t^2,\ t\in[0,1/2],\\ (1-t)^2,\ (1/2,1]\end{cases}=m(t)$$

and by (2.15) we deduce (2.13).

Now, if we take the integral in (2.13), then we get

(2.16)

$$\begin{aligned}
&0\le 3\int_0^1 m(t)\,dt\\
&\times\left\{\frac{f\left(|A|^2\right)+f(\operatorname{Re}(B^*A))+f\left(|B|^2\right)}{3}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\right\}
\end{aligned}$$

$$\leq \left(\int_0^1 (1-t)^2\,dt\right) f\left(|A|^2\right) + 2\left(\int_0^1 t(1-t)\,dt\right) f(\operatorname{Re}(B^*A))$$
$$+ \left(\int_0^1 t^2 dt\right) f\left(|B|^2\right) - \int_0^1 f\left(|(1-t)A + tB|^2\right) dt$$
$$\leq 3\int_0^1 M(t)\,dt$$
$$\times \left\{ \frac{f\left(|A|^2\right) + f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{3} - f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \right\}.$$

Since

$$\int_0^1 m(t)\,dt = \frac{1}{12},\quad \int_0^1 M(t)\,dt = \frac{17}{12}$$

and

$$\int_0^1 (1-t)^2\,dt = 2\int_0^1 t(1-t)\,dt = \int_0^1 t^2 dt = \frac{1}{3},$$

hence by (2.16) we derive (2.14). □

Theorem 3. *Let* $f : [0,\infty) \to \mathbb{R}$ *be operator convex on* $[0,\infty)$*. If* $A,\ B \in \mathcal{B}(H)$ *with* $\operatorname{Re}(B^*A) \geq 0$*, then for all* $t \in [0,1]$

$$0 \leq 4q(t)\left[\frac{f\left(|A|^2\right) + 2f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right] \tag{2.17}$$
$$\leq (1-t)^2 f\left(|A|^2\right) + 2t(1-t) f(\operatorname{Re}(B^*A)) + t^2 f\left(|B|^2\right)$$
$$- f\left(|(1-t)A + tB|^2\right)$$
$$\leq 4Q(t)\left[\frac{f\left(|A|^2\right) + 2f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right],$$

where

$$q(t) = \begin{cases} t^2,\ t \in [0,1/2], \\ (1-t)^2,\ t \in (1/2,1] \end{cases} \quad \text{and } Q(t) = \begin{cases} (1-t)^2,\ t \in [0,1/2], \\ t^2,\ t \in (1/2,1]. \end{cases}$$

We also have the integral inequality

$$0 \leq \frac{1}{3}\left[\frac{f\left(|A|^2\right) + 2f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right] \tag{2.18}$$
$$\leq \frac{f\left(|A|^2\right) + f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{3} - \int_0^1 f\left(|(1-t)A + tB|^2\right) dt$$
$$\leq \frac{7}{3}\left[\frac{f\left(|A|^2\right) + 2f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right].$$

Proof. Consider $q_1 = \frac{1}{4}$, $q_2 = \frac{1}{2}$, $q_3 = \frac{1}{4}$, then $q_1 + q_2 + q_3 = 1$. Also, if we put

$$p_1 = (1-t)^2, \; p_2 = 2t(1-t) \text{ and } p_3 = t^2, \; t \in [0,1].$$

Then

$$p_1 + p_2 + p_3 = (1-t)^2 + 2t(1-t) + t^2 = (1-t+t)^2 = 1.$$

Now if we consider $A_1 = |A|^2$, $A_2 = \operatorname{Re}(B^* A)$ and $A_3 = |B|^2$, then by (2.10) we derive

