NEW INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA JENSEN AND SLATER'S RESULTS

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we prove among others that, if $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$, then

$$\begin{aligned}
0 &\le \langle Ax, x\rangle - \Delta_x(A) \\
&\le \langle A \ln Ax, x\rangle - \langle \ln Ax, x\rangle \langle Ax, x\rangle \\
&\le \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned}$$

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [6].

For each unit vector $x \in H$, see also [9], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;

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(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [6] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [10]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Motivated by the above results, in this paper we prove among others that, if $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$, then

$$\begin{aligned} 0 &\leq \langle Ax, x\rangle - \Delta_x(A) \\ &\leq \langle A\ln Ax, x\rangle - \langle \ln Ax, x\rangle \langle Ax, x\rangle \\ &\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right). \end{aligned}$$

2. Main Result

The first result is as follows:

Theorem 1. *Assume that* $0 < mI \leq A \leq MI$ *and* $x \in H$, $\|x\| = 1$, *then*

$$\begin{aligned}
(2.1) \qquad & 0 \leq \langle Ax, x\rangle - \Delta_x(A) \\
& \leq \langle A \ln Ax, x\rangle - \langle \ln Ax, x\rangle \langle Ax, x\rangle \\
& \leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\|\ln Ax\|^2 - \langle \ln Ax, x\rangle^2\right]^{1/2} \end{cases} \\
& \leq \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned}$$

Proof. In [1] we obtained the following result.

Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If B is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(B) \subseteq [k, K] \subset \mathring{I}$, then

$$\begin{aligned}
(2.2) \qquad & (0 \leq) \langle f(B)x, x\rangle - f(\langle Bx, x\rangle) \\
& \leq \langle f'(B)Bx, x\rangle - \langle Bx, x\rangle \langle f'(B)x, x\rangle \\
& \leq \begin{cases} \frac{1}{2}(K-k)\left[\|f'(B)x\|^2 - \langle f'(B)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(f'(K) - f'(k))\left[\|Bx\|^2 - \langle Bx, x\rangle^2\right]^{1/2} \end{cases} \\
& \leq \frac{1}{4}(K-k)(f'(K) - f'(k)),
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

If we take $f(t) = \exp t$, $t \in \mathbb{R}$ and B is a selfadjoint operator with $\operatorname{Sp}(B) \subseteq [k, K]$, then by (2.2) we get

$$\begin{aligned}
(2.3) \qquad & 0 \leq \langle \exp(B)x, x\rangle - \exp(\langle Bx, x\rangle) \\
& \leq \langle B\exp(B)x, x\rangle - \langle Bx, x\rangle \langle \exp(B)x, x\rangle \\
& \leq \begin{cases} \frac{1}{2}(K-k)\left[\|\exp(B)x\|^2 - \langle \exp(B)x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(\exp(K) - \exp(k))\left[\|Bx\|^2 - \langle Bx, x\rangle^2\right]^{1/2} \end{cases} \\
& \leq \frac{1}{4}(K-k)(\exp(K) - \exp(k)),
\end{aligned}$$

for any $x \in H$ with $\|x\| = 1$.

If $0 < m \le A \le M$, then $\ln m \le \ln A \le \ln M$ and by taking $B = \ln A$, $k = \ln m$, $K = \ln M$ in (2.3) we get

$$\begin{aligned} 0 &\le \langle \exp(\ln A) x, x\rangle - \exp(\langle \ln A x, x\rangle) \\ &\le \langle \ln A \exp(\ln A) x, x\rangle - \langle \ln A x, x\rangle \langle \exp(\ln A) x, x\rangle \\ &\le \begin{cases} \frac{1}{2}(\ln M - \ln m)\left[\|\exp(\ln A) x\|^2 - \langle \exp(\ln A) x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(\exp(\ln M) - \exp(\ln m))\left[\|\ln A x\|^2 - \langle \ln A x, x\rangle^2\right]^{1/2} \end{cases} \\ &\le \frac{1}{4}(\ln M - \ln m)(\exp(\ln M) - \exp(\ln m)), \end{aligned}$$

namely

$$\begin{aligned} 0 &\le \langle Ax, x\rangle - \exp(\langle \ln A x, x\rangle) \\ &\le \langle A \ln A x, x\rangle - \langle \ln A x, x\rangle \langle Ax, x\rangle \\ &\le \begin{cases} \frac{1}{2}(\ln M - \ln m)\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m)\left[\|\ln A x\|^2 - \langle \ln A x, x\rangle^2\right]^{1/2} \end{cases} \\ &\le \frac{1}{4}(\ln M - \ln m)(M - m), \end{aligned}$$

which is the desired result (2.1). □

Remark 1. *We note that the first two inequalities hold for any positive definite operator $A > 0$.*

