# SOME DETERMINANT POWER INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA JENSEN'S INEQUALITY FOR EXPONENTIAL FUNCTION

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we prove among others that, if $(A_j)_{j=1,...,m}$ are positive definite matrices of order $n$ and $p_j \geq 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} p_j = 1$, then

$$0 \leq m \min_{i\in\{1,...,m\}} \{p_i\} \left( \frac{1}{m} \sum_{i=1}^{m} [\det (A_i)]^{-p} - \left[ \det \left( \frac{1}{m} \sum_{i=1}^{m} A_i \right) \right]^{-p} \right)$$
$$\leq \sum_{i=1}^{m} p_i [\det (A_i)]^{-p} - \left[ \det \left( \sum_{i=1}^{m} p_i A_i \right) \right]^{-p}$$
$$\leq m \max_{i\in\{1,...,m\}} \{p_i\} \left( \frac{1}{m} \sum_{i=1}^{m} [\det (A_i)]^{-p} - \left[ \det \left( \frac{1}{m} \sum_{i=1}^{m} A_i \right) \right]^{-p} \right)$$

for all natural number $p$.

## 1. Introduction

A real square matrix $A = (a_{ij})$, $i, j = 1, ..., n$ is *symmetric* provided $a_{ij} = a_{ji}$ for all $i, j = 1, ..., n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is positive for all $x = (x_1, ..., x_n) \in \mathbb{R}^n \setminus \{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix $A$ to be positive definite, and we write $A > 0$, is that all determinants

$$\det(A_k) = \det (a_{ij}),\ i,\ j = 1, ..., k;\ k = 1, ..., n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [9, pp. 211-212]

$$J_n(A) := \int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx := \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\langle Ax, x\rangle)dx = \frac{\pi^{n/2}}{[\det (A)]^{1/2}}, \tag{1.1}$$

where $A$ is a positive definite matrix of order $n$ and $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to

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Ky Fan ([1, p. 63] or [9, p. 212]), namely

$$\det\left((1-\lambda)A+\lambda B\right)\geq\left[\det(A)\right]^{1-\lambda}\left[\det(B)\right]^{\lambda} \tag{1.2}$$

for any positive definite matrices $A$, $B$ and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (1.2) which was obtained by L. Mirsky in [8], see also [9, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det(A_j)\right]^{\lambda_j},\ m\geq 2, \tag{1.3}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.3) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det(B_j)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det(A_j)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.4}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.1) one can also prove the result, see [9, p. 212],

$$\det(A)=\det(A_{1n})\leq\det(A_{1k})\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.5}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs})=\det(a_{ij}),i,\ j=r,...,s.$$

In particular,

$$\det(A)\leq a_{11}a_{22}...a_{nn}. \tag{1.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det(A+B)\right]^{1/n}\geq\left[\det(A)\right]^{1/n}+\left[\det(B)\right]^{1/n} \tag{1.7}$$

for $A$, $B$ positive definite matrices of order $n$. For other determinant inequalities see Chapter VIII of the classic book [9]. For some recent results see [3]-[7].

Motivated by the above results, we prove in the present paper that, if $A_i$, $i\in\{1,...,m\}$ are positive definite matrices, $\{p_i\}_{i\in\{1,...,m\}}$ are nonnegative numbers with $\sum_{i=1}^{m}p_i=1$, then for $p$ a natural number $\geq 1$ we have

$$\begin{aligned}0&\leq m\min_{i\in\{1,...,m\}}\{p_i\}\left(\frac{1}{m}\sum_{i=1}^{m}\left[\det(A_i)\right]^{-p}-\left[\det\left(\frac{1}{m}\sum_{i=1}^{m}A_i\right)\right]^{-p}\right)\\&\leq\sum_{i=1}^{m}p_i\left[\det(A_i)\right]^{-p}-\left[\det\quad\sum_{i=1}^{m}p_iA_i\right)\right]^{-p}\\&\leq m\max_{i\in\{1,...,m\}}\{p_i\}\left(\frac{1}{m}\sum_{i=1}^{m}\left[\det(A_i)\right]^{-p}-\left[\det\quad\frac{1}{m}\sum_{i=1}^{m}A_i\right)\right]^{-p}\right).\end{aligned}$$

