INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA ONE VARIABLE LOG INEQUALITIES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we prove among others that,

$$\exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax-x\|^2\right) \le \frac{\exp[\langle Ax,x\rangle - 1]}{\Delta_x(A)} \le \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax-x\|^2\right)$$

and

$$\exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax-x\|^2\right) \le \frac{\Delta_x(A)}{\exp[\langle A^{-1}x,x\rangle - 1]} \le \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax-x\|^2\right)$$

for $0 < mI \le A \le MI$ and $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [5], [6], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [5].

For each unit vector $x \in H$, see also [8], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

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(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [5] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [6], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Motivated by the above results, in this paper we prove among others that,

$$\exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax - x\|^2\right) \leq \frac{\exp[\langle Ax, x\rangle - 1]}{\Delta_x(A)} \leq \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax - x\|^2\right)$$

and

$$\exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax - x\|^2\right) \leq \frac{\Delta_x(A)}{\exp[\langle A^{-1}x, x\rangle - 1]} \leq \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax - x\|^2\right)$$

for $0 < mI \leq A \leq MI$ and $x \in H$ with $\|x\| = 1$.

2. Main Results

We have the following inequalities for logarithm:

Lemma 1. *For any $a,\ b>0$ we have*

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 &= \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}}\\ &\le \frac{b-a}{a}-\ln b+\ln a\\ &\le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}}=\frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.\end{aligned} \tag{2.1}$$

Proof. Observe that

$$\int_a^b \frac{b-t}{t^2}dt = b\int_a^b t^{-2}dt-\int_a^b \frac{1}{t}dt=\frac{b-a}{a}-\ln b+\ln a \tag{2.2}$$

for any $a,\ b>0$.

If $b>a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2}\ge\int_a^b\frac{b-t}{t^2}dt\ge\frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.3}$$

If $a>b$ then

$$\int_a^b\frac{b-t}{t^2}dt=-\int_b^a\frac{b-t}{t^2}dt=\int_b^a\frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2}\ge\int_b^a\frac{t-b}{t^2}dt\ge\frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.4}$$

Therefore, by (2.3) and (2.4) we have for any $a,\ b>0$ that

$$\int_a^b\frac{b-t}{t^2}dt\ge\frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}}=\frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b\frac{b-t}{t^2}dt\le\frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}}=\frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (2.2) we then get the desired result (2.1). □

When some bounds for $a,\ b$ are provided, then we have:

Corollary 1. *Assume that $a,\ b\in[m,M]\subset(0,\infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2}\le\frac{b-a}{a}-\ln b+\ln a\le\frac{1}{2}\frac{(b-a)^2}{m^2} \tag{2.5}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{M^2}\le\ln b-\ln a-\frac{b-a}{b}\le\frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{2.6}$$

Remark 1. *If we take in (2.1) $a=1$ and $b=t\in(0,\infty)$, then we get*

$$\begin{aligned}\frac{1}{2}\left(1-\frac{\min\{1,t\}}{\max\{1,t\}}\right)^2 &= \frac{1}{2}\frac{(t-1)^2}{\max^2\{1,t\}}\\ &\le t-1-\ln t\\ &\le \frac{1}{2}\frac{(t-1)^2}{\min^2\{1,t\}}=\frac{1}{2}\left(\frac{\max\{1,t\}}{\min\{1,t\}}-1\right)^2\end{aligned} \tag{2.7}$$

and if we take $a = t$ *and* $b = 1$, *then we also get*

$$\frac{1}{2}\left(1-\frac{\min\{1,t\}}{\max\{1,t\}}\right)^2 = \frac{1}{2}\frac{(t-1)^2}{\max^2\{1,t\}} \le \ln t - \frac{t-1}{t} \le \frac{1}{2}\frac{(t-1)^2}{\min^2\{1,t\}} = \frac{1}{2}\left(\frac{\max\{1,t\}}{\min\{1,t\}}-1\right)^2. \tag{2.8}$$

If $t \in [k, K] \subset (0, \infty)$, then by analyzing all possible locations of the interval $[k, K]$ and 1 we have

$$\min\{1,k\} \le \min\{1,t\} \le \min\{1,K\}$$

and

$$\max\{1,k\} \le \max\{1,t\} \le \max\{1,K\}.$$

By (2.7) and (2.8) we get the *local bounds*

$$\frac{1}{2}\frac{(t-1)^2}{\max^2\{1,K\}} \le t-1-\ln t \le \frac{1}{2}\frac{(t-1)^2}{\min^2\{1,k\}} \tag{2.9}$$

and

$$\frac{1}{2}\frac{(t-1)^2}{\max^2\{1,K\}} \le \ln t - \frac{t-1}{t} \le \frac{1}{2}\frac{(t-1)^2}{\min^2\{1,k\}} \tag{2.10}$$

for any $t \in [k, K]$.