Theorem 2. *Assume that $I < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$, then*

$$\begin{aligned} 0 &\le \langle Ax, x\rangle - \Delta_x(A) \\ &\le \begin{cases} \frac{1}{4}\frac{(\ln M - \ln m)(M-m)}{\sqrt{Mm \ln M \ln m}} \langle \ln A x, x\rangle \langle Ax, x\rangle, \\ \\ \left(\sqrt{\ln M} - \sqrt{\ln m}\right)\left(\sqrt{M} - \sqrt{m}\right)\left[\langle \ln A x, x\rangle \langle Ax, x\rangle\right]^{\frac{1}{2}}, \end{cases} \end{aligned} \tag{2.4}$$

or, equivalently

$$\begin{aligned} 0 &\le 1 - \frac{\Delta_x(A)}{\langle Ax, x\rangle} \\ &\le \begin{cases} \frac{1}{4}\frac{(\ln M - \ln m)(M-m)}{\sqrt{Mm \ln M \ln m}} \langle \ln A x, x\rangle, \\ \\ \left(\sqrt{\ln M} - \sqrt{\ln m}\right)\left(\sqrt{M} - \sqrt{m}\right)\left[\frac{\langle \ln A x, x\rangle}{\langle Ax, x\rangle}\right]^{\frac{1}{2}}. \end{cases} \end{aligned} \tag{2.5}$$

Proof. Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If B is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(B) \subseteq [k, K] \subset \mathring{I}$, then [1]

$$\begin{aligned}
&(0 \leq) \langle f(B)x, x\rangle - f(\langle Bx, x\rangle) \\
&\leq \langle f'(B)Bx, x\rangle - \langle Bx, x\rangle \langle f'(B)x, x\rangle \\
&\leq \begin{cases} \frac{1}{4}\frac{(K-k)\left(f'(K)-f'(k)\right)}{\sqrt{Kkf'(K)f'(k)}} \langle Bx, x\rangle \langle f'(B)x, x\rangle, \\ \\ \left(\sqrt{K}-\sqrt{k}\right)\left(\sqrt{f'(K)}-\sqrt{f'(k)}\right)\left[\langle Bx, x\rangle \langle f'(B)x, x\rangle\right]^{\frac{1}{2}}, \end{cases}
\end{aligned}$$

if $k > 0$ and $f'(k) > 0$, for any $x \in H$ with $\|x\| = 1$.

If we take $f(t) = \exp t$, $t \in \mathbb{R}$ and B is a selfadjoint operator with $\operatorname{Sp}(B) \subseteq [k, K]$, then

$$\begin{aligned}
&0 \leq \langle \exp(B)x, x\rangle - \exp(\langle Bx, x\rangle) \\
&\leq \begin{cases} \frac{1}{4}\frac{(K-k)(\exp(K)-\exp(k))}{\sqrt{Kk\exp(K)\exp(k)}} \langle Bx, x\rangle \langle \exp(B)x, x\rangle, \\ \\ \left(\sqrt{K}-\sqrt{k}\right)\left(\sqrt{\exp(K)}-\sqrt{\exp(k)}\right) \\ \times\left[\langle Bx, x\rangle \langle \exp(B)x, x\rangle\right]^{\frac{1}{2}}. \end{cases}
\end{aligned} \tag{2.6}$$

If $I < mI \leq A \leq MI$, then $0 < \ln m \leq \ln A \leq \ln M$ and by taking $B = \ln A$, $k = \ln m$, $K = \ln M$ in (2.6) we get

$$\begin{aligned}
&0 \leq \langle \exp(\ln A)x, x\rangle - \exp(\langle \ln Ax, x\rangle) \\
&\leq \begin{cases} \frac{1}{4}\frac{(\ln M-\ln m)(\exp(\ln M)-\exp(\ln m))}{\sqrt{\ln M\ln m\exp(\ln M)\exp(\ln m)}} \langle \ln Ax, x\rangle \langle \exp(\ln A)x, x\rangle, \\ \\ \left(\sqrt{\ln M}-\sqrt{\ln m}\right)\left(\sqrt{\exp(\ln M)}-\sqrt{\exp(\ln m)}\right) \\ \times\left[\langle \ln Ax, x\rangle \langle \exp(\ln A)x, x\rangle\right]^{\frac{1}{2}}, \end{cases}
\end{aligned}$$

