INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA OSTROWSKI TYPE RESULTS

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$. In this paper we prove among others that, if $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$, then

$$\begin{aligned}\frac{m}{M} &\leq \left(\frac{m}{M}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]} \\ &\leq \frac{\Delta_x(A)}{I_d(m,M)} \\ &\leq \left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]} \leq \frac{M}{m},\end{aligned}$$

where I_d is the *identric mean*.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [4].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

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(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A+\alpha B)\geq\Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0<\alpha<1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a,\\ \\ a \text{ if } b=a.\end{cases}$$

In [4] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0<mI\leq A\leq MI$, where m,M are positive numbers,

$$0\leq\langle Ax,x\rangle-\Delta_x(A)\leq L(m,M)\left[\ln L(m,M)+\frac{M\ln m-m\ln M}{M-m}-1\right] \tag{1.1}$$

for all $x\in H$, $\|x\|=1$.

We recall that *Specht's ratio* is defined by [9]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ \\ 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [5], the authors obtained the following multiplicative reverse inequality as well

$$1\leq\frac{\langle Ax,x\rangle}{\Delta_x(A)}\leq S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0<mI\leq A\leq MI$ and $x\in H$, $\|x\|=1$.

Motivated by the above results, in this paper we prove among others that, if $0<mI\leq A\leq MI$ and $x\in H$, $\|x\|=1$, then

$$\begin{aligned}\frac{m}{M}&\leq\left(\frac{m}{M}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]}\\&\leq\frac{\Delta_x(A)}{I_d(m,M)}\\&\leq\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]}\leq\frac{M}{m}.\end{aligned}$$

2. Main Results

Recall the *identric mean*

$$I_d(a,b):=\begin{cases}\dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b\neq a\\ \\ a & \text{if} \quad b=a\end{cases};\ a,b>0$$

It is easy to observe that connection between the integral mean of the logarithmic function and the logarithm of the identric mean,

$$\frac{1}{b-a}\int_a^b\ln t\,dt=\ln I_d(a,b)$$

for $a \neq b$ positive numbers.

Theorem 1. *Assume that the operator A satisfies the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers, then*

$$
\begin{aligned}
(2.1)\qquad & \exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right] \\
& \leq \exp\left(-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\left\langle\left(A-\frac{m+M}{2}I\right)^2 x, x\right\rangle\right]\right), \\
& \leq \frac{\Delta_x(A)}{I_d(m,M)} \\
& \leq \exp\left(\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\left\langle\left(A-\frac{m+M}{2}I\right)^2 x, x\right\rangle\right]\right) \\
& \leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right]
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

Also, we have

$$
\begin{aligned}
(2.2)\qquad & \exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right] \\
& \leq \exp\left(-\left(\frac{M}{m}-1\right)\right. \\
& \left.\times\left[\frac{1}{4}+\frac{1}{(m^{-1}-M^{-1})^2}\left\langle\left(A^{-1}-\frac{m^{-1}+M^{-1}}{2}I\right)^2 x, x\right\rangle\right]\right), \\
& \leq \frac{\left[I_d(m^{-1},M^{-1})\right]^{-1}}{\Delta_x(A)} \\
& \leq \exp\left(-\left(\frac{M}{m}-1\right)\right. \\
& \left.\times\left[\frac{1}{4}+\frac{1}{(m^{-1}-M^{-1})^2}\left\langle\left(A^{-1}-\frac{m^{-1}+M^{-1}}{2}I\right)^2 x, x\right\rangle\right]\right) \\
& \leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right]
\end{aligned}
$$

for all $x \in H$, $\|x\| = 1$.

Proof. We use Ostrowski's inequality [8]:

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{s\in(a,b)} |f'(s)| < \infty$, then

$$
(2.3)\qquad \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a),
$$

for all $t \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $f(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$ in (2.3) and observe that

$$\|f'\|_\infty = \sup_{t\in[a,b]} t^{-1} = \frac{1}{a},$$

then we get

$$|\ln t - \ln I_d(a,b)| \le \left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right),$$

for all $t \in [a,b]$.