## 2. Main Results

We have the following representation result:

**Lemma 1.** *Assume that $A$ is a positive definite matrix of order $n \geq 2$ and $k \geq 2$ a natural number, then*

$$[\det(A)]^{-k/2} = \frac{1}{\pi^{kn/2}} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{k} \langle Ay_j, y_j \rangle) dy_1 \dots dy_k \tag{2.1}$$

*and*

$$[\det(A)]^{k/2} = \frac{1}{\pi^{kn/2}} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{k} \langle A^{-1} y_j, y_j \rangle) dy_1 \dots dy_k. \tag{2.2}$$

*Proof.* By taking the power $k$ in (1.1) we get

$$[\det(A)]^{-k/2} = \frac{1}{\pi^{kn/2}} \left( \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx \right)^k. \tag{2.3}$$

Using the multiple integrals, we have

$$\begin{aligned} &\left( \int_{\mathbb{R}^n} \exp(-\langle Ax, x \rangle) dx \right)^k \\ &= \int_{\mathbb{R}^n} \exp(-\langle Ay_1, y_1 \rangle) dy_1 \dots \int_{\mathbb{R}^n} \exp(-\langle Ay_k, y_k \rangle) dy_k \\ &= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{k} \langle Ay_j, y_j \rangle) dy_1 \dots dy_k. \end{aligned} \tag{2.4}$$

where $y_1, \dots, y_k \in \mathbb{R}^n$.

By utilizing (2.3) and (2.4) we derive (2.1).

Since, by the properties of determinants and by (2.1) we derive

$$\begin{aligned} [\det(A)]^{k/2} &= \left[\det\left(A^{-1}\right)\right]^{-k/2} = \frac{1}{\pi^{kn/2}} \left( \int_{\mathbb{R}^n} \exp(-\left\langle A^{-1} x, x \right\rangle) dx \right)^k \\ &= \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{k} \left\langle A^{-1} y_j, y_j \right\rangle) dy_1 \dots dy_k, \end{aligned}$$

which proves (2.2). □

**Remark 1.** *If $k = 2p$, $p$ a natural number $\geq 1$, then we have*

$$[\det(A)]^{-p} = \frac{1}{\pi^{pn}} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{2p} \langle Ay_j, y_j \rangle) dy_1 \dots dy_{2p} \tag{2.5}$$

*and*

$$[\det(A)]^{p} = \frac{1}{\pi^{pn}} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{2p} \left\langle A^{-1} y_j, y_j \right\rangle) dy_1 \dots dy_{2p}. \tag{2.6}$$

*If $k = 2p + 1$, $p$ a natural number, then we have*

$$[\det(A)]^{-p-1/2} = \frac{1}{\pi^{kn/2}} \int_{\mathbb{R}^n} \dots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{2p+1} \langle Ay_j, y_j \rangle) dy_1 \dots dy_{2p+1} \tag{2.7}$$

*and*

$$[\det(A)]^{p+1/2} = \frac{1}{\pi^{kn/2}} \int_{\mathbb{R}^n} \ldots \int_{\mathbb{R}^n} \exp(-\sum_{j=1}^{2p+1} \left\langle A^{-1} y_j, y_j \right\rangle) dy_1 \ldots dy_{2p+1}. \tag{2.8}$$

*We observe that for $p = 1$ we get*

$$[\det(A)]^{-1} = \frac{1}{\pi^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle Ay, y\rangle - \langle Az, z\rangle) dy dz \tag{2.9}$$

*and*

$$\det(A) = \frac{1}{\pi^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\left\langle A^{-1} y, y\right\rangle - \left\langle A^{-1} z, z\right\rangle) dy dz. \tag{2.10}$$

Our main result is as follows:

