On Certain Properties of sub E-functions

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ABSTRACT: The aim of this work is to study the standard functional operations of sub E-functions. Furthermore, we introduce a class BE[a, b] of functions representable as a difference of two sub E-functions.

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1 Introduction

Suppose that $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ along with $a < b$. There are a lot of generalizations of the concept of convex functions see [1, 3, 4, 5]. One way to generalize the concept of convex functions is to replace linear functions by another family of functions in the sense of Beckenbach [2]. In 2016, Mohamed S. S. Ali [1] introduced sub E-functions by dealing with a family $\{E(x)\}$ of exponential functions

$$E(x) = A \exp Bx,$$

where A, B are arbitrary constants. More precisely, [1, 6] a positive function $f : I \to (0, \infty)$ is said to be a sub E-function on I, if for all $x \in [a, b] \subset I$,

$$f(x) \leq E(x)$$

where A and B are chosen such that $E(a) = f(a)$, and $E(b) = f(b)$. In this paper, we deal with this family $\{E(x)\}$ of exponential functions which is called sub E-functions.

2 Definitions and Preliminary Results

This section is devoted to introduce the main definitions and results about sub E-functions see [1, 6, 7], which will be used in the following.

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Definition 2.1. *A positive function $f : I \to (0,\infty)$ is said to be a sub E-function on I, if for all $a, b \in I$ with $a < b$, the graph of $f(x)$ for $a \le x \le b$, lies on or under the graph of a function*

$$E(x) = Ae^{Bx},$$

where A and B are taken so that $E(a) = f(a)$, and $E(b) = f(b)$.
Equivalently, for all $x \in [a, b]$

$$\begin{aligned} f(x) &\le E(x) \\ &= exp\left[\frac{(b-x)\ln f(a) + (x-a)\ln f(b)}{b-a}\right]. \end{aligned} \tag{2.1}$$

There is more than one form for the function $E(x)$ other than that stated in (2.1); for example,

$$E(x) = f(a)e^{B(x-a)};\ B = \frac{\ln f(b) - \ln f(a)}{b-a}$$

or in a multiplicative formula

$$E(x) = [f(a)]^{\frac{b-x}{b-a}}.[f(b)]^{\frac{x-a}{b-a}}.$$

Remark 2.1. *The sub E-functions possess a number of properties analogous to those of convex functions. For example: Let $f : I \to (0,\infty)$ be a sub E-function, then for all $a, b \in I$, the inequality $f(x) \ge E(x)$ holds outside the interval $[a, b]$.*

Definition 2.2. *Assume that $f : I \to (0,\infty)$ is a sub E-function, then a function*

$$T_u(x) = Ae^{Bx}$$

is called a supporting function for $f(x)$ at the point $u \in (a, b)$ if

1. $T_u(u) = f(u)$
2. $T_u(x) \le f(x)\ \ \forall x \in I$.

That is, if $f(x)$ and $T_u(x)$ agree at $x = u$, then the graph of $f(x)$ lies on or above the support curve.

Definition 2.3. *Let $f : I \to \mathbb{R}$, f is said to satisfy a Lipschitz condition if there exists a constant $K > 0$ such that for every $x, y \in I$ we have*

$$|f(x) - f(y)| \le K|x - y|.$$

Theorem 2.1. *A function f that is F ρ-convex satisfies a Lipschitz condition in every compact subinterval J of (a, b), and thus is absolutely continuous and has a derivative almost everywhere that is bounded in J.*

Theorem 2.2. *If $f : [a, b] \to \mathbb{R}$ satisfies a Lipschitz condition on $[a, b]$ with constant K, then $f \in V[a, b]$ and $V_a^b(f) \le K(b-a)$.*

Proposition 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable sub E-function, then the supporting function for $f(x)$ at the point $u \in I$ has the formula*

$$T_u(x) = f(u) \exp\left[(x-u)\frac{f'(u)}{f(u)}\right]. \tag{2.2}$$

Proposition 2.2. *For a sub E-function $f : I \to (0,\infty)$, the supporting function at $u \in I$ is written in the following formula*

$$T_u(x) = f(u) \exp\left[(x-u)\frac{M_{u,f}}{f(u)}\right].$$

The constant $M_{u,f}$ is equal to $f'(u)$ if f is differentiable at the point $u \in I$; otherwise $f'_-(u) \leq M_{u,f} \leq f'_+(u)$.

Theorem 2.3. *If $f : I \to (0,\infty)$ is a two-times continuously differentiable function. The function f is a sub E-function on I if and only if $f(x)f''(x) - (f'(x))^2 \geq 0$ for all x in I.*

Theorem 2.4. *Suppose that a function $f : I \to (0,\infty)$ is a sub E-function on I if and only if there exist a supporting function for $f(x)$ at each point $x \in I$.*

3 Main Results

Theorem 3.1. *If $f : I \to (0,\infty)$ and $g : I \to (0,\infty)$ are sub E-functions and $\alpha \geq 0$ then $f+g$ and αf are sub E-functions on I.*

Proof.