This inequality is equivalent to

$$-\left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right), \le \ln t - \ln I_d(a,b) \le \left[\frac{1}{4} + \left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right), \tag{2.4}$$

for all $t \in [a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.4) that

$$-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}I + \frac{1}{(M-m)^2}\left(I - \frac{m+M}{2}\right)^2\right], \le \ln A - \ln I_d(m,M)I \le \left(\frac{M}{m}-1\right)\left[\frac{1}{4}I + \frac{1}{(M-m)^2}\left(A - \frac{m+M}{2}I\right)^2\right],$$

which is equivalent to

$$-\left(\frac{M}{m}-1\right)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left\langle\left(A - \frac{m+M}{2}I\right)^2 x, x\right\rangle\right], \le \langle \ln Ax, x\rangle - \ln I_d(m,M) \le \left(\frac{M}{m}-1\right)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left\langle\left(A - \frac{m+M}{2}I\right)^2 x, x\right\rangle\right] \tag{2.5}$$

for all $x \in H$, $\|x\| = 1$.

By taking the exponential in (2.5) we obtain

$$\exp\left(-\left(\frac{M}{m}-1\right)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left\langle\left(A - \frac{m+M}{2}I\right)^2 x, x\right\rangle\right]\right), \le \frac{\exp\langle \ln Ax, x\rangle}{I_d(m,M)} \le \exp\left(\frac{M}{m}-1\right)\left[\frac{1}{4} + \frac{1}{(M-m)^2}\left\langle\left(A - \frac{m+M}{2}I\right)^2 x, x\right\rangle\right]$$

for all $x \in H$, $\|x\| = 1$.

Since

$$\left\langle \left(A - \frac{m+M}{2} I\right)^2 x, x \right\rangle = \left\| Ax - \frac{m+M}{2} x \right\|^2 \le \frac{1}{4} (M-m)^2,$$

hence

$$\frac{1}{4} + \frac{1}{(M-m)^2} \left\langle \left(A - \frac{m+M}{2} I\right)^2 x, x \right\rangle \le \frac{1}{2}$$

and

$$-\frac{1}{2} \le -\left(\frac{1}{4} + \frac{1}{(M-m)^2} \left\langle \left(A - \frac{m+M}{2} I\right)^2 x, x \right\rangle\right)$$

for all $x \in H$, $\|x\| = 1$.

These prove the desired result (2.1).

If $0 < mI \le A \le MI$, then $0 < M^{-1} I \le A^{-1} \le m^{-1} I$ and if we write the inequality (2.1) for A^{-1}, we derive (2.2). $\square$

Theorem 2. *With the assumptions of Theorem 1, we have the inequalities*

$$\frac{m}{M} \le \left(\frac{m}{M}\right)^{\left[\frac{1}{2} + \frac{1}{M-m}\left\langle \left|A - \frac{m+M}{2} I\right| x, x\right\rangle\right]} \le \frac{\Delta_x(A)}{I_d(m, M)} \le \left(\frac{M}{m}\right)^{\left[\frac{1}{2} + \frac{1}{M-m}\left\langle \left|A - \frac{m+M}{2} I\right| x, x\right\rangle\right]} \le \frac{M}{m} \tag{2.6}$$

and

$$\frac{m}{M} \le \left(\frac{m}{M}\right)^{\left[\frac{1}{2} + \frac{1}{m^{-1}-M^{-1}}\left\langle \left|A - \frac{M^{-1}+m^{-1}}{2} I\right| x, x\right\rangle\right]} \le \frac{\left[I_d\left(m^{-1}, M^{-1}\right)\right]^{-1}}{\Delta_x(A)} \le \left(\frac{M}{m}\right)^{\left[\frac{1}{2} + \frac{1}{m^{-1}-M^{-1}}\left\langle \left|A - \frac{M^{-1}+m^{-1}}{2} I\right| x, x\right\rangle\right]} \le \frac{M}{m} \tag{2.7}$$

for all $x \in H$, $\|x\| = 1$.

