INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA TOMINAGA AND FURUICHI RESULTS

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ABSTRACT. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$. In this paper we prove among others that, if $0 < mI \le A \le MI$, then

$$1 \le \exp\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle \le \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}} \le S\left(\frac{M}{m}\right),$$

for $x \in H$, $\|x\| = 1$, where $S(\cdot)$ is *Specht's ratio*.

1. INTRODUCTION

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [1], [2], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [1].

For each unit vector $x \in H$, see also [5], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

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(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A+\alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [1] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [6]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [2], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu} \leq\right) S\left(\left(\frac{a}{b}\right)^{r}\right) a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \leq S\left(\frac{a}{b}\right) a^{1-\nu}b^{\nu}, \tag{1.6}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.6) is due to Tominaga [7] while the first one is due to Furuichi [3].

2. Main Results

Our first main result is as follows:

Theorem 1. *If* $0 < mI \leq A \leq MI$ *for positive numbers* m, M, *then*

$$1 \leq \exp \left\langle \ln S\left(\left(\frac{M}{m} \right)^{\frac{1}{2}I - \frac{1}{M-m}\left| A - \frac{1}{2}(m+M)I \right|} \right) x, x \right\rangle \tag{2.1}$$

$$\leq \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}} \leq S\left(\frac{M}{m} \right)$$

for $x \in H$, $\|x\| = 1$.

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0,1]$. Then

$$\min\{1-\nu, \nu\} = \frac{1}{2} - \left| \nu - \frac{1}{2} \right| = \frac{1}{2} - \left| \frac{t-m}{M-m} - \frac{1}{2} \right|$$

$$= \frac{1}{2} - \frac{1}{M-m} \left| t - \frac{1}{2}(m+M) \right|,$$

$$(1-\nu) m + \nu M = \frac{M-t}{M-m} m + \frac{t-m}{M-m} M = t$$

and

$$m^{1-\nu} M^{\nu} = m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}.$$

By using the inequality (1.6) we deduce

$$m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \leq S\left(\left(\frac{M}{m} \right)^{\frac{1}{2} - \frac{1}{M-m}\left| t - \frac{1}{2}(m+M) \right|} \right) m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \tag{2.2}$$

$$\leq t \leq S\left(\frac{M}{m} \right) m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}$$

for $t \in [m, M]$.

By taking the log in (2.2) we get

$$\frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M \tag{2.3}$$

$$\leq \ln S\left(\left(\frac{M}{m} \right)^{\frac{1}{2} - \frac{1}{M-m}\left| t - \frac{1}{2}(m+M) \right|} \right) + \frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M$$

$$\leq \ln t \leq \ln S\left(\frac{M}{m} \right) + \frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M,$$

for $t \in [m, M]$.

If $0 < mI \leq A \leq MI$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$\ln m \frac{MI - A}{M-m} + \ln M \frac{A - mI}{M-m}$$

$$\leq \ln S\left(\left(\frac{M}{m} \right)^{\frac{1}{2}I - \frac{1}{M-m}\left| A - \frac{1}{2}(m+M)I \right|} \right) + \ln m \frac{MI - A}{M-m} + \ln M \frac{A - mI}{M-m}$$

$$\leq \ln A \leq \ln S\left(\frac{M}{m} \right) I + \ln m \frac{MI - A}{M-m} + \ln M \frac{t-m}{M-m},$$

which is equivalent to

$$\begin{aligned}
&\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m}\\
&\leq\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle\\
&+\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m}\\
&\leq\langle \ln Ax,x\rangle\leq \ln S\left(\frac{M}{m}\right)+\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\frac{\langle Ax,x\rangle-m}{M-m}\ln M
\end{aligned}$$

for $x\in H$, $\|x\|=1$.

