INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA REFINEMENTS AND REVERSES OF YOUNG'S RESULT

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x\in H$, define the normalized determinant by $\Delta_x(A):=\exp\langle \ln Ax,x\rangle$. In this paper we prove among others that, if $0<mI\le A\le MI$ and $x\in H$, $\|x\|=1$, then

$$1\le \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}\le \exp\left[\frac{1}{Mm}\langle (MI-A)(A-mI)x,x\rangle\right]$$
$$\le \exp\left[\frac{1}{Mm}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)\right]\le \exp\left[\frac{1}{4Mm}(M-m)^2\right].$$

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A\ge 0$) if $\langle Ax,x\rangle\ge 0$ for all $x\in H$. In particular, $A>0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A\ge B$ means as usual that $A-B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x\in H$, namely $\|x\|=1$, defined by $\Delta_x(A):=\exp\langle \ln Ax,x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [4].

For each unit vector $x\in H$, see also [8], we have:

(i) *continuity*: the map $A\to\Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x,x\rangle^{-1}\le\Delta_x(A)\le\langle Ax,x\rangle$;
(iii) *continuous mean*: $\langle A^px,x\rangle^{1/p}\downarrow\Delta_x(A)$ for $p\downarrow 0$ and $\langle A^px,x\rangle^{1/p}\uparrow\Delta_x(A)$ for $p\uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t)=\Delta_x(A)^t$ for all $t>0$;
(v) *homogeneity*: $\Delta_x(tA)=t\Delta_x(A)$ and $\Delta_x(tI)=t$ for all $t>0$;
(vi) *monotonicity*: $0<A\le B$ implies $\Delta_x(A)\le\Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB)=\Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A+\alpha B)\ge\Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0<\alpha<1$.

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We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [4] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [10]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [5], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right). \tag{1.5}$$

Kittaneh and Manasrah [11], [12] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.6}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$. The case $\nu = \frac{1}{2}$ reduces (1.6) to an identity.

For some operator versions of (1.6) see [11] and [12].

We also have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}} \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq \frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.7}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

In this paper, motivated by the above results, we provide upper an lower bounds for the quantities

$$\frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}$$

and

$$\ln\Delta_x(A)-(\ln m)^{\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}}(\ln M)^{\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}}$$

under various assumptions for the positive operator A with spectrum in $[m,M]$ and $x\in H$, $\|x\|=1$.

2. Main Results

The first result is as follows:

Theorem 1. *Assume that* $0<mI\le A\le MI$ *and* $x\in H$, $\|x\|=1$, *then*

$$1\le\frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}\le\exp\left[\frac{1}{Mm}\langle(MI-A)(A-mI)x,x\rangle\right]\tag{2.1}$$

$$\le\exp\left[\frac{1}{Mm}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)\right]\le\exp\left[\frac{1}{4Mm}(M-m)^2\right].$$

Proof. In [2] we obtained the following reverses of Young's inequality:

$$1\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right],$$

where $a,\ b>0$, $\nu\in[0,1]$.

This is equivalent, by taking the logarithm, with

$$0\le\ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b\le\nu(1-\nu)\frac{(b-a)^2}{ba}$$

where $a,\ b>0$, $\nu\in[0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$0\le\ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M\le\frac{(M-t)(t-m)}{(M-m)^2}\frac{(M-m)^2}{Mm}$$
$$=\frac{(M-t)(t-m)}{Mm}.$$

Using the continuous functional calculus for selfadjoint operators, we have

$$0\le\ln A-\frac{MI-A}{M-m}\ln m-\frac{AI-m}{M-m}\ln M\le\frac{(MI-A)(A-mI)}{Mm},$$

which is equivalent to

$$0\le\langle\ln Ax,x\rangle-\frac{M-\langle Ax,x\rangle}{M-m}\ln m-\frac{\langle Ax,x\rangle-m}{M-m}\ln M$$
$$\le\frac{1}{Mm}\langle(MI-A)(A-mI)x,x\rangle,$$

for all $x\in H$, $\|x\|=1$.

If we take the exponential, then we get

$$1 \leq \frac{\exp \langle \ln Ax, x \rangle}{\exp \left[\frac{M - \langle Ax, x \rangle}{M-m} \ln m + \frac{\langle Ax, x \rangle - m}{M-m} \ln M \right]} \leq \exp \left[\frac{1}{Mm} \langle (MI - A)(A - mI) x, x \rangle \right], \tag{2.2}$$

for all $x \in H$, $\|x\| = 1$.