Proof. In 1997, Dragomir and Wang proved the following Ostrowski type inequality [1]:

Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$, then

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| \le \left[\frac{1}{2} + \frac{\left|t - \frac{a+b}{2}\right|}{b-a}\right] \|f'\|_{[a,b],1}, \tag{2.8}$$

for all $t \in [a, b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a, b]$, i.e., we recall it

$$\|g\|_{[a,b],1} := \int_a^b |g(t)|\, dt.$$

The constant $\frac{1}{2}$ is best possible.

If we take $f(t) = \ln t$, $t \in [a, b] \subset (0, \infty)$ in (2.8) and observe that

$$\|f'\|_{[a,b],1} = \ln b - \ln a,$$

then by (2.8) we get

$$|\ln t-\ln I_d(a,b)|\leq\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a),$$

for all $t\in[a,b]$.

This inequality is equivalent to

$$\begin{aligned}&-\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a)\\&\leq\ln t-\ln I_d(a,b)\\&\leq\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a),\end{aligned}\tag{2.9}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.4) that

$$\begin{aligned}&-(\ln M-\ln m)\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{m+M}{2}I\right|\right]\\&\leq\ln A-\ln I_d(m,M)I\\&\leq(\ln M-\ln m)\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{m+M}{2}I\right|\right],\end{aligned}$$

which is equivalent to

$$\begin{aligned}&-(\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]\\&\leq\langle\ln Ax,x\rangle-\ln I_d(m,M)\\&\leq(\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]\end{aligned}$$

for all $x\in H$, $\|x\|=1$.

By taking the exponential, we derive

$$\begin{aligned}&\exp\left(-(\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]\right)\\&\leq\frac{\exp\langle\ln Ax,x\rangle}{I_d(m,M)}\\&\leq\exp\left((\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\right]\right)\end{aligned}$$

Also

$$\left\langle\left|A-\frac{m+M}{2}I\right|x,x\right\rangle\leq\frac{1}{2}(M-m)$$

for all $x\in H$, $\|x\|=1$.

These prove the desired result (2.6).

The inequality (2.7) follows by (2.6) applied for A^{-1}. □

Theorem 3. *With the assumptions of Theorem 1, we have the inequalities*

$$\exp\left(-\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right) \tag{2.10}$$
$$\leq -\exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right.$$
$$\left.\times\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]x,x\right\rangle^{1/q}\right)$$
$$\leq -\exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right.$$
$$\left.\times\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}x,x\right\rangle\right)$$

$$\leq\frac{\Delta_x(A)}{I_d(m,M)}$$
$$\leq \exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right.$$
$$\left.\times\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}x,x\right\rangle\right)$$
$$\leq \exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right.$$
$$\left.\times\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]x,x\right\rangle^{1/q}\right)$$
$$\leq \exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right)$$

Proof. In 1998, Dragomir and Wang proved the following Ostrowski type inequality [2]:

Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,dt\right| \tag{2.11}$$
$$\leq\frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p},$$

for all $t \in [a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p dt\right)^{1/p}.$$

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (2.11) and observe that

$$\begin{aligned}\|f'\|_{[a,b],p} &:= \left(\int_a^b t^{-p}dt\right)^{1/p} = \left(\frac{b^{-p+1}-a^{-p+1}}{1-p}\right)^{1/p}\\ &= \left(\frac{\frac{1}{b^{p-1}}-\frac{1}{a^{p-1}}}{1-p}\right)^{1/p} = \left[\frac{b^{p-1}-a^{p-1}}{(p-1)\,a^{p-1}b^{p-1}}\right]^{1/p}\\ &= \frac{\left(b^{p-1}-a^{p-1}\right)^{1/p}}{(p-1)^{1/p}\,a^{1-1/p}b^{1-1/p}} = \frac{\left(b^{p-1}-a^{p-1}\right)^{1/p}}{(p-1)^{1/p}\,a^{1/q}b^{1/q}},\end{aligned}$$

then we get

$$\begin{aligned}&|\ln t - \ln I_d(a,b)|\\ &\le \frac{1}{(q+1)^{1/q}(p-1)^{1/p}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{1/q}\\ &\times \frac{(b-a)^{1/q}\left(b^{p-1}-a^{p-1}\right)^{1/p}}{a^{1/q}b^{1/q}}\\ &\le \frac{1}{(q+1)^{1/q}(p-1)^{1/p}}\frac{(b-a)^{1/q}\left(b^{p-1}-a^{p-1}\right)^{1/p}}{a^{1/q}b^{1/q}}\end{aligned} \tag{2.12}$$

for $t\in[a,b]$, since

$$\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\le 1$$

for $t\in[a,b]$.

