INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA SOME INEQUALITIES IN TERMS OF KANTOROVICH RATIO

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$. In this paper we prove among others that, if $0 < mI \leq A \leq MI$, then

$$\begin{aligned} 1 &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \\ &\leq \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}} \\ &\leq \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \leq K\left(\frac{M}{m}\right), \end{aligned}$$

for $x \in H$, $\|x\| = 1$, where $K(\cdot)$ is *Kantorovich's ratio.*

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [1], [2], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [1].

For each unit vector $x \in H$, see also [5], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;

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(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [1] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [7]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [2], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu}\leq\right)K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.7}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.7) was obtained by Zou et al. in [9] while the second by Liao et al. [6].

2. Main Results

Our first main result is as follows:

Theorem 1. *If $0<mI\leq A\leq MI$ for positive numbers m, M, then*

$$\begin{aligned}1&\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]}\\&\leq\frac{\Delta_{x}(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}\\&\leq\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]}\leq K\left(\frac{M}{m}\right)\end{aligned} \tag{2.1}$$

for $x\in H$, $\|x\|=1$.

Proof. Assume that $t\in[m,M]$ and consider $\nu=\frac{t-m}{M-m}\in[0,1]$. Then

$$\begin{aligned}\min\{1-\nu,\nu\}&=\frac{1}{2}-\left|\nu-\frac{1}{2}\right|=\frac{1}{2}-\left|\frac{t-m}{M-m}-\frac{1}{2}\right|\\&=\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|,\end{aligned}$$

$$\begin{aligned}\max\{1-\nu,\nu\}&=\frac{1}{2}+\left|\nu-\frac{1}{2}\right|=\frac{1}{2}+\left|\frac{t-m}{M-m}-\frac{1}{2}\right|\\&=\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|,\end{aligned}$$

$$(1-\nu)\,m+\nu M=\frac{M-t}{M-m}m+\frac{t-m}{M-m}M=t$$

and

$$m^{1-\nu}M^{\nu}=m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.$$

By using (1.7) we get

$$\begin{aligned}m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}&\leq\left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|}m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}\\&\leq t\leq\left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|}m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}\end{aligned} \tag{2.2}$$

for $t\in[m,M]$.

By taking the log in (2.2) we get

$$
\begin{aligned}
(2.3)\qquad & \frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
& \leq\left[\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
& +\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
& \leq\ln t\leq\left[\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
& +\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
& \leq\ln K\left(\frac{M}{m}\right)+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M
\end{aligned}
$$

for $t\in[m,M]$.

If $0<mI\leq A\leq MI$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$
\begin{aligned}
& \ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
& \leq\left[\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|\right]\ln K\left(\frac{M}{m}\right) \\
& +\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
& \leq\ln A\leq\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|\right]\ln K\left(\frac{M}{m}\right) \\
& +\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
& \leq\ln K\left(\frac{M}{m}\right)I+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& \ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m} \\
& \leq\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right] \\
& +\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m} \\
& \leq\langle\ln Ax,x\rangle \\
& \leq\ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right] \\
& \ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m} \\
& \leq\ln K\left(\frac{M}{m}\right)+\ln m\frac{M-\langle Ax,x\rangle}{M-m}+\ln M\frac{\langle Ax,x\rangle-m}{M-m},
\end{aligned}
$$

for $x\in H$, $\|x\|=1$.