Theorem 1. *Assume that the operator* A *satisfies the condition* $0 < mI \le A \le MI$, *where* m, M *are positive numbers, then*

$$(1 \le) \exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax-x\|^2\right) \le \frac{\exp[\langle Ax,x\rangle - 1]}{\Delta_x(A)} \le \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax-x\|^2\right) \tag{2.11}$$

and

$$(1 \le) \exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax-x\|^2\right) \le \frac{\Delta_x(A)}{\exp[\langle A^{-1}x,x\rangle - 1]} \le \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax-x\|^2\right) \tag{2.12}$$

for all $x \in H$, $\|x\| = 1$.

Proof. Using the continuous functional calculus for A and the inequality (2.9) we have

$$\frac{1}{2\max^2\{1,M\}}(A-I)^2 \le A - I - \ln A \le \frac{1}{2\min^2\{1,m\}}(A-I)^2,$$

which is equivalent to

$$\frac{1}{2\max^2\{1,M\}}\left\langle (A-I)^2 x,x\right\rangle \le \langle Ax,x\rangle - 1 - \langle \ln Ax,x\rangle \le \frac{1}{2\min^2\{1,m\}}\left\langle (A-I)^2 x,x\right\rangle$$

for all $x\in H$, $\|x\|=1$.

This is equivalent to

$$\exp\left(\frac{1}{2\max^2\{1,M\}}\|Ax-x\|^2\right) \le \frac{\exp\left[\langle Ax,x\rangle -1\right]}{\exp\langle \ln Ax,x\rangle} \le \exp\left(\frac{1}{2\min^2\{1,m\}}\|Ax-x\|^2\right)$$

which proves (2.11).

From (2.10) we also get

$$\frac{1}{2\max^2\{1,M\}}(A-I)^2 \le \ln A - I + A^{-1} \le \frac{1}{2\min^2\{1,m\}}(A-I)^2,$$

namely

$$\frac{1}{2\max^2\{1,M\}}\|Ax-x\|^2 \le \langle \ln Ax,x\rangle - 1 + \left\langle A^{-1}x,x\right\rangle \le \frac{1}{2\min^2\{1,m\}}\|Ax-x\|^2,$$

which gives in (2.12). □

Corollary 2. *With the assumptions of Theorem 1,*

$$1 \le \exp\left(\frac{\min^2\{1,m\}}{2}\left\|A^{-1}x-x\right\|^2\right) \le \frac{\Delta_x(A)}{\exp\left[1-\langle A^{-1}x,x\rangle\right]} \le \exp\left(\frac{\max^2\{1,M\}}{2}\left\|A^{-1}x-x\right\|^2\right) \tag{2.13}$$

and

$$1 \le \exp\left(\frac{\min^2\{1,m\}}{2}\left\|A^{-1}x-x\right\|^2\right) \le \frac{\exp\left[1-\langle Ax,x\rangle\right]}{\Delta_x(A)} \le \exp\left(\frac{\max^2\{1,M\}}{2}\left\|A^{-1}x-x\right\|^2\right) \tag{2.14}$$

for all $x\in H$, $\|x\|=1$.

Proof. If $0<mI\le A\le MI$, where m,M are positive numbers, then $0<M^{-1}I\le A^{-1}\le m^{-1}I$. If we write the inequality (2.11) for A^{-1}, then we can state that

$$1 \le \exp\left(\frac{1}{2\max^2\{1,m^{-1}\}}\left\|A^{-1}x-x\right\|^2\right) \le \frac{\exp\left[\left\langle A^{-1}x,x\right\rangle -1\right]}{\Delta_x(A^{-1})} \le \exp\left(\frac{1}{2\min^2\{1,M^{-1}\}}\left\|A^{-1}x-x\right\|^2\right) \tag{2.15}$$

for all $x\in H$, $\|x\|=1$.

Since

$$\max\left\{1,m^{-1}\right\} = \frac{1}{\min\{1,m\}},\quad \min\left\{1,M^{-1}\right\} = \frac{1}{\max\{1,M\}}$$

and

$$\frac{\exp\left[\left\langle A^{-1}x,x\right\rangle -1\right]}{\Delta_x(A^{-1})} = \frac{\Delta_x(A)}{\exp\left[1-\langle A^{-1}x,x\rangle\right]},$$

hence by (2.15) we get (2.13). □

Let a, $b>0$ and such that $\frac{b}{a}\in[k,K]\subset(0,\infty)$, then by (2.9) and (2.10):

$$\frac{1}{2}\frac{(b-a)^2}{a^2\max^2\{1,K\}}\le\frac{b-a}{a}-\ln b+\ln a\le\frac{1}{2}\frac{(b-a)^2}{a^2\min^2\{1,k\}} \tag{2.16}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{a^2\max^2\{1,K\}}\le\ln b-\ln a-\frac{b-a}{b}\le\frac{1}{2}\frac{(b-a)^2}{a^2\min^2\{1,k\}}. \tag{2.17}$$