This inequality can also be written as

$$\begin{aligned}
&\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)\\
&\leq\ln\quad\exp\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle\Bigg)\\
&+\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)\\
&\leq\langle \ln Ax,x\rangle\leq \ln S\left(\frac{M}{m}\right)+\ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)
\end{aligned}\tag{2.4}$$

for $x\in H$, $\|x\|=1$.

If we take the exponential in (2.4), then we get

$$\begin{aligned}
&m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\\
&\leq\quad\exp\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)x,x\right\rangle\Bigg)m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\\
&\leq\exp\langle \ln Ax,x\rangle\leq S\left(\frac{M}{m}\right)m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}
\end{aligned}$$

and the inequality (2.1) is proved. □

Remark 1. *From (1.4) and (2.1) we derive the following inequalities in terms of Specht's ratio*

$$\frac{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}{S\left(\frac{M}{m}\right)}\leq\frac{\langle Ax,x\rangle}{S\left(\frac{M}{m}\right)}\leq\Delta_x(A)\leq S\left(\frac{M}{m}\right)m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}\tag{2.5}$$

for $x\in H$, $\|x\|=1$.

Corollary 1. *With the assumption of Theorem 1, we get*

$$\begin{aligned}
1&\leq\exp\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x,x\right\rangle\\
&\leq\frac{M^{\frac{m^{-1}-\left\langle A^{-1}x,x\right\rangle}{m^{-1}-M^{-1}}}m^{\frac{\left\langle A^{-1}x,x\right\rangle-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_x(A)}\leq S\left(\frac{M}{m}\right)
\end{aligned}\tag{2.6}$$

for $x\in H$, $\|x\|=1$.

Proof. If we write the inequality for A^{-1} that satisfies the condition $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then

$$1 \leq \exp\left\langle \ln S\left(\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}I - \frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x, x\right\rangle$$
$$\leq \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}} \leq S\left(\frac{m^{-1}}{M^{-1}}\right),$$

namely

$$1 \leq \exp\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I - \frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x, x\right\rangle$$
$$\leq \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}} \leq S\left(\frac{M}{m}\right),$$

or

$$1 \leq \exp\left\langle \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I - \frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)x, x\right\rangle$$
$$\leq \frac{[\Delta_x(A)]^{-1}}{\left(M^{\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \leq S\left(\frac{M}{m}\right),$$

which is equivalent to the desired result (2.6). □

Corollary 2. *If $0 < mI \leq A,\ B \leq MI$ for positive numbers m, M, then*

$$\frac{\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right)}{S\left(\frac{M}{m}\right)} \leq \frac{\left\langle\frac{A+B}{2}x,x\right\rangle}{S\left(\frac{M}{m}\right)} \leq \int_0^1 \Delta_x((1-t)A+tB)dt \leq S\left(\frac{M}{m}\right)\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right), \tag{2.7}$$

where

$$\Theta\left(A,B,m,M,x\right) := \begin{cases} \dfrac{\left(\frac{M}{m}\right)^{\frac{\langle(B-A)x,x\rangle}{M-m}}-1}{\frac{\langle(B-A)x,x\rangle}{M-m}} \ \text{if} \ \langle(B-A)x,x\rangle \neq 0, \\ 1 \ \text{if} \ \langle(B-A)x,x\rangle = 0, \end{cases}$$

for $x \in H$, $\|x\| = 1$.

Proof. From (2.5) we get

$$\frac{m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}}{S\left(\frac{M}{m}\right)}$$
$$\leq \frac{\langle[(1-t)A+tB]x,x\rangle}{S\left(\frac{M}{m}\right)} \leq \Delta_x((1-t)A+tB)$$
$$\leq S\left(\frac{M}{m}\right)m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}$$

for $t\in[0,1]$.