namely

$$\begin{aligned}
&0 \leq \langle Ax, x\rangle - \exp(\langle \ln Ax, x\rangle) \\
&\leq \begin{cases} \frac{1}{4}\frac{(\ln M-\ln m)(M-m)}{\sqrt{Mm\ln M\ln m}} \langle \ln Ax, x\rangle \langle Ax, x\rangle, \\ \\ \left(\sqrt{\ln M}-\sqrt{\ln m}\right)\left(\sqrt{M}-\sqrt{m}\right)\left[\langle \ln Ax, x\rangle \langle Ax, x\rangle\right]^{\frac{1}{2}}, \end{cases}
\end{aligned}$$

which proves (2.4). □

Theorem 3. *Assume that* $0 < mI \leq A \leq MI$ *and* $x \in H$, $\|x\| = 1$, *then*

$$0 \leq \ln \Delta_x(A) - \ln\left(\left\langle A^{-1}x, x\right\rangle^{-1}\right) \leq \langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1 \tag{2.7}$$
$$\leq \begin{cases} \frac{1}{2}\frac{M-m}{mM}\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}, \\ \\ \frac{1}{2}(M-m)\left[\left\|A^{-1}x\right\|^2 - \left\langle A^{-1}x, x\right\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}\frac{(M-m)^2}{mM},$$

or, equivalently

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq \exp\left[\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1\right] \tag{2.8}$$
$$\leq \begin{cases} \exp\left(\frac{1}{2}\frac{M-m}{mM}\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{1}{2}(M-m)\left[\left\|A^{-1}x\right\|^2 - \left\langle A^{-1}x, x\right\rangle^2\right]^{1/2}\right) \end{cases}$$
$$\leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right].$$

Proof. Assume that A is a selfadjoint operator with spectrum $\operatorname{Sp}(A) \subset I$ and $x \in H$, $\|x\| = 1$. In [2] we proved the following reverse of Slater's inequality

$$0 \leq f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle$$
$$\leq f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\frac{\langle f'(A)Ax, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle}{\langle f'(A)x, x\rangle}$$
$$\leq \frac{1}{\langle f'(A)x, x\rangle} f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)$$
$$\times \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right]\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}, \\ \\ \frac{1}{2}(M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x, x\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4}\frac{(M-m)\left[f'(M) - f'(m)\right]}{\langle f'(A)x, x\rangle} f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right),$$

where $f : I \to \mathbb{R}$ is a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$ and

$$\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle} \in \mathring{I} \text{ for any } x \in H, \|x\| = 1.$$

Now, if we write this inequality for the convex function $f(t) = -\ln t$, $t > 0$, then we have for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$, that

$$\begin{aligned}
0 &\le \langle \ln Ax, x\rangle - \ln\left(\left\langle A^{-1}x, x\right\rangle^{-1}\right) \le \langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1 \\
&\le \begin{cases} \frac{1}{2}\frac{M-m}{mM}\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}, \\ \\ \frac{1}{2}(M-m)\left[\left\|A^{-1}x\right\|^2 - \left\langle A^{-1}x, x\right\rangle^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}\frac{(M-m)^2}{mM},
\end{aligned}$$

which is equivalent to (2.7). □

Remark 2. *The first two inequalities hold for any $A > 0$.*

3. Related Results

We also have:

Theorem 4. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
0 &\le \frac{m(\ln M - \langle \ln Ax, x\rangle) + M(\langle \ln Ax, x\rangle - \ln m)}{\ln M - \ln m} - \Delta_x(A) \quad (3.1)\\
&\le \frac{M-m}{\ln M - \ln m}(\langle \ln Ax, x\rangle - \ln m)(\ln M - \langle \ln Ax, x\rangle) \\
&\le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right)
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

Proof. In [4] we obtained the following inequality for convex functions $f : [a, b] \to \mathbb{R}$

$$0 \le (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \le (1-\nu)\nu(b-a)\left[f'_-(b) - f'_+(a)\right]$$

for $\nu \in [0, 1]$, provided the lateral derivatives $f'_-(b)$, $f'_+(a)$ are finite.