If we assume that a, $b\in[m,M]\subset(0,\infty)$, then by taking $k=\frac{m}{M}<1<\frac{M}{m}=K$ in (2.16) and (2.17) we get

$$\frac{1}{2}\frac{m^2}{M^2}\left(\frac{b}{a}-1\right)^2\le\frac{b-a}{a}-\ln b+\ln a\le\frac{1}{2}\frac{M^2}{m^2}\left(\frac{b}{a}-1\right)^2 \tag{2.18}$$

and

$$\frac{1}{2}\frac{m^2}{M^2}\left(\frac{b}{a}-1\right)^2\le\ln b-\ln a-\frac{b-a}{b}\le\frac{1}{2}\frac{M^2}{m^2}\left(\frac{b}{a}-1\right)^2. \tag{2.19}$$

Observe also that for $t\in[k,K]$ we have

$$1-\frac{\min\{1,t\}}{\max\{1,t\}}\ge1-\frac{\min\{1,K\}}{\max\{1,k\}}\ge0$$

and

$$0\le\frac{\max\{1,t\}}{\min\{1,t\}}-1\le\frac{\max\{1,K\}}{\min\{1,k\}}-1.$$

Now, by (2.7) and (2.8) we get the *global bounds*

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\le t-1-\ln t\le\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.20}$$

and

$$\frac{1}{2}\left(1-\frac{\min\{1,K\}}{\max\{1,k\}}\right)^2\le\ln t-\frac{t-1}{t}\le\frac{1}{2}\left(\frac{\max\{1,K\}}{\min\{1,k\}}-1\right)^2 \tag{2.21}$$

By the use of (2.20), (2.21) and a similar argument to the one in the proof of Theorem 1 we can also state:

Theorem 2. *With the assumptions of Theorem 1 we have*

$$\begin{aligned}1&\le\exp\left(\frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2\right)\le\frac{\exp[\langle Ax,x\rangle-1]}{\Delta_x(A)}\\&\le\exp\left(\frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2\right)\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}1&\le\exp\left(\frac{1}{2}\left(1-\frac{\min\{1,M\}}{\max\{1,m\}}\right)^2\right)\le\frac{\Delta_x(A)}{\exp[\langle A^{-1}x,x\rangle-1]}\\&\le\exp\left(\frac{1}{2}\left(\frac{\max\{1,M\}}{\min\{1,m\}}-1\right)^2\right)\end{aligned} \tag{2.23}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 3. *With the assumptions of Theorem 1 we have*

$$1 \le \exp\left(\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right) \le \frac{\Delta_x(A)}{\exp\left[1 - \langle A^{-1}x, x\rangle\right]} \le \exp\left(\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right) \tag{2.24}$$

and

$$1 \le \exp\left(\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right) \le \frac{\exp\left[1 - \langle Ax, x\rangle\right]}{\Delta_x(A)} \le \exp\left(\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right) \tag{2.25}$$

for all $x \in H$, $\|x\| = 1$.

Remark 2. *If we multiply (2.23) and (2.24) we get*

$$1 \le \exp\left(\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right) \le \Delta_x^2(A) \le \exp\left(\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right),$$

which is equivalent to

$$1 \le \exp\left(\frac{1}{2}\left(1 - \frac{\min\{1, M\}}{\max\{1, m\}}\right)^2\right) \le \Delta_x(A) \le \exp\left(\frac{1}{2}\left(\frac{\max\{1, M\}}{\min\{1, m\}} - 1\right)^2\right), \tag{2.26}$$

for all $x \in H$, $\|x\| = 1$.

3. Related Results

We also have:

Lemma 2. *For any* a, $b > 0$ *we have*

$$(0 \le) \frac{b-a}{a} - \ln b + \ln a \le \frac{(b-a)^2}{ab} \tag{3.1}$$

and

$$(0 \le) \ln b - \ln a - \frac{b-a}{b} \le \frac{(b-a)^2}{ab}. \tag{3.2}$$

Proof. If $b > a$, then

$$\int_a^b \frac{b-t}{t^2} dt \le (b-a) \int_a^b \frac{1}{t^2} dt = (b-a) \frac{b-a}{ab} = \frac{(b-a)^2}{ab}.$$