Observe that

$$\exp \left[\frac{M - \langle Ax, x \rangle}{M-m} \ln m + \frac{\langle Ax, x \rangle - m}{M-m} \ln M \right] = \exp \left[\ln \left(m^{\frac{M - \langle Ax, x \rangle}{M-m}} M^{\frac{\langle Ax, x \rangle - m}{M-m}} \right) \right] = m^{\frac{M - \langle Ax, x \rangle}{M-m}} M^{\frac{\langle Ax, x \rangle - m}{M-m}}$$

and by (2.2) we obtain the first inequality in (2.1).

The function $g(t) = (M - t)(t - m)$ is concave on $[m, M]$ and by Jensen's inequality

$$\langle g(A) x, x \rangle \leq g(\langle Ax, x \rangle), \ x \in H, \|x\| = 1$$

we have

$$\langle (MI - A)(A - mI) x, x \rangle \leq ((M - \langle Ax, x \rangle)(\langle Ax, x \rangle - m))$$

for all $x \in H$, $\|x\| = 1$, which proves the third inequality in (2.1). □

Corollary 1. *With the assumptions of Theorem 1,*

$$1 \leq \frac{M^{\frac{m^{-1} - \langle A^{-1}x, x \rangle}{m^{-1} - M^{-1}}} m^{\frac{\langle A^{-1}x, x \rangle - M^{-1}}{m^{-1} - M^{-1}}}}{\Delta_x(A)} \leq \exp \left[mM \left\langle \left(m^{-1}I - A^{-1} \right) \left(A^{-1} - M^{-1}I \right) x, x \right\rangle \right] \leq \exp \left[mM \left(m^{-1} - \langle A^{-1}x, x \rangle \right) \left(\langle A^{-1}x, x \rangle - M^{-1} \right) \right] \leq \exp \left[\frac{1}{4} mM (M - m)^2 \right], \tag{2.3}$$

for $x \in H$, $\|x\| = 1$.

Proof. Observe that $0 < mI \leq A \leq MI$ implies that $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$. If we write the inequality (2.1) for A^{-1}, then we get

$$1 \leq \frac{\Delta_x(A^{-1})}{M^{-\frac{m^{-1} - \langle A^{-1}x, x \rangle}{m^{-1} - M^{-1}}} m^{-\frac{\langle A^{-1}x, x \rangle - M^{-1}}{m^{-1} - M^{-1}}}} \leq \exp \left[\frac{1}{m^{-1}M^{-1}} \left\langle \left(m^{-1}I - A^{-1} \right) \left(A^{-1} - M^{-1}I \right) x, x \right\rangle \right] \leq \exp \left[\frac{1}{m^{-1}M^{-1}} \left(m^{-1} - \langle A^{-1}x, x \rangle \right) \left(\langle A^{-1}x, x \rangle - M^{-1} \right) \right] \leq \exp \left[\frac{1}{4m^{-1}M^{-1}} \left(m^{-1} - M^{-1} \right)^2 \right],$$

for all $x \in H$, $\|x\| = 1$, which is equivalent to (2.3). □

In [3] we obtained the following refinement and reverse of Young's inequality:

$$\begin{aligned}\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] &\\ \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} &\\ \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right], &\end{aligned} \tag{2.4}$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Theorem 2. *Assume that* $0 < mI \leq A \leq MI$ *and* $x \in H$, $\|x\| = 1$, *then*

$$\begin{aligned}1 &\leq \exp\left[\frac{1}{2M^2}\langle(MI-A)(A-mI)x,x\rangle\right]\\ &\leq \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}}M^{\frac{\langle Ax,x\rangle-m}{M-m}}}\\ &\leq \exp\left[\frac{1}{2m^2}\langle(MI-A)(A-mI)x,x\rangle\right]\\ &\leq \exp\left[\frac{1}{2m^2}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)\right]\\ &\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right].\end{aligned} \tag{2.5}$$

Proof. From (2.4) we have

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\right]\\ &\leq \frac{(1-\nu)m+\nu M}{m^{1-\nu}M^{\nu}} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2\right],\end{aligned}$$

for $\nu \in [0,1]$.

By taking the logarithm, we obtain

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\\ &\leq \ln((1-\nu)m+\nu M)-(1-\nu)\ln m-\nu\ln M\\ &\leq \frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2,\end{aligned} \tag{2.6}$$

for $\nu \in [0,1]$.

If we take $a = m$, $b = M$, $t \in [m, M]$ and $\nu = \frac{t-m}{M-m} \in [0,1]$, then we get

$$\begin{aligned}\frac{(M-t)(t-m)}{2M^2} &\leq \ln t - \frac{M-t}{M-m}\ln m - \frac{t-m}{M-m}\ln M\\ &\leq \frac{(M-t)(t-m)}{2m^2}\end{aligned}$$

$t \in [m, M]$.

As above, we get the vector inequality

$$\begin{aligned}
&\frac{1}{2M^2}\langle (MI-A)(A-mI)x,x\rangle \\
&\le \langle \ln Ax,x\rangle - \frac{M-\langle Ax,x\rangle}{M-m}\ln m - \frac{\langle Ax,x\rangle - m}{M-m}\ln M \\
&\le \frac{1}{2m^2}\langle (MI-A)(A-mI)x,x\rangle,
\end{aligned}$$

for $x\in H$, $\|x\|=1$.

If we take the exponential, then we derive

$$\begin{aligned}
&\exp\left[\frac{1}{2M^2}\langle (MI-A)(A-mI)x,x\rangle\right] \\
&\le \frac{\exp\langle \ln Ax,x\rangle}{\exp\left[\frac{M-\langle Ax,x\rangle}{M-m}\ln m + \frac{\langle Ax,x\rangle - m}{M-m}\ln M\right]} \\
&\le \exp\left[\frac{1}{2m^2}\langle (MI-A)(A-mI)x,x\rangle\right],
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves the first part of (2.5).

The second part is obvious. □

Corollary 2. *With the assumptions of Theorem 1,*

$$\begin{aligned}
1 &\le \exp\left[\frac{1}{2}m^2\left\langle \left(m^{-1}I-A^{-1}\right)\left(A^{-1}-M^{-1}I\right)x,x\right\rangle\right] \qquad (2.7)\\
&\le \frac{M^{\frac{m^{-1}-\left\langle A^{-1}x,x\right\rangle}{m^{-1}-M^{-1}}}\, m^{\frac{\left\langle A^{-1}x,x\right\rangle - M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_x(A)} \\
&\le \exp\left[\frac{1}{2}M^2\left\langle \left(m^{-1}I-A^{-1}\right)\left(A^{-1}-M^{-1}I\right)x,x\right\rangle\right] \\
&\le \exp\left[\frac{1}{2}M^2\left(m^{-1}-\left\langle A^{-1}x,x\right\rangle\right)\left(\left\langle A^{-1}x,x\right\rangle - M^{-1}\right)\right] \\
&\le \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right],
\end{aligned}$$

for $x\in H$, $\|x\|=1$.

3. Related Results

We also have:

Theorem 3. *Assume that* $I < mI \le A \le MI$ *and* $x\in H$, $\|x\|=1$, *then*

$$\begin{aligned}
0 &\le \ln\Delta_x(A) - (\ln m)^{\frac{\ln M - \langle \ln Ax,x\rangle}{\ln M - \ln m}}(\ln M)^{\frac{\langle \ln Ax,x\rangle - \ln m}{\ln M-\ln m}} \qquad (3.1)\\
&\le \frac{(\ln M - \langle \ln Ax,x\rangle)(\langle \ln Ax,x\rangle - \ln m)}{\ln M - \ln m}\ln\left(\frac{\ln M}{\ln m}\right) \\
&\le \frac{1}{4}(\ln M - \ln m)\ln\left(\frac{\ln M}{\ln m}\right)
\end{aligned}$$

for $x\in H$, $\|x\|=1$.

Proof. In the recent paper [2] we obtained the following reverses of Young's inequality as well:

$$0 \le (1-\nu)\,a + \nu b - a^{1-\nu} b^{\nu} \le \nu (1-\nu)(a-b)(\ln a - \ln b) \tag{3.2}$$

where $a,\ b > 0$, $\nu \in [0,1]$.

If we take the exponential in (3.2), then we get

$$\begin{aligned} 1 &\le \frac{\exp\left[(1-\nu)\,a + \nu b\right]}{\exp\left(a^{1-\nu} b^{\nu}\right)} \le \exp\left[\nu (1-\nu)(a-b)(\ln a - \ln b)\right] \\ &= \exp\left[\ln\left(\frac{b}{a}\right)^{\nu(1-\nu)(b-a)}\right] = \left(\frac{b}{a}\right)^{\nu(1-\nu)(b-a)}. \end{aligned} \tag{3.3}$$

If we put $(1-\nu)\,a + \nu b = s > 0$, then $\nu = \frac{s-a}{b-a}$, $1-\nu = \frac{b-s}{b-a}$ and by (3.3) we obtain

$$1 \le \frac{\exp s}{\exp\left(a^{\frac{b-s}{b-a}} b^{\frac{s-a}{b-a}}\right)} \le \left(\frac{b}{a}\right)^{\frac{(s-a)(b-s)}{b-a}} \le \left(\frac{b}{a}\right)^{\frac{1}{4}(b-a)}. \tag{3.4}$$