**Theorem 1.** *Assume that $A_i$, $i \in \{1, ..., m\}$ are positive definite matrices and $\{p_i\}_{i\in\{1,...,m\}}$ are nonnegative numbers with $\sum_{i=1}^m p_i = 1$. Then for $p$ a natural number $\geq 1$ we have*

$$\begin{aligned} 0 &\leq m \min_{i\in\{1,...,m\}} \{p_i\} \left( \frac{1}{m} \sum_{i=1}^m [\det(A_i)]^{-p} - \left[\det \left( \frac{1}{m}\sum_{i=1}^m A_i\right)\right]^{-p}\right) \\ &\leq \sum_{i=1}^m p_i [\det(A_i)]^{-p} - \left[\det \left(\sum_{i=1}^m p_i A_i\right)\right]^{-p} \\ &\leq m \max_{i\in\{1,...,m\}} \{p_i\} \left( \frac{1}{m} \sum_{i=1}^m [\det(A_i)]^{-p} - \left[\det \left( \frac{1}{m}\sum_{i=1}^m A_i\right)\right]^{-p}\right). \end{aligned} \tag{2.11}$$

*Also,*

$$\begin{aligned} 0 &\leq m \min_{i\in\{1,...,m\}} \{p_i\} \left( \frac{1}{m} \sum_{i=1}^m [\det(A_i)]^{p} - \left[\det \left( \frac{1}{m}\sum_{i=1}^m A_i^{-1}\right)\right]^{-p}\right) \\ &\leq \sum_{i=1}^m p_i [\det(A_i)]^{p} - \left[\det \left(\sum_{i=1}^m p_i A_i^{-1}\right)\right]^{-p} \\ &\leq m \max_{i\in\{1,...,m\}} \{p_i\} \left( \frac{1}{m} \sum_{i=1}^m [\det(A_i)]^{p} - \left[\det \left( \frac{1}{m}\sum_{i=1}^m A_i^{-1}\right)\right]^{-p}\right). \end{aligned} \tag{2.12}$$

*Proof.* Observe that, by (2.5) we have

$$\begin{aligned}(2.13)\qquad & \left[\det\left(\sum_{i=1}^{m} p_i A_i\right)\right]^{-p} \\ &= \frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}...\int_{\mathbb{R}^n}\exp(-\sum_{j=1}^{2p}\left\langle\left(\sum_{i=1}^{m} p_i A_i\right)y_j, y_j\right\rangle)dy_1...dy_{2p} \\ &= \frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}...\int_{\mathbb{R}^n}\exp(-\sum_{i=1}^{m} p_i\left(\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right))dy_1...dy_{2p} \\ &= \frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}...\int_{\mathbb{R}^n}\exp(\sum_{i=1}^{m} p_i\left(\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right))dy_1...dy_{2p}.\end{aligned}$$

We recall the following result obtained by the author in [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned}(2.14)\qquad 0 &\le m\min_{i\in\{1,...,m\}}\{p_i\}\left[\frac{1}{m}\sum_{i=1}^{m}\Phi(z_i) - \Phi\left(\frac{1}{m}\sum_{i=1}^{m} z_i\right)\right] \\ &\le \sum_{i=1}^{m} p_i\Phi(z_i) - \Phi\left(\sum_{i=1}^{m} p_i z_i\right) \\ &\le m\max_{i\in\{1,...,m\}}\{p_i\}\left[\frac{1}{m}\sum_{i=1}^{m}\Phi(z_i) - \Phi\left(\frac{1}{m}\sum_{i=1}^{m} z_i\right)\right],\end{aligned}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset $C$ of the linear space $X$, $\{z_i\}_{i\in\{1,...,m\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,...,m\}}$ are nonnegative numbers with $\sum_{i=1}^{m} p_i = 1$.

Now, if we take $\Phi(z) = \exp z$, $z_i := \sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle$ where $y_1, ..., y_k \in \mathbb{R}^n$ in (2.14) then we get

$$\begin{aligned}0 &\le m\min_{i\in\{1,...,m\}}\{p_i\} \\ &\times\left[\frac{1}{m}\sum_{i=1}^{m}\exp\left(\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right) - \exp\left(\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right)\right] \\ &\le \sum_{i=1}^{m} p_i\exp\left(\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right) - \exp\left(\sum_{i=1}^{m} p_i\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right) \\ &\le m\max_{i\in\{1,...,m\}}\{p_i\} \\ &\times\left[\frac{1}{m}\sum_{i=1}^{m}\exp\left(\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right) - \exp\left(\frac{1}{m}\sum_{i=1}^{m}\sum_{j=1}^{2p}\langle -A_i y_j, y_j\rangle\right)\right],\end{aligned}$$

namely

$$\begin{aligned}
0 &\le m \min_{i\in\{1,\dots,m\}}\{p_i\} \\
&\times\left[\frac{1}{m}\sum_{i=1}^{m}\exp\left(-\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right)-\exp\left(-\sum_{j=1}^{2p}\left\langle\left(\frac{1}{m}\sum_{i=1}^{m}A_i\right)y_j, y_j\right\rangle\right)\right] \\
&\le \sum_{i=1}^{m}p_i\exp\left(-\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right)-\exp\left(-\sum_{j=1}^{2p}\left\langle\left(\sum_{i=1}^{m}p_iA_i\right)y_j, y_j\right\rangle\right) \\
&\le m \max_{i\in\{1,\dots,m\}}\{p_i\} \\
&\times\left[\frac{1}{m}\sum_{i=1}^{m}\exp\left(-\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right)-\exp\left(-\sum_{j=1}^{2p}\left\langle\left(\frac{1}{m}\sum_{i=1}^{m}A_i\right)y_j, y_j\right\rangle\right)\right],
\end{aligned}$$

for all $y_1, \dots, y_k \in \mathbb{R}^n$.

If we take the multiple integral on $\mathbb{R}^n \times \dots \times \mathbb{R}^n$, then we get

$$\begin{aligned}
0 &\le m \min_{i\in\{1,\dots,m\}}\{p_i\} \\
&\times\left[\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}\dots\int_{\mathbb{R}^n}\exp\left(-\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right)dy_1\dots dy_{2p}\right. \\
&\left.-\frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}\dots\int_{\mathbb{R}^n}\exp\left(-\sum_{j=1}^{2p}\left\langle\left(\frac{1}{m}\sum_{i=1}^{m}A_i\right)y_j, y_j\right\rangle dy_1\dots dy_{2p}\right)\right] \\
&\le \frac{1}{\pi^{pn}}\sum_{i=1}^{m}p_i\int_{\mathbb{R}^n}\dots\int_{\mathbb{R}^n}\exp\left(-\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right)dy_1\dots dy_{2p} \\
&-\frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}\dots\int_{\mathbb{R}^n}\exp\left(-\sum_{j=1}^{2p}\left\langle\left(\sum_{i=1}^{m}p_iA_i\right)y_j, y_j\right\rangle\right)dy_1\dots dy_{2p} \\
&\le m \max_{i\in\{1,\dots,m\}}\{p_i\} \\
&\times\left[\frac{1}{m}\sum_{i=1}^{m}\frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}\dots\int_{\mathbb{R}^n}\exp\left(-\sum_{j=1}^{2p}\langle A_i y_j, y_j\rangle\right)dy_1\dots dy_{2p}\right. \\
&\left.-\frac{1}{\pi^{pn}}\int_{\mathbb{R}^n}\dots\int_{\mathbb{R}^n}\exp\left(-\sum_{j=1}^{2p}\left\langle\left(\frac{1}{m}\sum_{i=1}^{m}A_i\right)y_j, y_j\right\rangle dy_1\dots dy_{2p}\right)\right],
\end{aligned}$$

and by representation (2.5) we get (2.11).