If we take the integral over $t\in[0,1]$, then we get

$$\frac{\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt}{S\left(\frac{M}{m}\right)} \tag{2.8}$$
$$\leq \frac{\left\langle\frac{A+B}{2}x,x\right\rangle}{S\left(\frac{M}{m}\right)} \leq \int_0^1 \Delta_x((1-t)A+tB)dt$$
$$\leq S\left(\frac{M}{m}\right)\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt.$$

Observe that

$$\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}dt$$
$$= m^{\frac{M}{M-m}}M^{\frac{-m}{M-m}}\int_0^1\left(\frac{M}{m}\right)^{\frac{\langle[(1-t)A+tB]x,x\rangle}{M-m}}dt$$
$$= m^{\frac{M}{M-m}}M^{\frac{-m}{M-m}}\left(\frac{M}{m}\right)^{\frac{1}{M-m}}\int_0^1\left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}}dt$$
$$= m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}\int_0^1\left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}}dt.$$

Since for $a>0$, $a\neq 1$ and $b\in\mathbb{R}$ we have

$$\int_0^1 a^{bx}dx = \frac{a^b-1}{b\ln a},$$

then for $\langle(B-A)x,x\rangle\neq 0$

$$\int_0^1\left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}}dt = \frac{\left(\frac{M}{m}\right)^{\frac{\langle(B-A)x,x\rangle}{M-m}}-1}{\frac{\langle(B-A)x,x\rangle}{M-m}\ln\left(\frac{M}{m}\right)}$$

and by (2.8) we derive (2.7). □

3. Related Results

We also have:

Theorem 2. *With the assumption of Theorem 1, we have that*

$$1 \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}\right) \le \frac{\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m+\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{3.1}$$

for $x\in H$, $\|x\|=1$.

Proof. Assume that $m^{1-\nu}M^{\nu}=\exp s$, then $s=(1-\nu)\ln m+\nu\ln M\in[\ln m,\ln M]$, which gives that

$$\nu=\frac{s-\ln m}{\ln M-\ln m}.$$

Also

$$\min\{1-\nu,\nu\}=\frac{1}{2}-\left|\frac{s-\ln m}{\ln M-\ln m}-\frac{1}{2}\right| =\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|.$$

From (2.1) we get

$$\exp s\le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\right)\exp s \le \frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M \le S\left(\frac{M}{m}\right)\exp s,$$

namely

$$1\le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\right) \le \frac{\frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M}{\exp s}\le S\left(\frac{M}{m}\right)$$

for $s\in[\ln m,\ln M]$.

If $0<m\le A\le M$ and $x\in H$, $\|x\|=1$, then $\ln m\le\langle \ln Ax,x\rangle\le\ln M$ and for $s=\langle \ln Ax,x\rangle$ we deduce

$$1 \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}\right) \le \frac{\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m+\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M}{\exp\langle \ln Ax,x\rangle} \le S\left(\frac{M}{m}\right),$$

which is equivalent to (3.1). □

Corollary 3. *With the assumption of Theorem 1, we get*

$$1 \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|}\right) \tag{3.2}$$
$$\le \frac{\Delta_x(A)}{\left(\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m^{-1}\right)^{-1}} \le S\left(\frac{M}{m}\right),$$

for $x\in H$, $\|x\|=1$.

Proof. If we write the inequality (3.1) for A^{-1} that satisfies the condition $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then we obtain

$$1 \le S\left(\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}-\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\left\langle \ln A^{-1}x,x\right\rangle-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|}\right)$$
$$\le \frac{\frac{\ln m^{-1}-\left\langle \ln A^{-1}x,x\right\rangle}{\ln m^{-1}-\ln M^{-1}}M^{-1}+\frac{\left\langle \ln A^{-1}x,x\right\rangle-\ln M^{-1}}{\ln m^{-1}-\ln M^{-1}}m^{-1}}{\Delta_x(A^{-1})} \le S\left(\frac{m^{-1}}{M^{-1}}\right),$$

namely

$$1 \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|}\right)$$
$$\le \frac{\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m^{-1}}{\Delta_x(A^{-1})} \le S\left(\frac{M}{m}\right)$$

for $x\in H$, $\|x\|=1$.

This proves (3.2). □

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