If we take $\nu = \frac{t-a}{b-a}$, $t \in [a, b]$, then we get

$$0 \le \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \le \frac{f'_-(b) - f'_+(a)}{b-a}(t-a)(b-t)$$

for $t \in [a, b]$.

If we take $f(t) = \exp t$, $t \in [a, b]$, then we get

$$0 \le \frac{(b-t)\exp a + (t-a)\exp b}{b-a} - \exp t \le \frac{\exp b - \exp a}{b-a}(t-a)(b-t) \quad (3.2)$$

for $t \in [a, b]$.

Since $0 < mI \le A \le M$, hence $\ln m \le \ln A \le \ln M$ and $\ln m \le \langle \ln Ax, x\rangle \le \ln M$ for $x \in H$, $\|x\| = 1$. By taking $a = \ln m$, $t = \langle \ln Ax, x\rangle$ and $b = \ln M$ in (3.2) we get

$$\begin{aligned}
0 &\le \frac{(\ln M - \langle \ln Ax, x\rangle)\exp \ln m + (\langle \ln Ax, x\rangle - \ln m)\exp \ln M}{\ln M - \ln m} \\
&- \exp\langle \ln Ax, x\rangle \\
&\le \frac{\exp \ln M - \exp \ln m}{\ln M - \ln m}(\langle \ln Ax, x\rangle - \ln m)(\ln M - \langle \ln Ax, x\rangle),
\end{aligned}$$

namely

$$\begin{aligned} 0 &\leq \frac{m\left(\ln M-\langle\ln Ax,x\rangle\right)+M\left(\langle\ln Ax,x\rangle-\ln m\right)}{\ln M-\ln m} \\ &-\exp\langle\ln Ax,x\rangle \\ &\leq \frac{M-m}{\ln M-\ln m}\left(\langle\ln Ax,x\rangle-\ln m\right)\left(\ln M-\langle\ln Ax,x\rangle\right), \end{aligned}$$

which gives the first inequality in (3.1). The second inequality is obvious. □

Theorem 5. *With the assumptions of Theorem 3 we have*

$$\begin{aligned} 1 &\leq \exp\left[\frac{1}{2M^2}\langle(A-mI)(MI-A)x,x\rangle\right] \\ &\leq \frac{\Delta_x(A)}{\exp\left[\frac{\ln m(M-\langle Ax,x\rangle)+\ln M(\langle Ax,x\rangle-m)}{M-m}\right]} \\ &\leq \exp\left[\frac{1}{2m^2}\langle(A-mI)(MI-A)x,x\rangle\right] \\ &\leq \exp\left[\frac{1}{2m^2}\left(\langle Ax,x\rangle-m\right)\left(M-\langle Ax,x\rangle\right)\right] \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right] \end{aligned} \tag{3.3}$$

for $x\in H,\ \|x\|=1$.

Proof. In [5] we proved the following double inequality. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that

$$d\leq f''(t)\leq D \text{ for any } t\in\mathring{I}, \tag{3.4}$$

then

$$\begin{aligned} \frac{1}{2}\nu(1-\nu)d(b-a)^2 &\leq (1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b) \\ &\leq \frac{1}{2}\nu(1-\nu)D(b-a)^2 \end{aligned} \tag{3.5}$$

for any $a,\ b\in\mathring{I}$ and $\nu\in[0,1]$.

If we take $\nu=\frac{t-a}{b-a}$, $t\in[a,b]$, then we get

$$\begin{aligned} \frac{1}{2}(t-a)(b-t)d &\leq \frac{(b-t)f(a)+(t-a)f(b)}{b-a}-f(t) \\ &\leq \frac{1}{2}(t-a)(b-t)D \end{aligned} \tag{3.6}$$

for $t\in[a,b]$.

If we use the continuous functional calculus, we have

$$\begin{aligned} \frac{1}{2}(B-aI)(bI-B)d &\leq \frac{f(a)(bI-B)+f(b)(B-aI)}{b-a}-f(A) \\ &\leq \frac{1}{2}(B-aI)(bI-B)D \end{aligned} \tag{3.7}$$

for $aI\leq B\leq bI$.

If we consider the convex function $f(t) = -\ln t$, $t \in [m, M] \subset (0, \infty)$, then

$$\frac{1}{M^2} \le f''(t) = \frac{1}{t^2} \le \frac{1}{m^2}$$

and by (3.7) we get

$$\frac{1}{2M^2}(A - mI)(MI - A) \le \ln(A) - \frac{\ln m (MI - A) + \ln M (A - mI)}{M - m} \le \frac{1}{2m^2}(A - mI)(MI - A),$$

which is equivalent to

$$\begin{aligned}(3.8)\qquad & \frac{1}{2M^2}\langle (A - mI)(MI - A)x, x\rangle \\ & \le \langle \ln(A)x, x\rangle - \frac{\ln m (MI - \langle Ax, x\rangle) + \ln M (\langle Ax, x\rangle - mI)}{M - m} \\ & \le \frac{1}{2m^2}\langle (A - mI)(MI - A)x, x\rangle\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By taking the exponential in (3.8), then we get the second and the third inequalities in (3.3).

The function $g(t) = (M - t)(t - m)$ is concave on $[m, M]$ and by Jensen's inequality

$$\langle g(A)x, x\rangle \le g(\langle Ax, x\rangle), \ x \in H, \|x\| = 1$$

we have

$$\langle (MI - A)(A - mI)x, x\rangle \le ((M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m))$$

for all $x \in H, \|x\| = 1$, which proves the fourth inequality in (3.3).

The last part is obvious. □

Theorem 6. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(3.9)\qquad 0 & \le \frac{1}{2}m(\langle \ln Ax, x\rangle - \ln m)(\ln M - \langle \ln Ax, x\rangle), \\ & \le \frac{(\ln M - \langle \ln Ax, x\rangle)m + (\langle \ln Ax, x\rangle - \ln m)M}{\ln M - \ln m} \\ & - \Delta_x(A) \\ & \le \frac{1}{2}M(\langle \ln Ax, x\rangle - \ln m)(\ln M - \langle \ln Ax, x\rangle), \\ & \le \frac{1}{8}M(\ln M - \ln m)\end{aligned}$$

for all $x \in H, \|x\| = 1$

Proof. If we take $f(t) = \exp t$, $t \in [a, b]$, in (3.6) then we get

$$\begin{aligned}(3.10)\qquad \frac{1}{2}(t - a)(b - t)\exp a & \le \frac{(b - t)\exp a + (t - a)\exp b}{b - a} - \exp t \\ & \le \frac{1}{2}(t - a)(b - t)\exp b.\end{aligned}$$

Since $0<mI\leq A\leq MI$, hence $\ln m\leq \ln A\leq \ln M$ and $\ln m\leq \langle \ln Ax,x\rangle \leq \ln M$ for $x\in H$, $\|x\|=1$. By taking $a=\ln m$, $t=\langle \ln Ax,x\rangle$ and $b=\ln M$ in (3.10) we get

$$\begin{aligned}0&\leq \frac{1}{2}m\left(\langle \ln Ax,x\rangle -\ln m\right)\left(\ln M-\langle \ln Ax,x\rangle\right),\\&\leq \frac{\left(\ln M-\langle \ln Ax,x\rangle\right)\exp \ln m+\left(\langle \ln Ax,x\rangle -\ln m\right)\exp \ln M}{\ln M-\ln m}-\exp\langle \ln Ax,x\rangle\\&\leq \frac{1}{2}M\left(\langle \ln Ax,x\rangle -\ln m\right)\left(\ln M-\langle \ln Ax,x\rangle\right),\end{aligned}$$

which proves (3.9). □

Finally, we can state the following

Proposition 1. *Assume that* $0<mI\leq A$, $B\leq MI$ *and* $x\in H$, $\|x\|=1$, *then*

$$0\leq \left\langle \frac{A+B}{2}x,x\right\rangle -\int_0^1\Delta_x((1-t)\,A+tB)dt\leq \frac{1}{4}\left(M-m\right)\ln\left(\frac{M}{m}\right). \tag{3.11}$$

Proof. We observe that $mI\leq (1-t)\,A+tB\leq MI$, $t\in[0,1]$ and by (2.1) we get

$$0\leq \langle [(1-t)\,A+tB]\,x,x\rangle -\Delta_x((1-t)\,A+tB)\leq \frac{1}{4}\left(M-m\right)\ln\left(\frac{M}{m}\right)$$

for all $t\in[0,1]$. By taking the integral over $t\in[0,1]$ we derive the desired result (3.11). □

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