Now, we take $a = \ln m$, $s = \langle \ln Ax, x\rangle$ and $b = \ln M$, $x \in H$, $\|x\| = 1$ in (3.4) to get

$$\begin{aligned} 1 &\le \frac{\exp\langle \ln Ax, x\rangle}{\exp\left((\ln m)^{\frac{\ln M - \langle \ln Ax, x\rangle}{\ln M - \ln m}} (\ln M)^{\frac{\langle \ln Ax, x\rangle - \ln m}{\ln M - \ln m}}\right)} \\ &\le \left(\frac{\ln M}{\ln m}\right)^{\frac{(\ln M - \langle \ln Ax, x\rangle)(\langle \ln Ax, x\rangle - \ln m)}{\ln M - \ln m}} \le \left(\frac{\ln M}{\ln m}\right)^{\frac{1}{4}(\ln M - \ln m)}. \end{aligned}$$

By taking the logarithm we then obtain (3.1). □

We also have:

Theorem 4. *Assume that $I < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$, then*

$$\begin{aligned} 0 &\le \left(\frac{1}{2} - \frac{1}{\ln M - \ln m}\left|\langle \ln Ax, x\rangle - \frac{\ln m + \ln M}{2}\right|\right) \\ &\quad\times \left(\sqrt{\ln M} - \sqrt{\ln m}\right)^2 \\ &\le \ln \Delta_x(A) - (\ln m)^{\frac{\ln M - \langle \ln Ax, x\rangle}{\ln M - \ln m}} (\ln M)^{\frac{\langle \ln Ax, x\rangle - \ln m}{\ln M - \ln m}} \\ &\le \left(\frac{1}{2} + \frac{1}{\ln M - \ln m}\left|\langle \ln Ax, x\rangle - \frac{\ln m + \ln M}{2}\right|\right) \\ &\quad\times \left(\sqrt{\ln M} - \sqrt{\ln m}\right)^2 \\ &\le \left(\sqrt{\ln M} - \sqrt{\ln m}\right)^2. \end{aligned} \tag{3.5}$$

Proof. If we take the exponential in (1.6) we get

$$\begin{aligned} 1 &\le \exp\left[\min\{1-\nu, \nu\}\left(\sqrt{a} - \sqrt{b}\right)^2\right] \\ &\le \frac{\exp\left[(1-\nu)\,a + \nu b\right]}{\exp\left(a^{1-\nu} b^{\nu}\right)} \\ &\le \exp\left[\max\{1-\nu, \nu\}\left(\sqrt{a} - \sqrt{b}\right)^2\right] \end{aligned} \tag{3.6}$$

for a, $b>0$, $\nu\in[0,1]$.

If we put $(1-\nu)a+\nu b=s>0$, then $\nu=\frac{s-a}{b-a}$,

$$\min\{1-\nu,\nu\}=\frac{1}{2}-\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|,$$

$$\max\{1-\nu,\nu\}=\frac{1}{2}+\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|,$$

and by (3.6) we get

$$\begin{aligned} 1 &\le \exp\left[\left(\frac{1}{2}-\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|\right)\left(\sqrt{a}-\sqrt{b}\right)^2\right] \\ &\le \frac{\exp s}{\exp\left(a^{\frac{b-s}{b-a}}b^{\frac{s-a}{b-a}}\right)} \\ &\le \exp\left[\left(\frac{1}{2}+\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|\right)\left(\sqrt{a}-\sqrt{b}\right)^2\right] \end{aligned} \tag{3.7}$$

for $s\in[a,b]$.

Now, we take $a=\ln m$, $s=\langle\ln Ax,x\rangle$ and $b=\ln M$, $x\in H$, $\|x\|=1$ in (3.7) to get

$$\begin{aligned} 1 &\le \exp\left[\left(\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln m}-\sqrt{\ln M}\right)^2\right] \\ &\le \frac{\exp\langle\ln Ax,x\rangle}{\exp\left((\ln m)^{\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}}(\ln M)^{\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}}\right)} \\ &\le \exp\left[\left(\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln m}-\sqrt{\ln M}\right)^2\right]. \end{aligned}$$

Taking the logarithm, we obtain

$$\begin{aligned} 0 &\le \left(\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2 \\ &\le \ln\Delta_x(A)-(\ln m)^{\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}}(\ln M)^{\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}} \\ &\le \left(\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\langle\ln Ax,x\rangle-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2 \\ &\le \left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2 \end{aligned}$$

for $x\in H$, $\|x\|=1$, which proves the desired result. □

We also have:

Theorem 5. *With the assumptions of Theorem 4,*

$$\begin{aligned} 0 &\le \frac{1}{2}\frac{(\langle\ln Ax,x\rangle-\ln m)(\ln M-\langle\ln Ax,x\rangle)}{\ln M} \\ &\le \ln\Delta_x(A)-(\ln m)^{\frac{\ln M-\langle\ln Ax,x\rangle}{\ln M-\ln m}}(\ln M)^{\frac{\langle\ln Ax,x\rangle-\ln m}{\ln M-\ln m}} \\ &\le \frac{1}{2}\frac{(\langle\ln Ax,x\rangle-\ln m)(\ln M-\langle\ln Ax,x\rangle)}{\ln m} \\ &\le \frac{1}{8\ln m}(\ln M-\ln m)^2 \end{aligned} \tag{3.8}$$

for $x \in H$, $\|x\| = 1$.

Proof. If we take the exponential in (1.7), then we get

$$1 \le \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}}\right] \le \frac{\exp[(1-\nu)a+\nu b]}{\exp(a^{1-\nu}b^{\nu})} \le \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}}\right] \tag{3.9}$$

for a, $b > 0$, $\nu \in [0,1]$.

If we put $(1-\nu)a + \nu b = s > 0$, then $\nu = \frac{s-a}{b-a}$, $1-\nu = \frac{b-s}{b-a}$ and by (3.9) we derive

$$\exp\left[\frac{1}{2}\frac{(s-a)(b-s)}{\max\{a,b\}}\right] \le \frac{\exp s}{\exp\left(a^{\frac{b-s}{b-a}}b^{\frac{s-a}{b-a}}\right)} \le \exp\left[\frac{1}{2}\frac{(s-a)(b-s)}{\min\{a,b\}}\right]. \tag{3.10}$$

Now, we put $a = \ln m$, $s = \langle \ln Ax, x\rangle$ and $b = \ln M$, $x \in H$, $\|x\| = 1$ in (3.10) to get

$$1 \le \exp\left[\frac{1}{2}\frac{(\langle \ln Ax, x\rangle - \ln m)(\ln M - \langle \ln Ax, x\rangle)}{\ln M}\right] \le \frac{\exp\langle \ln Ax, x\rangle}{\exp\left((\ln m)^{\frac{\ln M - \langle \ln Ax, x\rangle}{\ln M - \ln m}}(\ln M)^{\frac{\langle \ln Ax, x\rangle - \ln m}{\ln M - \ln m}}\right)} \le \exp\left[\frac{1}{2}\frac{(\langle \ln Ax, x\rangle - \ln m)(\ln M - \langle \ln Ax, x\rangle)}{\ln m}\right]$$

and by taking the logarithm we obtain the first part of (3.8).

The second part is obvious. □

In [3] we also obtained the following result

$$\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\} \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\} \tag{3.11}$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Theorem 6. *With the assumptions of Theorem 4,*

$$\begin{aligned}
0 &\leq \frac{1}{2}\left(\langle \ln Ax, x\rangle - \ln m\right)\left(\ln M - \langle \ln Ax, x\rangle\right) \\
&\quad \times \left[\frac{\ln(\ln M) - \ln(\ln m)}{\ln M - \ln m}\right]^2 \ln(\ln m) \\
&\leq \ln \Delta_x(A) - (\ln m)^{\frac{\ln M - \langle \ln Ax, x\rangle}{\ln M - \ln m}} (\ln M)^{\frac{\langle \ln Ax, x\rangle - \ln m}{\ln M - \ln m}} \\
&\leq \frac{1}{2}\left(\langle \ln Ax, x\rangle - \ln m\right)\left(\ln M - \langle \ln Ax, x\rangle\right) \\
&\quad \times \left[\frac{\ln(\ln M) - \ln(\ln m)}{\ln M - \ln m}\right]^2 \ln(\ln M) \\
&\leq \frac{1}{8}\left[\ln(\ln M) - \ln(\ln m)\right]^2 \ln(\ln M)
\end{aligned} \tag{3.12}$$

for $x \in H$, $\|x\| = 1$.

Proof. If we take the exponential in (3.11), then we get

$$\begin{aligned}
1 &\leq \exp\left[\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a, b\}\right] \\
&\leq \frac{\exp\left[(1-\nu)a + \nu b\right]}{\exp\left(a^{1-\nu} b^{\nu}\right)} \\
&\leq \exp\left[\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a, b\}\right]
\end{aligned}$$

for any a, $b > 0$ and $\nu \in [0, 1]$.

By utilizing a similar argument to the one from Theorem **??** we deduce the desired result (3.12).

The details are omitted. □

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