$$\begin{aligned}
0 &\le \min\left\{4(1-t)^2, 4t(1-t), 4t^2\right\} \\
&\times \left[\frac{1}{4} f\left(|A|^2\right) + \frac{1}{2} f\left(\operatorname{Re}(B^* A)\right) + \frac{1}{4} f\left(|B|^2\right)\right. \\
&\left. - f\left(\frac{1}{4}|A|^2 + \frac{1}{2}\operatorname{Re}(B^* A) + \frac{1}{4}|B|^2\right)\right] \\
&\le (1-t)^2 f\left(|A|^2\right) + 2t(1-t) f\left(\operatorname{Re}(B^* A)\right) + t^2 f\left(|B|^2\right) \\
&- f\left(|(1-t)A + tB|^2\right) \\
&\le \max\left\{4(1-t)^2, 4t(1-t), 4t^2\right\} \\
&\times \left[\frac{1}{4} f\left(|A|^2\right) + \frac{1}{2} f\left(\operatorname{Re}(B^* A)\right) + \frac{1}{4} f\left(|B|^2\right)\right. \\
&\left. - f\left(\frac{1}{4}|A|^2 + \frac{1}{2}\operatorname{Re}(B^* A) + \frac{1}{4}|B|^2\right)\right],
\end{aligned}$$

namely

$$\begin{aligned}
0 &\le 4\min\left\{(1-t)^2, t(1-t), t^2\right\} \\
&\times \left[\frac{1}{4} f\left(|A|^2\right) + \frac{1}{2} f\left(\operatorname{Re}(B^* A)\right) + \frac{1}{4} f\left(|B|^2\right) - f\left(\left|\frac{A+B}{2}\right|^2\right)\right] \\
&\le (1-t)^2 f\left(|A|^2\right) + 2t(1-t) f\left(\operatorname{Re}(B^* A)\right) + t^2 f\left(|B|^2\right) \\
&- f\left(|(1-t)A + tB|^2\right) \\
&\le 4\max\left\{(1-t)^2, t(1-t), t^2\right\} \\
&\times \left[\frac{1}{4} f\left(|A|^2\right) + \frac{1}{2} f\left(\operatorname{Re}(B^* A)\right) + \frac{1}{4} f\left(|B|^2\right) - f\left(\left|\frac{A+B}{2}\right|^2\right)\right]
\end{aligned}$$

for all $t \in [0,1]$.

We observe that

$$\begin{aligned}
\min\left\{(1-t)^2, t(1-t), t^2\right\} &= \begin{cases} t^2, \; t \in [0, 1/2], \\[2ex] (1-t)^2, \; t \in (1/2, 1] \end{cases} \\
&= q(t)
\end{aligned}$$

and

$$\max\left\{(1-t)^2, t(1-t), t^2\right\} = \begin{cases} (1-t)^2, \ t \in [0,1/2], \\ \\ t^2, \ t \in (1/2,1] \end{cases}$$
$$= Q(t)$$

for all $t \in [0,1]$, which proves (2.17).

If we take the integral over t in (2.17) and take into account that

$$\int_0^1 Q(t)dt = \frac{7}{12} \text{ and } \int_0^1 q(t)\,dt = \frac{1}{12},$$

then we obtain the desired inequality (2.18). □

3. Some Examples

Assume that $A, B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$, then by writing inequality (2.18) for the operator convex function $f(t) = t^r$ for $r \in [1,2]$ we get

(3.1)
$$0 \leq \frac{1}{3}\left[\frac{|A|^{2r} + 2[\operatorname{Re}(B^*A)]^r + |B|^{2r}}{4} - \left|\frac{A+B}{2}\right|^{2r}\right]$$
$$\leq \frac{|A|^{2r} + [\operatorname{Re}(B^*A)]^r + |B|^{2r}}{3} - \int_0^1 |(1-t)A + tB|^{2r}\,dt$$
$$\leq \frac{7}{3}\left[\frac{|A|^{2r} + 2[\operatorname{Re}(B^*A)]^r + |B|^{2r}}{4} - \left|\frac{A+B}{2}\right|^{2r}\right].$$

If $p \in (0,1)$ then by (2.18) for the operator concave function $f(t) = t^p$ we get

(3.2)
$$0 \leq \frac{1}{3}\left[\left|\frac{A+B}{2}\right|^{2p} - \frac{|A|^{2p} + 2[\operatorname{Re}(B^*A)]^p + |B|^{2p}}{4}\right]$$
$$\leq \int_0^1 |(1-t)A + tB|^{2p}\,dt - \frac{|A|^{2p} + [\operatorname{Re}(B^*A)]^p + |B|^{2p}}{3}$$
$$\leq \frac{7}{3}\left[\left|\frac{A+B}{2}\right|^{2p} - \frac{|A|^{2p} + 2[\operatorname{Re}(B^*A)]^p + |B|^{2p}}{4}\right].$$