Let $h(x) = f(x) + g(x)$, $h(x)h''(x) - (h'(x))^2 \geq 0$, hence

$$\begin{aligned}
&[f(x)+g(x)][f''(x)+g''(x)] - [f'(x)+g'(x)]^2 = \\
&= f(x)f''(x) + g(x)g''(x) + g(x)f''(x) + f(x)g''(x) - f'^2(x) - g'^2(x) - 2f'(x)g'(x) \\
&= f(x)f''(x) - f'^2(x) + g(x)g''(x) - g'^2(x) + g(x)f''(x) + f(x)g''(x) - 2f'(x)g'(x) \\
&\geq f(x)f''(x) - f'^2(x) + g(x)g''(x) - g'^2(x) + g(x)\frac{f'^2(x)}{f(x)} + f(x)\frac{g'^2(x)}{g(x)} - 2f'(x)g'(x) \\
&= f(x)f''(x) - f'^2(x) + g(x)g''(x) - g'^2(x) + \frac{g^2(x)f'^2(x) + f^2(x)g'^2(x) - 2f'(x)g'(x)f(x)g(x)}{f(x)g(x)} \\
&= f(x)f''(x) - f'^2(x) + g(x)g''(x) - g'^2(x) + \frac{(g(x)f'(x) - f(x)g'(x))^2}{f(x)g(x)} \\
&\geq 0.
\end{aligned}$$

Hence, $f+g$ is a sub E-function.

$$\begin{aligned}
f(x) &\leq \exp[\frac{(v-x)\ln f(u) + (x-u)\ln f(v)}{v-u}] \\
&\leq \exp[\frac{v-x}{v-u}\ln f(u) + \frac{x-u}{v-u}\ln f(v)] \\
&\leq \exp[\ln f(u)^{\frac{v-x}{v-u}} + \ln f(v)^{\frac{x-u}{v-u}}] \\
&\leq \exp[\ln f(u)^{\frac{v-x}{v-u}} f(v)^{\frac{x-u}{v-u}}] \\
&\leq f(u)^{\frac{v-x}{v-u}} f(v)^{\frac{x-u}{v-u}}.
\end{aligned}$$

Since $\alpha \geq 0$, then

$$\begin{aligned}
\alpha f(x) &\leq \alpha f(u)^{\frac{v-x}{v-u}} f(v)^{\frac{x-u}{v-u}} \\
&= \alpha^{\frac{v-x}{v-u}} f(u)^{\frac{v-x}{v-u}} \alpha^{\frac{x-u}{v-u}} f(v)^{\frac{x-u}{v-u}} \\
&= (\alpha f(u))^{\frac{v-x}{v-u}} (\alpha f(v))^{\frac{x-u}{v-u}} \\
&= \exp[\ln(\alpha f(u))^{\frac{v-x}{v-u}} (\alpha f(v))^{\frac{x-u}{v-u}}] \\
&= \exp[\ln(\alpha f(u))^{\frac{v-x}{v-u}} + \ln(\alpha f(v))^{\frac{x-u}{v-u}}] \\
&= \exp[\frac{v-x}{v-u} \ln(\alpha f(u)) + \frac{x-u}{v-u} \ln(\alpha f(v))] \\
&= \exp\left[\frac{(v-x)\ln(\alpha f(u)) + (x-u)\ln(\alpha f(v))}{(v-u)}\right]
\end{aligned}$$

Hence, $\alpha f(x)$ is a sub E-function.

Theorem 3.2. *If $f : I \to (0, \infty)$ is a sub E-function then f is convex.*
Proof.
Since, f is a sub E-function, then

$$f(x) \leq [f(u)]^{\frac{v-x}{v-u}} [f(v)]^{\frac{x-u}{v-u}}, \quad u \leq x \leq v$$

Let $x = \lambda u + (1-\lambda)v$, $\lambda \in [0, 1]$. Since by arithmetic-geometric mean inequality, we have

$$\begin{aligned}
f(\lambda u + (1-\lambda)v) &\leq [f(u)]^{\frac{v-\lambda u-(1-\lambda v)}{v-u}} [f(v)]^{\frac{\lambda u+(1-\lambda)v-u}{v-u}} \\
&= [f(u)]^{\lambda} [f(v)]^{1-\lambda} \\
&\leq \lambda f(u) + (1-\lambda) f(v).
\end{aligned}$$

Hence, f is convex function.