This implies as above

$$\begin{aligned}&-\frac{1}{(q+1)^{1/q}(p-1)^{1/p}}\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\\ &\times \frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{m^{1/q}M^{1/q}}\\ &\le \ln A - \ln I_d(a,b)\, I\\ &\le \frac{1}{(q+1)^{1/q}(p-1)^{1/p}}\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\\ &\times \frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{m^{1/q}M^{1/q}},\end{aligned}$$

namely

$$
\begin{aligned}
&-\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}} \\
&\times\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}x,x\right\rangle \\
&\leq\langle\ln Ax,x\rangle-\ln I_d(a,b) \\
&\leq\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}} \\
&\times\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}x,x\right\rangle
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$.

By Jensen's inequality for concave functions, we also have that

$$
\begin{aligned}
&\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}x,x\right\rangle \\
&\leq\left\langle\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}x,x\right\rangle^{1/q}
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$ and also

$$
\left\langle\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}x,x\right\rangle^{1/q}\leq 1
$$

for all $x\in H$, $\|x\|=1$.

Now, by taking the exponential and making use of a similar argument as above, we derive the desired result (2.10). □

Corollary 1. *With the assumption of Theorem 1 we have*

$$
\begin{aligned}
&\exp\left(-\frac{M-m}{\sqrt{3mM}}\right) \\
&\leq\exp\left(-\frac{M-m}{\sqrt{3mM}}\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{3}+\left(\frac{MI-A}{M-m}\right)^{3}\right]x,x\right\rangle^{1/2}\right) \\
&\leq\exp\left(-\frac{M-m}{\sqrt{3mM}}\left\langle\left[\left(\frac{A-mI}{M-m}\right)^{3}+\left(\frac{MI-A}{M-m}\right)^{3}\right]^{1/3}x,x\right\rangle\right)
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&\leq \frac{\Delta_x(A)}{I_d\left(m,M\right)}\\
&\leq \exp\left(\frac{M-m}{\sqrt{3mM}}\left\langle\left[\left(\frac{A-mI}{M-m}\right)^3+\left(\frac{MI-A}{M-m}\right)^3\right]^{1/2}x,x\right\rangle\right)\\
&\leq \exp\left(\frac{M-m}{\sqrt{3mM}}\left\langle\left[\left(\frac{A-mI}{M-m}\right)^3+\left(\frac{MI-A}{M-m}\right)^3\right]x,x\right\rangle^{1/q}\right)\\
&\leq \exp\left(\frac{M-m}{\sqrt{3mM}}\right)
\end{aligned}
$$

for all $x\in H$, $\|x\|=1$.

Remark 1. *If we apply the inequality (2.10) for A^{-1}, then we get*

$$
\begin{aligned}
&\exp\left(-\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}M^{-1/q}m^{-1/q}}\right)\\
&\leq -\exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}M^{-1/q}m^{-1/q}}\right.\\
&\quad\left.\times\left\langle\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]x,x\right\rangle^{1/q}\right)\\
&\leq -\exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}M^{-1/q}m^{-1/q}}\right.\\
&\quad\left.\times\left\langle\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]^{1/q}x,x\right\rangle\right)\\
&\leq \frac{\left[I_d\left(m^{-1},M^{-1}\right)\right]^{-1}}{\Delta_x(A)}\\
&\leq \exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}M^{-1/q}m^{-1/q}}\right.\\
&\quad\left.\times\left\langle\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]^{1/q}x,x\right\rangle\right)\\
&\leq \exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}M^{-1/q}m^{-1/q}}\right.\\
&\quad\left.\times\left\langle\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]x,x\right\rangle^{1/q}\right)\\
&\leq \exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}M^{-1/q}m^{-1/q}}\right)
\end{aligned}
\tag{2.14}
$$

for all $x \in H$, $\|x\| = 1$.