The inequality (2.12) follows by (2.11) by replacing $A_i$ with $A_i^{-1}$, $i \in \{1, \dots, n\}$. □

**Remark 2.** *Assume that* $A$, $B$ *are positive definite matrices and* $p \geq 1$, *then for all* $t \in [0,1]$

$$\begin{aligned}(2.15)\qquad & 0 \leq 2\min\{t, 1-t\} \\ & \times \left(\frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \left[\det\left(\frac{1}{2}(A+B)\right)\right]^{-p}\right) \\ & \leq (1-t)[\det(A)]^{-p} + t[\det(B)]^{-p} - [\det((1-t)A + tB)]^{-p} \\ & \leq 2\max\{t, 1-t\} \\ & \times \left(\frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \left[\det\left(\frac{1}{2}(A+B)\right)\right]^{-p}\right).\end{aligned}$$

*Also,*

$$\begin{aligned}(2.16)\qquad & 0 \leq 2\min\{t, 1-t\} \\ & \times \left(\frac{1}{2}([\det(A)]^{p} + [\det(B)]^{p}) - \left[\det\left(\frac{1}{2}\left(A^{-1}+B^{-1}\right)\right)\right]^{-p}\right) \\ & \leq (1-t)[\det(A)]^{p} + t[\det(B)]^{p} - \left[\det\left((1-t)A^{-1} + tB^{-1}\right)\right]^{-p} \\ & \leq 2\max\{t, 1-t\} \\ & \times \left(\frac{1}{2}([\det(A)]^{p} + [\det(B)]^{p}) - \left[\det\left(\frac{1}{2}\left(A^{-1}+B^{-1}\right)\right)\right]^{-p}\right).\end{aligned}$$

**Corollary 1.** *Assume that* $A$, $B$ *are positive definite matrices and* $p \geq 1$, *then for all* $t \in [0,1]$

$$\begin{aligned}(2.17)\qquad & 0 \leq \frac{1}{2}\left(\frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \left[\det\left(\frac{1}{2}(A+B)\right)\right]^{-p}\right) \\ & \leq \frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \int_0^1 [\det((1-t)A + tB)]^{-p}\,dt \\ & \leq \frac{3}{2}\left(\frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \left[\det\left(\frac{1}{2}(A+B)\right)\right]^{-p}\right).\end{aligned}$$

*Also,*

$$\begin{aligned}(2.18)\qquad & 0 \leq \frac{1}{2}\left(\frac{1}{2}([\det(A)]^{p} + [\det(B)]^{p}) - \left[\det\left(\frac{1}{2}\left(A^{-1}+B^{-1}\right)\right)\right]^{-p}\right) \\ & \leq \frac{1}{2}([\det(A)]^{p} + [\det(B)]^{p}) - \int_0^1 \left[\det\left((1-t)A^{-1} + tB^{-1}\right)\right]^{-p}\,dt \\ & \leq \frac{3}{2}\left(\frac{1}{2}([\det(A)]^{p} + [\det(B)]^{p}) - \left[\det\left(\frac{1}{2}\left(A^{-1}+B^{-1}\right)\right)\right]^{-p}\right).\end{aligned}$$

*Proof.* We take the integral over $t \in [0,1]$ to get Assume that $A$, $B$ are positive definite matrices and $p \geq 1$, then for all $t \in [0,1]$

$$
\begin{aligned}
(2.19)\qquad 0 &\leq 2\int_0^1 \min\{t, 1-t\}\, dt \\
&\times \frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \left[\det\left(\frac{1}{2}(A+B)\right)\right]^{-p}\Bigg) \\
&\leq \left(\int_0^1 (1-t)\, dt\right)[\det(A)]^{-p} + \left(\int_0^1 t dt\right)[\det(B)]^{-p} \\
&- \int_0^1 [\det((1-t)A + tB)]^{-p}\, dt \\
&\leq 2\int_0^1 \max\{t, 1-t\}\, dt \\
&\times \frac{1}{2}\left([\det(A)]^{-p} + [\det(B)]^{-p}\right) - \left[\det\left(\frac{1}{2}(A+B)\right)\right]^{-p}\Bigg).
\end{aligned}
$$