In particular, for $p = 1/2$ we get

(3.3)
$$0 \leq \frac{1}{3}\left[\left|\frac{A+B}{2}\right| - \frac{|A| + 2[\operatorname{Re}(B^*A)]^{1/2} + |B|}{4}\right]$$
$$\leq \int_0^1 |(1-t)A + tB|\,dt - \frac{|A| + [\operatorname{Re}(B^*A)]^{1/2} + |B|}{3}$$
$$\leq \frac{7}{3}\left[\left|\frac{A+B}{2}\right|^{2p} - \frac{|A|^{2p} + 2[\operatorname{Re}(B^*A)]^p + |B|^{2p}}{4}\right]$$

provided that $A, B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$.

If $|A|^2,\ |B|^2,\ \mathrm{Re}(B^*A)>0$ then we have for the operator concave function $f(t)=\ln t$ on $(0,\infty)$ that

$$0\le\frac{1}{3}\left[\ln\left(\left|\frac{A+B}{2}\right|^2\right)-\frac{\ln\left(|A|^2\right)+2\ln\left(\mathrm{Re}(B^*A)\right)+\ln\left(|B|^2\right)}{4}\right] \tag{3.4}$$
$$\le\int_0^1\ln\left(|(1-t)A+tB|^2\right)dt-\frac{\ln\left(|A|^2\right)+\ln\left(\mathrm{Re}(B^*A)\right)+\ln\left(|B|^2\right)}{3}$$
$$\le\frac{7}{3}\left[\ln\left(\left|\frac{A+B}{2}\right|^2\right)-\frac{\ln\left(|A|^2\right)+2\ln\left(\mathrm{Re}(B^*A)\right)+\ln\left(|B|^2\right)}{4}\right].$$

If $|A|^2,\ |B|^2,\ \mathrm{Re}(B^*A)>0$ then we have for the operator convex function $f(t)=t^{-p}$ on $(0,\infty)$ with $p\in(0,1]$ that

$$0\le\frac{1}{3}\left[\frac{|A|^{-2p}+2\left(\mathrm{Re}(B^*A)\right)^{-p}+|B|^{-2p}}{4}-\left|\frac{A+B}{2}\right|^{-2p}\right] \tag{3.5}$$
$$\le\frac{|A|^{-2p}+\left(\mathrm{Re}(B^*A)\right)^{-p}+|B|^{-2p}}{3}-\int_0^1|(1-t)A+tB|^{-2p}dt$$
$$\le\frac{7}{3}\left[\frac{|A|^{-2p}+2\left(\mathrm{Re}(B^*A)\right)^{-p}+|B|^{-2p}}{4}-\left|\frac{A+B}{2}\right|^{-2p}\right].$$

In particular, we have

$$0\le\frac{1}{3}\left[\frac{|A|^{-2}+2\left(\mathrm{Re}(B^*A)\right)^{-1}+|B|^{-2}}{4}-\left|\frac{A+B}{2}\right|^{-2}\right] \tag{3.6}$$
$$\le\frac{|A|^{-2}+\left(\mathrm{Re}(B^*A)\right)^{-1}+|B|^{-2}}{3}-\int_0^1|(1-t)A+tB|^{-2}dt$$
$$\le\frac{7}{3}\left[\frac{|A|^{-2}+2\left(\mathrm{Re}(B^*A)\right)^{-1}+|B|^{-2}}{4}-\left|\frac{A+B}{2}\right|^{-2}\right]$$

and

$$0\le\frac{1}{3}\left[\frac{|A|^{-1}+2\left(\mathrm{Re}(B^*A)\right)^{-1/2}+|B|^{-1}}{4}-\left|\frac{A+B}{2}\right|^{-1}\right] \tag{3.7}$$
$$\le\frac{|A|^{-1}+\left(\mathrm{Re}(B^*A)\right)^{-1/2}+|B|^{-1}}{3}-\int_0^1|(1-t)A+tB|^{-1}dt$$
$$\le\frac{7}{3}\left[\frac{|A|^{-1}+2\left(\mathrm{Re}(B^*A)\right)^{-1/2}+|B|^{-1}}{4}-\left|\frac{A+B}{2}\right|^{-1}\right],$$

provided that $|A|^2,\ |B|^2,\ \mathrm{Re}(B^*A)>0$.

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