3. Related Results

The following results of Ostrowski type holds, see [3]:

Lemma 1. *Let* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on* $[a, b]$. *Then for any* $t \in [a, b]$ *one has the inequality*

$$\begin{aligned} &\frac{1}{2}\left[(b-t)^2 f'_+(t) - (t-a)^2 f'_-(t)\right] \\ &\leq \int_a^b f(s)\,ds - (b-a) f(t) \\ &\leq \frac{1}{2}\left[(b-t)^2 f'_-(b) - (t-a)^2 f'_+(a)\right]. \end{aligned} \tag{3.1}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities. The second inequality also holds for* $t = a$ *or* $t = b$.

If the function is differentiable in $t \in (a, b)$ then the first inequality in (3.1) becomes

$$\left(\frac{a+b}{2} - t\right) f'(t) \leq \frac{1}{b-a}\int_a^b f(s)\,ds - f(t). \tag{3.2}$$

We also have:

Theorem 4. *Assume that the operator* A *satisfies the condition* $0 < mI \leq A \leq MI$, *where* m, M *are positive numbers, then*

$$\begin{aligned} &\exp\left(1 - \frac{m+M}{2}\left\langle A^{-1}x, x\right\rangle\right) \\ &\leq \frac{\Delta_x(A)}{I_d(m, M)} \\ &\leq \exp\left(\frac{1}{m}\|Ax - mx\|^2 - \frac{1}{M}\|Mx - Ax\|^2\right) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\exp\left(1 - \frac{m^{-1}+M^{-1}}{2}\langle Ax, x\rangle\right) \\ &\leq \frac{\left[I_d\left(m^{-1}, M^{-1}\right)\right]^{-1}}{\Delta_x(A)} \\ &\leq \exp\left(M\left\|A^{-1}x - M^{-1}x\right\|^2 - m\left\|m^{-1}x - A^{-1}x\right\|^2\right) \end{aligned} \tag{3.4}$$

for all $x \in H$, $\|x\| = 1$.

Proof. Writing (3.1) and (3.2) for the convex function $f(t) = -\ln t$, then we get

$$1 - \frac{a+b}{2}t^{-1} \leq \ln t - \ln I_d(a, b) \leq \frac{(t-a)^2}{a} - \frac{(b-t)^2}{b},$$

for all $t \in [a, b] \subset (0, \infty)$.

If we use the functional calculus, we get

$$1 - \frac{m+M}{2}A^{-1} \leq \ln A - \ln I_d(m, M) \leq \frac{(A-mI)^2}{m} - \frac{(MI-A)^2}{M},$$

which is equivalent to

$$\begin{aligned}
&1-\frac{m+M}{2}\left\langle A^{-1}x,x\right\rangle \\
&\leq \langle \ln Ax,x\rangle -\ln I_d\left(m,M\right) \\
&\leq \frac{1}{m}\left\langle \left(A-mI\right)^2x,x\right\rangle -\frac{1}{M}\left\langle \left(MI-A\right)^2x,x\right\rangle \\
&=\frac{1}{m}\left\Vert Ax-mx\right\Vert ^2-\frac{1}{M}\left\Vert Mx-Ax\right\Vert ^2
\end{aligned}$$

for all $x\in H$, $\Vert x\Vert =1$.

If we take the exponential, then we get

$$\begin{aligned}
&\exp \left(1-\frac{m+M}{2}\left\langle A^{-1}x,x\right\rangle \right) \\
&\leq \frac{\exp \langle \ln Ax,x\rangle }{I_d\left(m,M\right)} \\
&\leq \exp \left(\frac{1}{m}\left\Vert Ax-mx\right\Vert ^2-\frac{1}{M}\left\Vert Mx-Ax\right\Vert ^2\right)
\end{aligned}$$

for all $x\in H$, $\Vert x\Vert =1$.

The inequality (3.4) follows by (3.1) applied for A^{-1}. □

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