Theorem 3.3. *Assume that $f : I \to (0, \infty)$ and $g : J \to (0, \infty)$ where $range(f) \subseteq J$, Let f and g are both non-negative, sub E-functions, two times continuously differentiable and g is increasing, then the composite function gof is sub E-functions on I.*
Proof.
Since, g is increasing, then

$$g'(x) \geq 0 \; \forall x \in J \tag{3.1}$$

Since, f and g are sub E-functions, then by using Theorem 2.3, we have

$$f(x)f''(x) - (f'(x))^2 \geq 0, \; \forall x \in I \tag{3.2}$$
$$g(x)g''(x) - (g'(x))^2 \geq 0, \; \forall x \in J. \tag{3.3}$$

We have,

$$h(x) = g(f(x)), \tag{3.4}$$
$$h'(x) = g'(f(x))f'(x), \tag{3.5}$$
$$h''(x) = g''(f(x))(f'(x))^2 + g'(f(x))f''(x). \tag{3.6}$$

Then,

$$\begin{aligned} h(x)h''(x) - (h'(x))^2 &= g(f(x))g''(f(x))(f'(x))^2 + g(f(x)g'(f(x)))f''(x) - [g'(f(x))f'(x)]^2 \\ &= [g(f(x))g''(f(x) - g'(f(x)))][f'(x)]^2 + g'(f(x))f''(x)g(f(x)) \end{aligned}$$

Now using (3.1), (3.2), (3.3), we conclude that

$$h(x)h''(x) - (h'(x))^2 \geq 0.$$

Hence, h(x) is a sub E-function.

Definition 3.1. *Let $BE[a,b]$ be the class of functions $f : [a,b] \to (0,\infty)$ representable as difference of two sub E-functions in the the form $f = g - h$ where g and h are sub E-functions on $[a,b]$ and $g'_+(a)$, $g'_-(b)$, $h'_+(a)$, $h'_-(b)$ are all finite.*

Theorem 3.4. *The Class $BE[a,b]$ is closed under addition, subtraction and scalar multiplication.*

Proof.

Let $f, g \in BE[a,b], f = f_1 - f_2, g = g_1 - g_2$
For addition:
$f + g = f_1 - f_2 + g_1 - g_2 = (f_1 + g_1) - (f_2 + g_2)$
Since $f_1 + g_1, f_2 + g_2$ are sub E-functions by Theorem 3.1.
Then $f + g \in BE[a,b]$.
For subtraction:
$f - g = f_1 - f_2 - g_1 + g_2 = (f_1 + g_2) - (f_2 + g_1)$.
Since $f_1 + g_2, f_2 + g_1$ are sub E-functions.
Then $f - g \in BE[a,b]$.
For scalar multiplication: Case(1) let $\alpha \geq 0$
$\alpha f = \alpha f_1 - \alpha f_2$.
Since $\alpha f_1, \alpha f_2$ are sub E-functions by Theorem 3.1.
Then $\alpha f \in BE[a,b]$.
Case(2) let $\alpha < 0$.
$\alpha f = \alpha f_1 - \alpha f_2 = -\alpha f_2 - (-\alpha) f_1$.
Since $-\alpha f_2, -\alpha f_1$ are sub E-functions.
Then $\alpha f \in BE[a,b]$.
It is clear that all the previous functions have finite endpoint derivatives.

Corollary 3.1. *$BE[a,b]$ is a linear space.*

Theorem 3.5. *If $f \in BE[a,b]$, then f satisfies lipschitz condition and consequently absolutely continuous on $[a,b]$.*

Proof.

Let $f \in BE[a,b], f = f_1 - f_2$. Since f_1 and f_2 are sub E-functions.
Then from Theorem 2.1 f_1, f_2 satisfy lipschitz condition.

$$|f_1(x) - f_1(y)| \leq k|x - y| \quad \text{and} \quad |f_2(x) - f_2(y)| \leq m|x - y|.$$

$\forall x, y \in [a, b]$,

$$\begin{aligned} f(x) - f(y) &= f_1(x) - f_2(x) - (f_1(y) - f_2(y)) \\ &= f_1(x) - f_1(y) - (f_2(x) - f_2(y)) \\ |f(x) - f(y)| &= |f_1(x) - f_1(y) - (f_2(x) - f_2(y))| \\ &\leq |f_1(x) - f_1(y)| + |-(f_2(x) - f_2(y)| \\ &\leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \end{aligned}$$

$$\begin{aligned} |f(x) - f(y)| &\leq k|x - y| + m|x - y| \\ &= (k + m)|x - y| \\ &= h|x - y|, \end{aligned}$$

where, $h = k + m$

$$\textit{then} \qquad |f(x) - f(y)| \leq h|x - y| \qquad \forall x, y \in [a, b]$$

Hence, f satisfies lipschitz condition.
let $\epsilon > 0$, choose $\delta = \frac{\epsilon}{h}$ such that for any collection $\{(x_i, y_i) : i = 1, 2, ..., n\}$ of disjoint open subinterval of $[a, b]$ with $\sum_{i=1}^{n} |x_i - y_i| < \delta$

$$\textit{then} \sum_{1}^{n} |f(x_i) - f(y_i)| < \sum_{1}^{n} h|x_i - y_i| = h \sum_{1}^{n} |x_i - y_i| < h\frac{\epsilon}{h} = \epsilon$$

Hence, f is absolutely continuous on $[a, b]$.

Corollary 3.2. *If $f \in BE[a, b]$ then $V_a^b(f) < \infty$.*
Proof.
Let $f \in BE[a, b]$.
Then, from Theorem 2.1 f satisfies lipschitz condition.
Then, from Theorem 2.2 $f \in V[a, b]$.
Hence, $V_a^b(f) < \infty$.

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