This inequality can also be written as

$$
\begin{aligned}
(2.4)\qquad & \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right) \\
& \le \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \\
& + \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right) \\
& \le \langle \ln Ax, x\rangle \\
& \le \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \\
& + \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right) \\
& \le \ln K\left(\frac{M}{m}\right) + \ln\left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right)
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential in (2.4), then we get

$$
\begin{aligned}
& m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}} \\
& \le \left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right) K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \\
& \le \exp\langle \ln Ax, x\rangle \\
& \le \left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right) \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|x,x\right\rangle\right]} \\
& \le \left(m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle-m}{M-m}}\right) K\left(\frac{M}{m}\right),
\end{aligned}
$$

and the inequality (2.1) is proved. □

Corollary 1. *With the assumption of Theorem 1, we have the alternative inequality*

$$
\begin{aligned}
(2.5)\qquad 1 & \le K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]} \\
& \le \frac{M^{\frac{m^{-1}-\left\langle A^{-1}x,x\right\rangle}{m^{-1}-M^{-1}}} m^{\frac{\left\langle A^{-1}x,x\right\rangle-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_x(A)} \\
& \le \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]} \le K\left(\frac{M}{m}\right)
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality for A^{-1} that satisfies the condition $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then

$$\begin{aligned}
1 &\leq K\left(\frac{m^{-1}}{M^{-1}}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\leq \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}}\\
&\leq \left[K\left(\frac{m^{-1}}{M^{-1}}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\leq K\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}$$

namely

$$\begin{aligned}
1 &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]}\\
&\leq \frac{\left[\Delta_x(A)\right]^{-1}}{\left(M^{\frac{m^{-1}-\langle A^{-1}x,x\rangle}{m^{-1}-M^{-1}}}m^{\frac{\langle A^{-1}x,x\rangle-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}}\\
&\leq \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|x,x\right\rangle\right]} \leq K\left(\frac{M}{m}\right),
\end{aligned}$$

which is equivalent to the desired result (2.5). □

Corollary 2. *If $0 < mI \leq A,\ B \leq MI$ for positive numbers $m,\ M$, then*

$$\begin{aligned}
&\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right)\\
&\leq \int_0^1 \Delta_x((1-t)\,A+tB)dt\\
&\leq K\left(\frac{M}{m}\right)\frac{m^{\frac{M-1}{M-m}}M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}\Theta\left(A,B,m,M,x\right),
\end{aligned}\tag{2.6}$$

where

$$\Theta\left(A,B,m,M,x\right) := \begin{cases} \dfrac{\left(\frac{M}{m}\right)^{\frac{\langle(B-A)x,x\rangle}{M-m}}-1}{\frac{\langle(B-A)x,x\rangle}{M-m}} & \text{if } \langle\left(B-A\right)x,x\rangle \neq 0\\[2ex] 1 \text{ if } \langle\left(B-A\right)x,x\rangle = 0, \end{cases}$$

for $x \in H$, $\|x\| = 1$.

Proof. From (2.5) we get

$$\begin{aligned}
&m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}\\
&\leq \Delta_x((1-t)\,A+tB)\\
&\leq K\left(\frac{M}{m}\right)m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}}M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}}
\end{aligned}$$

for $t \in [0,1]$.

If we take the integral over $t \in [0,1]$, then we get

$$\begin{aligned}(2.7)\quad & \int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}} M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}} dt \\ & \leq \int_0^1 \Delta_x((1-t)A+tB)dt \\ & \leq K\left(\frac{M}{m}\right)\int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}} M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}} dt.\end{aligned}$$

Observe that

$$\begin{aligned} & \int_0^1 m^{\frac{M-\langle[(1-t)A+tB]x,x\rangle}{M-m}} M^{\frac{\langle[(1-t)A+tB]x,x\rangle-m}{M-m}} dt \\ & = m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{\frac{\langle[(1-t)A+tB]x,x\rangle}{M-m}} dt \\ & = m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \left(\frac{M}{m}\right)^{\frac{1}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}} dt \\ & = m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}} dt.\end{aligned}$$

Since for $a>0$, $a \neq 1$ and $b \in \mathbb{R}$ we have

$$\int_0^1 a^{bx}dx = \frac{a^b-1}{b\ln a},$$

then for $\langle (B-A)x,x\rangle \neq 0$

$$\int_0^1 \left(\frac{M}{m}\right)^{t\frac{\langle(B-A)x,x\rangle}{M-m}} dt = \frac{\left(\frac{M}{m}\right)^{\frac{\langle(B-A)x,x\rangle}{M-m}}-1}{\frac{\langle(B-A)x,x\rangle}{M-m}\ln\left(\frac{M}{m}\right)}$$

and by (2.7) we derive (2.6). □

3. Related Results

We also have:

Theorem 2. *With the assumption of Theorem 1, we get*

$$\begin{aligned}(3.1)\quad 1 & \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|} \\ & \leq \frac{\frac{\ln M-\langle \ln Ax,x\rangle}{\ln M-\ln m}m+\frac{\langle \ln Ax,x\rangle-\ln m}{\ln M-\ln m}M}{\Delta_x(A)} \\ & \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\langle \ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|} \leq K\left(\frac{M}{m}\right)\end{aligned}$$

for $x \in H$, $\|x\|=1$.

Proof. Assume that $m^{1-\nu}M^{\nu}=\exp s$, then $s=(1-\nu)\ln m+\nu\ln M \in [\ln m,\ln M]$ which gives that

$$\nu = \frac{s-\ln m}{\ln M-\ln m}.$$

Also

$$\min\{1-\nu,\nu\}=\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|$$

and

$$\max\{1-\nu,\nu\}=\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|$$

From (1.7) we have

$$\begin{aligned}\exp s &\le \exp s\left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\\ &\le \frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M\\ &\le \exp s\left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\end{aligned}$$

namely

$$\begin{aligned}1 &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\\ &\le \frac{\frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M}{\exp s}\\ &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\end{aligned}$$

for $s\in[\ln m,\ln M]$.

If $0<m\le A\le M$ and $x\in H$, $\|x\|=1$, then $\ln m\le\langle\ln Ax,x\rangle\le\ln M$ and for $s=\langle\ln Ax,x\rangle$ we deduce

$$\begin{aligned}1 &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}\\ &\le \frac{\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}m+\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}M}{\exp\langle\ln Ax,x\rangle}\\ &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}\end{aligned}$$

and the inequality (3.1) is proved. □

Corollary 3. *With the assumption of Theorem 1, we have*

$$\begin{aligned}1 &\le \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln M+\ln m}{2}\right|}\\ &\le \frac{\Delta_x(A)}{\left(\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}m^{-1}\right)^{-1}}\le K\left(\frac{M}{m}\right)\end{aligned}\tag{3.2}$$

for $x\in H$, $\|x\|=1$.

Proof. If we write the inequality (3.1) for A^{-1} that satisfies the condition $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then we obtain

$$\begin{aligned}
1 &\le K\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}-\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\left\langle \ln A^{-1}x,x\right\rangle-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|} \\
&\le \frac{\frac{\ln m^{-1}-\left\langle \ln A^{-1}x,x\right\rangle}{\ln m^{-1}-\ln M^{-1}}M^{-1}+\frac{\left\langle \ln A^{-1}x,x\right\rangle-\ln M^{-1}}{\ln m^{-1}-\ln M^{-1}}m^{-1}}{\Delta_x(A^{-1})} \\
&\le K\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}+\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\left\langle \ln A^{-1}x,x\right\rangle-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|} \le M\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}$$

namely

$$\begin{aligned}
1 &\le K\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\left\langle \ln Ax,x\right\rangle-\frac{\ln m+\ln M}{2}\right|} \\
&\le \frac{\frac{\left\langle \ln Ax,x\right\rangle-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\left\langle \ln Ax,x\right\rangle}{\ln M-\ln m}m^{-1}}{\Delta_x(A^{-1})} \\
&\le K\left(\frac{M}{m}\right)^{\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\left\langle \ln Ax,x\right\rangle-\frac{\ln m+\ln M}{2}\right|} \le K\left(\frac{M}{m}\right),
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

This proves (3.2). □

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