Since

$$
\int_0^1 \min\{t, 1-t\}\, dt = \frac{1}{4} \text{ and } \int_0^1 \max\{t, 1-t\}\, dt = \frac{3}{4},
$$

hence by (2.19) we derive (2.17). □

We also have:

**Theorem 2.** *Assume that $A_i$, $i \in \{1, ..., m\}$ are positive definite matrices and $\{p_i\}_{i\in\{1,...,m\}}$ are nonnegative numbers with $\sum_{i=1}^m p_i = 1$. Then for $p$ a natural number, we have*

$$
\begin{aligned}
(2.20)\qquad 0 &\leq m \min_{i\in\{1,...,m\}}\{p_i\} \\
&\times \left(\frac{1}{m}\sum_{i=1}^m [\det(A_i)]^{-p-1/2} - \left[\det \quad \frac{1}{m}\sum_{i=1}^m A_i\right)\right]^{-p-1/2}\right) \\
&\leq \sum_{i=1}^m p_i [\det(A_i)]^{-p-1/2} - \left[\det \quad \sum_{i=1}^m p_i A_i\right)\right]^{-p-1/2} \\
&\leq m \max_{i\in\{1,...,m\}}\{p_i\} \\
&\times \left(\frac{1}{m}\sum_{i=1}^m [\det(A_i)]^{-p-1/2} - \left[\det \quad \frac{1}{m}\sum_{i=1}^m A_i\right)\right]^{-p-1/2}\right).
\end{aligned}
$$

*Also,*

$$0 \leq m \min_{i\in\{1,...,m\}} \{p_i\} \times \left( \frac{1}{m} \sum_{i=1}^{m} [\det(A_i)]^{p+1/2} - \left[ \det \left( \frac{1}{m} \sum_{i=1}^{m} A_i^{-1} \right) \right]^{p+1/2} \right) \leq \sum_{i=1}^{m} p_i [\det(A_i)]^{p+1/2} - \left[ \det \left( \sum_{i=1}^{m} p_i A_i^{-1} \right) \right]^{p+1/2} \leq m \max_{i\in\{1,...,m\}} \{p_i\} \times \left( \frac{1}{m} \sum_{i=1}^{m} [\det(A_i)]^{p+1/2} - \left[ \det \left( \frac{1}{m} \sum_{i=1}^{m} A_i^{-1} \right) \right]^{p+1/2} \right). \tag{2.21}$$

The proof is similar to the one above by utilizing the representation (2.7) and we omit the details.

## 3. The Case of Hermitian Matrices

A complex square matrix $H = (h_{ij})$, $i, j = 1, ..., n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all $i$, $j = 1, ..., n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^{n} a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1, ..., z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [9, p. 215], for a positive definite Hermitian matrix $H$, we have

$$K_n(H) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp(-\langle \overline{z}, Hz \rangle) \, dxdy = \frac{\pi^n}{\det(H)}, \tag{3.1}$$

where $z = x + iy$ and $dx$ and $dy$ denote integration over real $n$-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$.

As shown in Lemma 1 we can show that if $H$ is a positive definite Hermitian matrix of order $n \geq 2$ and $k \geq 2$ a natural number, then

$$[\det(H)]^{-k} = \frac{1}{\pi^{kn}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ... \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left( -\sum_{j=1}^{k} \langle \overline{z}_j, Hz_j \rangle \right) dx_1 dy_1 ... dx_k dy_k \tag{3.2}$$

where $z_k = x_k + iy_k$ and $dx_k$ and $dy_k$ denote integration over real $n$-dimensional space $\mathbb{R}^n$.

Also, we have

$$[\det(H)]^{k} = \frac{1}{\pi^{kn}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ... \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left( -\sum_{j=1}^{k} \langle \overline{z}_j, H^{-1} z_j \rangle \right) dx_1 dy_1 ... dx_k dy_k. \tag{3.3}$$

By utilizing these representations we can obtain the corresponding inequalities for positive definite Hermitian matrices. However the details are nor provided here.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

*E-mail address*: sever.dragomir@vu.edu.au

*URL*: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa