INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA JENSEN AND SLATER'S RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we prove among others that,

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq \exp\left[\langle Ax, x\rangle \langle A^{-1}x, x\rangle - 1\right]$$

for $A > 0$ and $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [5], [6], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [5].

For each unit vector $x \in H$, see also [8], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.
Key words and phrases. Positive operators, Normalized determinants, Inequalities.

RGMIA Res. Rep. Coll. **25** (2022), Art. 6,11 pp. Received 28/01/22

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [5] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [6], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Motivated by the above results, in this paper we prove among others that,

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)}, \ \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq \exp\left[\langle Ax, x\rangle \langle A^{-1}x, x\rangle - 1\right]$$

for $A > 0$ and $x \in H$ with $\|x\| = 1$.

2. Main Results

Our first main result is as follows:

Theorem 1. *For all $A > 0$ and $x \in H$, $\|x\| = 1$,*

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq \exp\left[\langle Ax, x\rangle \langle A^{-1}x, x\rangle - 1\right]. \tag{2.1}$$

Proof. In [1] we proved the following reverse of Jensen's inequality

$$0 \leq \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle,$$

where $f: I \to \mathbb{R}$ is a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$, A is a selfadjoint operator with spectrum $\operatorname{Sp}(A) \subset I$ and $x \in H$, $\|x\| = 1$.

If we take in this inequality $f(t) = -\ln t$, $t > 0$, then we get [1]

$$0 \leq \ln\langle Ax, x\rangle - \langle \ln Ax, x\rangle \leq \langle Ax, x\rangle \langle A^{-1}x, x\rangle - 1$$

for $A > 0$ and $x \in H$, $\|x\| = 1$.

Therefore

$$\ln\langle Ax, x\rangle \leq \langle \ln Ax, x\rangle + \langle Ax, x\rangle \langle A^{-1}x, x\rangle - 1$$

and by taking the exponential, we get

$$\begin{aligned}\exp\left(\ln\langle Ax,x\rangle\right) &\leq \exp\left[\langle \ln Ax,x\rangle+\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\ &=\exp\langle \ln Ax,x\rangle\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\end{aligned}$$

namely

$$\langle Ax,x\rangle\leq\Delta_x(A)\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right],$$

which is equivalent to (2.1). □

Corollary 1. *Assume that $0<mI\leq A\leq MI$, where m, M are positive numbers, then*

$$\begin{aligned}1\leq\frac{\langle Ax,x\rangle}{\Delta_x(A)}&\leq\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\ &\leq\exp\left[\frac{(M-m)^2}{4mM}\right]\end{aligned}\tag{2.2}$$

for all $x\in H$, $\|x\|=1$.

Proof. We use Kantorovich inequality, see for instance [7, p. 30]

$$\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle\leq\frac{(m+M)^2}{4mM}$$

for all $x\in H$, $\|x\|=1$.

Then

$$\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\leq\frac{(m+M)^2}{4mM}-1=\frac{(M-m)^2}{4mM}$$

and the third inequality in (2.2) is obtained. □

Corollary 2. *With the assumptions of Corollary 1,*

$$\begin{aligned}1\leq\frac{\langle Ax,x\rangle}{\Delta_x(A)}&\leq\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\ &\leq\exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\langle Ax,x\rangle\right]\leq\exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right]\end{aligned}\tag{2.3}$$

for all $x\in H$, $\|x\|=1$.

Proof. We use the additive inequality [7, p. 30]

$$\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}$$

for all $x\in H$, $\|x\|=1$.

If we multiply by $\langle Ax,x\rangle>0$ we get

$$\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\langle Ax,x\rangle\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m},$$

which proves the last part of (2.3). □

Theorem 2. *For all $A>0$ and $x\in H$, $\|x\|=1$,*

$$1\leq\frac{\Delta_x(A)}{\left\langle A^{-1}x,x\right\rangle^{-1}}\leq\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]. \tag{2.4}$$

Proof. Assume that A is a selfadjoint operator with spectrum $\operatorname{Sp}(A)\subset I$ and $x\in H$, $\|x\|=1$. In [2] we proved the following reverse of Slater's inequality

$$\begin{aligned}0&\leq f\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)-\langle f(A)x,x\rangle\\&\leq f'\left(\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\right)\frac{\langle f'(A)Ax,x\rangle-\langle Ax,x\rangle\langle f'(A)x,x\rangle}{\langle f'(A)x,x\rangle},\end{aligned}$$

where $f:I\to\mathbb{R}$ is a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$ and

$$\frac{\langle Af'(A)x,x\rangle}{\langle f'(A)x,x\rangle}\in\mathring{I}\text{ for any }x\in H,\|x\|=1.$$

Now, if we write this inequality for the convex function $f(t)=-\ln t$, $t>0$, then we have [2]

$$0\leq\langle\ln Ax,x\rangle-\ln\left(\left\langle A^{-1}x,x\right\rangle^{-1}\right)\leq\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1$$

for $A>0$ and $x\in H$, $\|x\|=1$.

Therefore

$$\langle\ln Ax,x\rangle\leq\ln\left(\left\langle A^{-1}x,x\right\rangle^{-1}\right)+\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1$$

and by taking the exponential, we get

$$\begin{aligned}\Delta_x(A)&\leq\exp\left[\ln\left(\left\langle A^{-1}x,x\right\rangle^{-1}\right)+\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\&=\exp\left[\ln\left(\left\langle A^{-1}x,x\right\rangle^{-1}\right)\right]\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\&=\left\langle A^{-1}x,x\right\rangle^{-1}\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right],\end{aligned}$$

which proves the second part of (2.4). □

Finally, we have:

Corollary 3. *With the assumptions of Corollary 1,*

$$\begin{aligned}1&\leq\frac{\Delta_x(A)}{\left\langle A^{-1}x,x\right\rangle^{-1}}\leq\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\&\leq\exp\left[\frac{(M-m)^2}{4mM}\right]\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}1&\leq\frac{\Delta_x(A)}{\left\langle A^{-1}x,x\right\rangle^{-1}}\leq\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]\\&\leq\exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\langle Ax,x\rangle\right]\leq\exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right]\end{aligned} \tag{2.6}$$

for all $x\in H$, $\|x\|=1$.

Theorem 3. *Assume that $0 < mI \leq A \leq MI$, where m, M are positive numbers, then*

$$1 \leq \frac{\Delta_x(A)}{m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}}} \leq \exp\left[\frac{\langle (MI-A)(A-mI)x,x\rangle}{mM}\right] \leq \exp\left[\frac{((M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m))}{mM}\right] \leq \exp\left[\frac{(M-m)^2}{4mM}\right] \tag{2.7}$$

for all $x \in H$, $\|x\| = 1$.

Proof. Let $f : [m, M] \to \mathbb{R}$ be a convex function with finite lateral derivatives $f'_+(m)$ and $f'_-(M)$ and A a selfadjoint operator satisfying the condition $mI \leq A \leq MI$. We have the following generalized trapezoid inequality [3]

$$\begin{aligned} 0 &\leq \frac{f(m)(MI-A) + f(M)(A-mI)}{M-m} - f(A) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M-m}(MI-A)(A-mI) \end{aligned}$$

in the operator order.

If we write this inequality for the convex function $f(t) = -\ln t$, $t > 0$, then we get

$$0 \leq \ln A - \frac{\ln m(MI-A) + \ln M(A-mI)}{M-m} \leq \frac{(MI-A)(A-mI)}{mM}$$

for all $x \in H$, $\|x\| = 1$.

Therefore

$$\begin{aligned} \langle \ln Ax, x\rangle &\leq \frac{\ln m(M-\langle Ax,x\rangle) + \ln M(\langle Ax,x\rangle - m)}{M-m} \\ &+ \frac{\langle (MI-A)(A-mI)x,x\rangle}{mM} \end{aligned}$$

and by taking the exponential, we get

$$\begin{aligned} \exp\langle \ln Ax, x\rangle &\leq \exp\left[\frac{\ln m(M-\langle Ax,x\rangle) + \ln M(\langle Ax,x\rangle - m)}{M-m}\right] \\ &\times \exp\left[\frac{\langle (MI-A)(A-mI)x,x\rangle}{mM}\right] \\ &= m^{\frac{M-\langle Ax,x\rangle}{M-m}} M^{\frac{\langle Ax,x\rangle - m}{M-m}} \exp\left[\frac{\langle (MI-A)(A-mI)x,x\rangle}{mM}\right] \end{aligned}$$

for all $x \in H$, $\|x\| = 1$, which proves the second inequality in (2.7).

The function $g(t) = (M-t)(t-m)$ is concave on $[m, M]$ and by Jensen's inequality

$$\langle g(A)x, x\rangle \leq g(\langle Ax,x\rangle),\ x \in H, \|x\| = 1$$

we have

$$\langle (MI-A)(A-mI)x,x\rangle \leq ((M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m))$$

for all $x \in H$, $\|x\| = 1$, which proves the third inequality in (2.7).

The last part is obvious. $\square$

3. Related results

We have:

Theorem 4. *For all* A, $B>0$ *and* $x\in H$, $\|x\|=1$,

$$\frac{2}{\langle A^{-1}x,x\rangle+\langle B^{-1}x,x\rangle}\leq\left\langle\left[\int_0^1((1-t)A+tB)^{-1}dt\right]x,x\right\rangle^{-1}\tag{3.1}$$
$$\leq\int_0^1\Delta_x((1-t)A+tB)dt\leq\left\langle\frac{Ax+Bx}{2},x\right\rangle.$$

Proof. By utilizing the inequalities from (ii) we get

$$\left\langle((1-t)A+tB)^{-1}x,x\right\rangle^{-1}\leq\Delta_x((1-t)A+tB)\leq\langle((1-t)A+tB)x,x\rangle$$

for all A, $B>0$, $x\in H$, $\|x\|=1$ and $t\in[0,1]$.

By taking the integral over $t\in[0,1]$, we get

$$\begin{aligned}\int_0^1\left\langle((1-t)A+tB)^{-1}x,x\right\rangle^{-1}dt&\leq\int_0^1\Delta_x((1-t)A+tB)dt\\&\leq\int_0^1\langle((1-t)A+tB)x,x\rangle dt\\&=\left\langle\frac{Ax+Bx}{2},x\right\rangle\end{aligned}$$

for all A, $B>0$, $x\in H$, $\|x\|=1$. This proves the last part of (3.1).

Using Jensen's integral inequality for the convex function $g(u)=u^{-1}$, $u>0$, namely

$$\int_0^1[f(t)]^{-1}\geq\left(\int_0^1f(t)dt\right)^{-1}$$

we obtain

$$\begin{aligned}\int_0^1\left\langle((1-t)A+tB)^{-1}x,x\right\rangle^{-1}dt&\geq\left(\int_0^1\left\langle((1-t)A+tB)^{-1}x,x\right\rangle dt\right)^{-1}\\&=\left\langle\left[\int_0^1((1-t)A+tB)^{-1}dt\right]x,x\right\rangle^{-1}\end{aligned}$$

for all A, $B>0$, $x\in H$, $\|x\|=1$.

The function $g(u)=u^{-1}$, $u>0$ is operator convex and by Hermite-Hadamard inequality for operator convex functions, we have, see [4]

$$\left(\frac{A+B}{2}\right)^{-1}\leq\int_0^1((1-t)A+tB)^{-1}dt\leq\frac{1}{2}\left(A^{-1}+B^{-1}\right)$$

for A, $B>0$.

Therefore

$$\left\langle\left[\int_0^1((1-t)A+tB)^{-1}dt\right]x,x\right\rangle\leq\frac{1}{2}\left\langle\left(A^{-1}+B^{-1}\right)x,x\right\rangle$$

$x \in H$, $\|x\| = 1$, which gives

$$\left\langle \left[\int_0^1 ((1-t)A + tB)^{-1} dt \right] x, x \right\rangle^{-1} \geq \left(\frac{1}{2} \left\langle \left(A^{-1} + B^{-1} \right) x, x \right\rangle \right)^{-1}$$
$$= \left(\frac{1}{2} \left[\left\langle A^{-1}x, x \right\rangle + \left\langle B^{-1}x, x \right\rangle \right] \right)^{-1}$$
$$= \frac{2}{\langle A^{-1}x, x\rangle + \langle B^{-1}x, x\rangle}$$

for $x \in H$, $\|x\| = 1$. This proves the first part of (3.1). □

Theorem 5. *For all A, $B > 0$ and $x \in H$, $\|x\| = 1$, then*

$$\int_0^1 \Delta_x((1-t)A + tB)dt \geq L(\Delta_x(A), \Delta_x(B)). \tag{3.2}$$

Also

$$\ln \Delta_x \left(\frac{A+B}{2} \right) \geq \int_0^1 \ln \Delta_x((1-t)A + tB)dt \geq \frac{1}{2} [\ln \Delta_x(A) + \ln \Delta_x(B)], \tag{3.3}$$

or, equivalently,

$$\Delta_x \left(\frac{A+B}{2} \right) \geq \exp \left[\int_0^1 \ln \Delta_x((1-t)A + tB)dt \right] \geq \sqrt{\Delta_x(A)\Delta_x(B)}. \tag{3.4}$$

Proof. Let $x \in H$, $\|x\| = 1$. From (viii) we have by taking the integral and assuming that $\Delta_x(A) \neq \Delta_x(B)$,

$$\int_0^1 \Delta_x((1-t)A + tB)dt \geq \int_0^1 \Delta_x(A)^{1-t} \Delta_x(B)^t dt = \frac{\Delta_x(A) - \Delta_x(B)}{\ln \Delta_x(A) - \ln \Delta_x(B)}$$
$$= L(\Delta_x(A), \Delta_x(B)).$$

If $\Delta_x(A) \neq \Delta_x(B)$, then the inequality (3.2) also holds.

If we take the log in (viii), then we get

$$\ln \Delta_x((1-t)A + tB) \geq (1-t) \ln \Delta_x(A) + t \ln \Delta_x(B) \tag{3.5}$$

for all A, $B > 0$ and $t \in [0,1]$.

By taking the integral over $t \in [0,1]$ in (3.5), we get

$$\int_0^1 \ln \Delta_x((1-t)A + tB)dt \geq \int_0^1 [(1-t) \ln \Delta_x(A) + t \ln \Delta_x(B)] dt$$
$$= \frac{1}{2} [\ln \Delta_x(A) + \ln \Delta_x(B)],$$

which proves the second inequality in (3.3).

From (3.5) we get

$$\ln \Delta_x \left(\frac{C+D}{2} \right) \geq \frac{1}{2} [\ln \Delta_x(C) + \ln \Delta_x(D)] \tag{3.6}$$

for all C, $D > 0$ and $x \in H$, $\|x\| = 1$.

Now, if we take $C = (1-t)A + tB$ and $D = tA + (1-t)B$ in (3.6), then we get

$$\ln \Delta_x \left(\frac{A+B}{2} \right) \geq \frac{1}{2} [\ln \Delta_x((1-t)A + tB) + \ln \Delta_x(tA + (1-t)B)] \tag{3.7}$$

for all A, $B > 0$ and $t \in [0,1]$.

If we take the integral over $t \in [0,1]$ in (3.7), we deduce

$$\begin{aligned}
&\ln \Delta_x \left(\frac{A+B}{2} \right) \\
&\geq \frac{1}{2} \int_0^1 \left[\ln \Delta_x ((1-t) A + tB) + \ln \Delta_x (tA + (1-t) B) \right] dt \\
&= \frac{1}{2} \left[\int_0^1 \ln \Delta_x ((1-t) A + tB) dt + \int_0^1 \ln \Delta_x (tA + (1-t) B) dt \right] \\
&= \int_0^1 \ln \Delta_x ((1-t) A + tB) dt,
\end{aligned}$$

which proves the first part of (3.3). □

We also have:

Proposition 1. *Assume that $0 < mI \leq A,\ B \leq MI$, where $m,\ M$ are positive numbers, then*

$$\begin{aligned}
0 &\leq \left\langle \frac{Ax+Bx}{2}, x \right\rangle - \int_0^1 \Delta_x ((1-t) A + tB) dt \\
&\leq L(m,M) \left[\ln L(m,M) + \frac{M \ln M - m \ln M}{M-m} - 1 \right]
\end{aligned} \tag{3.8}$$

and

$$1 \leq \frac{\left\langle \frac{Ax+Bx}{2}, x \right\rangle}{\int_0^1 \Delta_x ((1-t) A + tB) dt} \leq S\left(\frac{M}{m} \right) \tag{3.9}$$

for $x \in H$, $\|x\| = 1$.