If $a > b$, then

$$\int_a^b \frac{b-t}{t^2} dt = \int_b^a \frac{t-b}{t^2} dt \le (a-b) \int_b^a \frac{1}{t^2} dt = (a-b) \frac{a-b}{ab} = \frac{(b-a)^2}{ab}.$$

Therefore,

$$\int_a^b \frac{b-t}{t^2}dt \leq \frac{(b-a)^2}{ab}$$

for any $a, b > 0$ and by the representation (2.2) we get the desired result (3.1). $\square$

If we take in (3.1) and (3.2) $a = 1$ and $b = t \in (0, \infty)$, then we get

$$(0 \leq)\, t - 1 - \ln t \leq \frac{(t-1)^2}{t} \tag{3.3}$$

and

$$(0 \leq) \ln t - \frac{t-1}{t} \leq \frac{(t-1)^2}{t} \tag{3.4}$$

for any $t > 0$.

Corollary 4. *Let $a, b > 0$ and such that $\frac{b}{a} \in [k, K] \subset (0, \infty)$. Then we have*

$$\frac{b-a}{a} - \ln b + \ln a \leq U(k, K) \tag{3.5}$$

and

$$\ln b - \ln a - \frac{b-a}{b} \leq U(k, K), \tag{3.6}$$

where

$$U(k, K) := \begin{cases} \frac{(k-1)^2}{k} & \text{if } K < 1, \\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} & \text{if } k \leq 1 \leq K, \\ \frac{(K-1)^2}{K} & \text{if } 1 < k. \end{cases}$$

Proof. Consider the function $f(t) = \frac{(t-1)^2}{t}$, $t > 0$. We observe that

$$f'(t) = \frac{t^2-1}{t^2} \text{ and } f''(t) = \frac{2}{t^3},$$

which shows that f is strictly decreasing on $(0, 1)$, strictly increasing on $[1, \infty)$ and strictly convex for $t > 0$. We also have $f\left(\frac{1}{t}\right) = f(t)$ for $t > 0$.

By (3.3) and by the properties of f we then have that for any $t \in [k, K]$

$$\begin{aligned} t - 1 - \ln t &\leq \max_{t\in[k,K]} \frac{(t-1)^2}{t} \\ &= \begin{cases} \frac{(k-1)^2}{k} & \text{if } K < 1, \\ \max\left\{\frac{(k-1)^2}{k}, \frac{(K-1)^2}{K}\right\} & \text{if } k \leq 1 \leq K, \\ \frac{(K-1)^2}{K} & \text{if } 1 < k. \end{cases} \\ &= U(k, K). \end{aligned} \tag{3.7}$$

Now, put $t = \frac{b}{a} \in [k, K]$ in (3.7) to get the desired inequality (3.5).

Let $y=\frac{1}{t}$ with $t=\frac{b}{a}\in[k,K]$. Then $y\in\left[\frac{1}{K},\frac{1}{k}\right]$ and we have like in (3.7) that

$$\begin{aligned} y-1-\ln y &\leq \max_{y\in[K^{-1},k^{-1}]}\frac{(y-1)^2}{y} \\ &= \begin{cases} \frac{\left(K^{-1}-1\right)^2}{K^{-1}} \text{ if } k^{-1}<1, \\ \max\left\{\frac{\left(K^{-1}-1\right)^2}{K^{-1}},\frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}}\right\} \text{ if } k\leq 1\leq K^{-1}, \\ \frac{\left(\frac{1}{k^{-1}}-1\right)^2}{k^{-1}} \text{ if } 1<K^{-1}, \end{cases} \\ &= U(k,K), \end{aligned}$$

which implies (3.6). □

Remark 3. *If we take $a=1$ and $b=t$ in Corollary 4, then we get*

$$t-1-\ln t\leq U(k,K) \tag{3.8}$$

and

$$\ln t-\frac{t-1}{t}\leq U(k,K), \tag{3.9}$$

for $t\in[k,K]\subset(0,\infty)$.

Theorem 3. *Assume that the operator A satisfies the condition $0<mI\leq A\leq MI$, where m,M are positive numbers, then*

$$(1\leq)\ \frac{\exp\left[\langle Ax,x\rangle-1\right]}{\Delta_x(A)}\leq U(m,M) \tag{3.10}$$

and

$$(1\leq)\ \frac{\Delta_x(A)}{\exp\left[1-\langle A^{-1}x,x\rangle\right]}\leq U(m,M) \tag{3.11}$$

for all $x\in H$, $\|x\|=1$.

Proof. From (2.11) we have

$$0\leq A-I-\ln A\leq U(m,M),$$

namely

$$\langle Ax,x\rangle-1-\langle\ln Ax,x\rangle\leq U(m,M),$$

for all $x\in H$, $\|x\|=1$ and by taking the exponential, we derive (3.10).

The inequality (3.11) follows by (3.9). □

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