Proof. Since $0 < mI \leq A,\ B \leq MI$, hence $0 < mI \leq (1-t) A + tB \leq MI$ for all $t \in [0,1]$. By using (1.1) we get

$$\begin{aligned}
0 &\leq \langle ((1-t) A + tB) x, x \rangle - \Delta_x ((1-t) A + tB) \\
&\leq L(m,M) \left[\ln L(m,M) + \frac{M \ln M - m \ln M}{M-m} - 1 \right]
\end{aligned}$$

and by taking the integral over $t \in [0,1]$, we get (3.8).

The inequality (3.9) follows in a similar way from (3.9). □

Proposition 2. *With the assumptions of Proposition 1 we have*

$$1 \leq \frac{\left\langle \frac{Ax+Bx}{2}, x \right\rangle}{\int_0^1 \Delta_x ((1-t) A + tB) dt} \leq \exp \left[\frac{(M-m)^2}{4mM} \right] \tag{3.10}$$

and

$$1 \leq \frac{\left\langle \frac{Ax+Bx}{2}, x \right\rangle}{\int_0^1 \Delta_x ((1-t) A + tB) dt} \leq \exp \left[\frac{\left(\sqrt{M} - \sqrt{m} \right)^2}{m} \right] \tag{3.11}$$

for $x \in H$, $\|x\| = 1$.

Proof. If we use the inequality (2.2), then we can state for $x \in H$, $\|x\| = 1$ that

$$\langle ((1-t)A + tB)x, x\rangle \le \Delta_x(((1-t)A + tB)) \exp\left[\frac{(M-m)^2}{4mM}\right]$$

for all $t \in [0,1]$.

If we take the integral over $t \in [0,1]$, then we get (3.10).

Inequality (3.11) follows by (2.3). □

Proposition 3. *With the assumptions of Proposition 1 we have*

$$1 \le \frac{\int_0^1 \Delta_x((1-t)A + tB)\,dt}{\int_0^1 \left\langle ((1-t)A + tB)^{-1} x, x\right\rangle^{-1} dt} \le \exp\left[\frac{(M-m)^2}{4mM}\right] \tag{3.12}$$

and

$$1 \le \frac{\int_0^1 \Delta_x((1-t)A + tB)\,dt}{\int_0^1 \left\langle ((1-t)A + tB)^{-1} x, x\right\rangle^{-1} dt} \le \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}\right] \tag{3.13}$$

for $x \in H$, $\|x\| = 1$.

Proof. By (2.5) we have

$$\Delta_x((1-t)A + tB) \le \left\langle (((1-t)A + tB))^{-1} x, x\right\rangle^{-1} \exp\left[\frac{(M-m)^2}{4mM}\right]$$

for all for all $t \in [0,1]$.

If we take the integral over $t \in [0,1]$, then we get

$$\begin{aligned}&\int_0^1 \Delta_x((1-t)A + tB)\,dt \\ &\le \int_0^1 \left\langle ((1-t)A + tB)^{-1} x, x\right\rangle^{-1} dt \exp\left[\frac{(M-m)^2}{4mM}\right]\end{aligned}$$

for $x \in H$, $\|x\| = 1$, which proves (3.12).

The inequality (3.13) follows by (2.6). □

Proposition 4. *With the assumptions of Proposition 1 we have for* $x \in H$, $\|x\| = 1$ *that*

$$\begin{aligned}&\exp\left[\frac{M\ln m - m\ln M}{M-m} + \frac{\ln M - \ln m}{M-m}\right] L(A,B,x,m,M) \\ &\le \int_0^1 \Delta_x(((1-t)A + tB))\,dt \\ &\le \exp\left[\frac{M\ln m - m\ln M}{M-m} + \frac{\ln M - \ln m}{M-m} + \frac{(M-m)^2}{4mM}\right] \\ &\times L(A,B,x,m,M),\end{aligned} \tag{3.14}$$

where

$$L(A,B,x,m,M)$$
$$:= \begin{cases} \frac{M-m}{(\ln M-\ln m)\langle (B-A)x,x\rangle} \\ \times \left[\exp\left[\frac{\ln M-\ln m}{M-m}\langle (B-A)x,x\rangle\right]-1\right] \\ \textit{if } \langle (B-A)x,x\rangle \neq 0, \\ \\ 1 \textit{ if } \langle (B-A)x,x\rangle = 0. \end{cases}$$

Proof. From (2.7) we get

$$\begin{aligned} &\exp\left[\frac{\ln m\,(M-\langle((1-t)A+tB)x,x\rangle)+\ln M\,(\langle((1-t)A+tB)x,x\rangle-m)}{M-m}\right] \\ &\le \Delta_x(((1-t)A+tB)) \\ &\le \exp\left[\frac{\ln m\,(M-\langle((1-t)A+tB)x,x\rangle)+\ln M\,(\langle((1-t)A+tB)x,x\rangle-m)}{M-m}\right] \\ &\times \exp\left[\frac{(M-m)^2}{4mM}\right] \end{aligned}$$

namely

$$\begin{aligned} &\exp\left[\frac{M\ln m-m\ln M}{M-m}\right]\exp\left(\frac{\ln M-\ln m}{M-m}\langle((1-t)A+tB)x,x\rangle\right) \\ &\le \Delta_x(((1-t)A+tB)) \\ &\le \exp\left[\frac{M\ln m-m\ln M}{M-m}\right]\exp\left(\frac{\ln M-\ln m}{M-m}\langle((1-t)A+tB)x,x\rangle\right) \\ &\times \exp\left[\frac{(M-m)^2}{4mM}\right] \end{aligned}$$

for $x\in H$, $\|x\|=1$.

This is equivalent to

$$\begin{aligned} &\exp\left[\frac{M\ln m-m\ln M}{M-m}+\frac{\ln M-\ln m}{M-m}\right] \\ &\times \exp\left(t\frac{\ln M-\ln m}{M-m}\langle (B-A)x,x\rangle\right) \\ &\le \Delta_x(((1-t)A+tB)) \\ &\le \exp\left[\frac{M\ln m-m\ln M}{M-m}+\frac{\ln M-\ln m}{M-m}\right] \\ &\times \exp\left(t\frac{\ln M-\ln m}{M-m}\langle (B-A)x,x\rangle\right) \\ &\times \exp\left[\frac{(M-m)^2}{4mM}\right] \end{aligned}$$

for $x\in H$, $\|x\|=1$.

If we take the integral over $t \in [0,1]$, then we get

$$\begin{aligned}
&\exp\left[\frac{M\ln m - m\ln M}{M-m} + \frac{\ln M - \ln m}{M-m}\right] \\
&\times \int_0^1 \exp\left(t\frac{\ln M - \ln m}{M-m}\left\langle (B-A)x, x\right\rangle\right) dt \\
&\leq \int_0^1 \Delta_x\left(((1-t)A + tB)\right) dt \\
&\leq \exp\left[\frac{M\ln m - m\ln M}{M-m} + \frac{\ln M - \ln m}{M-m}\right] \\
&\times \int_0^1 \exp\left(t\frac{\ln M - \ln m}{M-m}\left\langle (B-A)x, x\right\rangle\right) dt \\
&\times \exp\left[\frac{(M-m)^2}{4mM}\right]
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

Now, observe that

$$\int_0^1 \exp\left(t\frac{\ln M - \ln m}{M-m}\left\langle (B-A)x, x\right\rangle\right) dt = L(A, B, x, m, M)$$

and the proof is completed. □

References

[1] S. S. Dragomir, Some Reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821.

[2] S. S. Dragomir, Some Slater's type inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Rev. Un. Mat. Argentina,* **52** (2011), no. 1, 109–120;

[3] S. S. Dragomir, Vector and operator trapezoid type inequalities for continuous functions of selfadjoint operators, *Elec. Lin. Alg.*, Volume **22**, pp. 161-178, March 2011.

[4] S. S. Dragomir, Hermite–Hadamard's type inequalities for operator convex functions, *Applied Mathematics and Computation*, **218** (2011), Issue 3, pp. 766–772.

[5] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[6] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310. ć

[7] T. Furuta, J. Mičić-Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Croatia.

[8] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.

[9] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa