INEQUALITIES FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA TWO VARIABLES LOG INEQUALITIES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$. In this paper we prove among others that

$$\exp\left(\frac{1}{2M^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right) \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le \exp\left(\frac{1}{2m^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right)$$

and

$$\exp\left(\frac{1}{2M^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right) \le \frac{\Delta_x(A)}{\langle Ax, x\rangle \exp\left(\langle Ax, x\rangle^{-1} - \langle A^{-1}x, x\rangle\right)} \le \exp\left(\frac{1}{2m^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right)$$

for $0 < mI \le A \le MI$ and $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [5], [6], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [5].

For each unit vector $x \in H$, see also [8], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

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(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [5] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [6], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

Motivated by the above results, in this paper we prove among others that

$$\begin{aligned}\exp\left(\frac{1}{2M^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right) &\le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \\ &\le \exp\left(\frac{1}{2m^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right)\end{aligned}$$

and

$$\begin{aligned}\exp\left(\frac{1}{2M^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right) &\le \frac{\Delta_x(A)}{\langle Ax, x\rangle\exp\left(\langle Ax, x\rangle^{-1} - \langle A^{-1}x, x\rangle\right)} \\ &\le \exp\left(\frac{1}{2m^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right)\end{aligned}$$

for $0 < mI \le A \le MI$ and $x \in H$ with $\|x\| = 1$.

2. Main Results

We have the following inequalities for logarithm:

Lemma 1. *For any $a,\ b>0$ we have*

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2=\frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \tag{2.1}$$
$$\leq\frac{b-a}{a}-\ln b+\ln a$$
$$\leq\frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}}=\frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

Proof. Observe that

$$\int_a^b\frac{b-t}{t^2}dt=b\int_a^b t^{-2}dt-\int_a^b\frac{1}{t}dt=\frac{b-a}{a}-\ln b+\ln a \tag{2.2}$$

for any $a,\ b>0$.

If $b>a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2}\geq\int_a^b\frac{b-t}{t^2}dt\geq\frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.3}$$

If $a>b$ then

$$\int_a^b\frac{b-t}{t^2}dt=-\int_b^a\frac{b-t}{t^2}dt=\int_b^a\frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2}\geq\int_b^a\frac{t-b}{t^2}dt\geq\frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.4}$$

Therefore, by (2.3) and (2.4) we have for any $a,\ b>0$ that

$$\int_a^b\frac{b-t}{t^2}dt\geq\frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}}=\frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$

and

$$\int_a^b\frac{b-t}{t^2}dt\leq\frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}}=\frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2.$$

By the representation (2.2) we then get the desired result (2.1). □

When some bounds for $a,\ b$ are provided, then we have:

Corollary 1. *Assume that $a,\ b\in[m,M]\subset(0,\infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2}\leq\frac{b-a}{a}-\ln b+\ln a\leq\frac{1}{2}\frac{(b-a)^2}{m^2} \tag{2.5}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{M^2}\leq\ln b-\ln a-\frac{b-a}{b}\leq\frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{2.6}$$

The first main result is as follows:

Theorem 1. *Assume that the operator A satisfies the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers, then for $a \in [m, M]$ we have*

$$(1 \leq) \exp\left(\frac{1}{2M^2}\|Ax - ax\|^2\right) \leq \frac{a\exp\left[\frac{\langle Ax,x\rangle - a}{a}\right]}{\Delta_x(A)} \leq \exp\left(\frac{1}{2m^2}\|Ax - ax\|^2\right) \tag{2.7}$$

and

$$(1 \leq) \exp\left(\frac{1}{2M^2}\|Ax - ax\|^2\right) \leq \frac{\Delta_x(A)}{a\exp\left(a^{-1} - \langle A^{-1}x, x\rangle\right)} \leq \exp\left(\frac{1}{2m^2}\|Ax - ax\|^2\right) \tag{2.8}$$

for all $x \in H$, $\|x\| = 1$.

Proof. By utilizing the continuous functional calculus for selfadjoint operators, we have by (2.5) that

$$\frac{1}{2}\frac{(A - aI)^2}{M^2} \leq \frac{A - aI}{a} - \ln A + \ln aI \leq \frac{1}{2}\frac{(A - aI)^2}{m^2}$$

for all $a \in [m, M]$.

For $x \in H$, $\|x\| = 1$ we then have

$$\frac{1}{2M^2}\left\langle (A - aI)^2 x, x\right\rangle \leq \frac{\langle Ax, x\rangle - a}{a} - \langle \ln Ax, x\rangle + \ln a \leq \frac{1}{2m^2}\left\langle (A - aI)^2 x, x\right\rangle,$$

namely

$$\frac{1}{2M^2}\|Ax - ax\|^2 \leq \frac{\langle Ax, x\rangle - a}{a} - \langle \ln Ax, x\rangle + \ln a \leq \frac{1}{2m^2}\|Ax - ax\|^2,$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential, then we get

$$\exp\left(\frac{1}{2M^2}\|Ax - ax\|^2\right) \leq \exp\left[\frac{\langle Ax, x\rangle - a}{a} + \ln a - \langle \ln Ax, x\rangle\right] \leq \exp\left(\frac{1}{2m^2}\|Ax - ax\|^2\right), \tag{2.9}$$

for $x \in H$, $\|x\| = 1$.

Now, observe that

$$\exp\left[\frac{\langle Ax, x\rangle - a}{a} + \ln a - \langle \ln Ax, x\rangle\right] = \frac{a\exp\left[\frac{\langle Ax,x\rangle - a}{a}\right]}{\exp\left(\langle \ln Ax, x\rangle\right)} \tag{2.10}$$

and by (2.9) we derive (2.7).

From (2.23) we get

$$\frac{1}{2}\frac{(A - a)^2}{M^2} \leq \ln A - \ln a - (A - a)A^{-1} \leq \frac{1}{2}\frac{(A - aI)^2}{m^2},$$

namely

$$\begin{aligned}\frac{1}{2M^2}\|Ax-ax\|^2 &\leq \langle \ln Ax, x\rangle - \ln a - 1 + a\left\langle A^{-1}x, x\right\rangle \\ &\leq \frac{1}{2m^2}\|Ax-ax\|^2,\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By taking the exponential, we get

$$\begin{aligned}\exp\left(\frac{1}{2M^2}\|Ax-ax\|^2\right) &\leq \exp\left[\langle \ln Ax, x\rangle - \ln a - 1 + a\left\langle A^{-1}x, x\right\rangle\right] \\ &\leq \exp\left(\frac{1}{2m^2}\|Ax-ax\|^2\right),\end{aligned} \tag{2.11}$$

for $x \in H$, $\|x\| = 1$.

Observe that

$$\begin{aligned}&\exp\left[\langle \ln Ax, x\rangle - \ln a - 1 + a\left\langle A^{-1}x, x\right\rangle\right] \\ &= \exp\left[\langle \ln Ax, x\rangle - \ln a - a\left(a^{-1} - \left\langle A^{-1}x, x\right\rangle\right)\right] \\ &= \frac{\exp\langle \ln Ax, x\rangle}{a\exp\left(a^{-1} - \langle A^{-1}x, x\rangle\right)}\end{aligned}$$

and by (2.11) we get (2.8). □

The following particular case is of interest:

Corollary 2. *Assume that the operator A satisfies the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers, then*

$$\begin{aligned}&(1 \leq) \exp\left(\frac{1}{2M^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right) \\ &\leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \\ &\leq \exp\left(\frac{1}{2m^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&(1 \leq) \exp\left(\frac{1}{2M^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right) \\ &\leq \frac{\Delta_x(A)}{\langle Ax, x\rangle \exp\left(\langle Ax, x\rangle^{-1} - \langle A^{-1}x, x\rangle\right)} \\ &\leq \exp\left(\frac{1}{2m^2}\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)\right)\end{aligned} \tag{2.13}$$

for all $x \in H$, $\|x\| = 1$.

Proof. If we take $a = \langle Ax, x\rangle \in [m, M]$, $x \in H$, $\|x\| = 1$ in (2.7), then we get

$$\exp\left(\frac{1}{2M^2}\|Ax - \langle Ax, x\rangle x\|^2\right) \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq \exp\left(\frac{1}{2m^2}\|Ax - \langle Ax, x\rangle x\|^2\right).$$

Observe that

$$\begin{aligned} \|Ax-\langle Ax,x\rangle x\|^{2} &= \|Ax\|^{2}-2\operatorname{Re}\langle Ax,\langle Ax,x\rangle x\rangle+\|\langle Ax,x\rangle x\|^{2} \\ &= \|Ax\|^{2}-2\left|\langle Ax,x\rangle\right|^{2}+\left|\langle Ax,x\rangle\right|^{2} \\ &= \|Ax\|^{2}-\langle Ax,x\rangle^{2} \end{aligned}$$

for $x\in H$, $\|x\|=1$ and the inequality (2.12) is obtained.

The inequality (2.13) follows by (2.8). □

Remark 1. *If we use the inequality, see for instance* [7, p. 27]

$$\|Ax\|^{2}=\left\langle A^{2}x,x\right\rangle \leq \frac{(M+m)^{2}}{4mM}\langle Ax,x\rangle^{2}$$

for $0<mI\leq A\leq MI$, *where* m,M *are positive numbers and* $x\in H$, $\|x\|=1$, *then*

$$\|Ax\|^{2}-\langle Ax,x\rangle^{2}\leq \frac{(M-m)^{2}}{4mM}\langle Ax,x\rangle^{2}$$

and by Corollary 2 we derive

$$\frac{\langle Ax,x\rangle}{\Delta_{x}(A)}\leq \exp\left(\frac{(M-m)^{2}}{8m^{3}M}\langle Ax,x\rangle^{2}\right) \tag{2.14}$$

and

$$\frac{\Delta_{x}(A)}{\langle Ax,x\rangle\exp\left(\langle Ax,x\rangle^{-1}-\langle A^{-1}x,x\rangle\right)}\leq \exp\left(\frac{(M-m)^{2}}{8m^{3}M}\langle Ax,x\rangle^{2}\right) \tag{2.15}$$

for $x\in H$, $\|x\|=1$.

Also, by using the inequality

$$\|Ax\|^{2}-\langle Ax,x\rangle^{2}\leq \frac{1}{4}(M-m)^{2}$$

for $x\in H$, $\|x\|=1$, *then by Corollary 2 we derive*

$$\frac{\langle Ax,x\rangle}{\Delta_{x}(A)}\leq \exp\left(\frac{1}{8}\left(\frac{M}{m}-1\right)^{2}\right) \tag{2.16}$$

and

$$\frac{\Delta_{x}(A)}{\langle Ax,x\rangle\exp\left(\langle Ax,x\rangle^{-1}-\langle A^{-1}x,x\rangle\right)}\leq \exp\left(\frac{1}{8}\left(\frac{M}{m}-1\right)^{2}\right) \tag{2.17}$$

for $x\in H$, $\|x\|=1$.

Corollary 3. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} &(1\leq)\exp\left(\frac{1}{2M^{2}}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}+\left(\langle Ax,x\rangle-\left\langle A^{-1}x,x\right\rangle^{-1}\right)^{2}\right)\right) \\ &\leq \frac{\left\langle A^{-1}x,x\right\rangle^{-1}\exp\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right]}{\Delta_{x}(A)} \\ &\leq \exp\left(\frac{1}{2m^{2}}\left(\|Ax\|^{2}-\langle Ax,x\rangle^{2}+\left(\langle Ax,x\rangle-\left\langle A^{-1}x,x\right\rangle^{-1}\right)^{2}\right)\right) \end{aligned} \tag{2.18}$$

and

$$(1\leq)\exp\left(\frac{1}{2M^2}\left(\|Ax\|^2-\langle Ax,x\rangle^2+\left(\langle Ax,x\rangle-\left\langle A^{-1}x,x\right\rangle^{-1}\right)^2\right)\right)\tag{2.19}$$
$$\leq\frac{\Delta_x(A)}{\left\langle A^{-1}x,x\right\rangle^{-1}}$$
$$\leq\exp\left(\frac{1}{2m^2}\left(\|Ax\|^2-\langle Ax,x\rangle^2+\left(\langle Ax,x\rangle-\left\langle A^{-1}x,x\right\rangle^{-1}\right)^2\right)\right)$$

for $x\in H$, $\|x\|=1$.

Proof. It follows by Theorem 1 on taking $a=\left\langle A^{-1}x,x\right\rangle^{-1}$

$$\begin{aligned}\left\|Ax-\left\langle A^{-1}x,x\right\rangle^{-1}x\right\|^2&=\|Ax\|^2-2\operatorname{Re}\left\langle Ax,\left\langle A^{-1}x,x\right\rangle^{-1}x\right\rangle+\left\langle A^{-1}x,x\right\rangle^{-2}\\&=\|Ax\|^2-2\left\langle A^{-1}x,x\right\rangle^{-1}\langle Ax,x\rangle+\left\langle A^{-1}x,x\right\rangle^{-2}\\&=\|Ax\|^2-\langle Ax,x\rangle^2+\left(\langle Ax,x\rangle-\left\langle A^{-1}x,x\right\rangle^{-1}\right)^2.\end{aligned}$$

□

Corollary 4. *With the assumptions of Theorem 1 and if* $m\leq 1\leq M$

$$\exp\left(\frac{1}{2M^2}\|Ax-x\|^2\right)\leq\frac{\exp\left[\langle Ax,x\rangle-1\right]}{\Delta_x(A)}\leq\exp\left(\frac{1}{2m^2}\|Ax-x\|^2\right)\tag{2.20}$$

and

$$\exp\left(\frac{1}{2M^2}\|Ax-x\|^2\right)\leq\frac{\Delta_x(A)}{\exp\left(1-\langle A^{-1}x,x\rangle\right)}\leq\exp\left(\frac{1}{2m^2}\|Ax-x\|^2\right)\tag{2.21}$$

for all $x\in H$, $\|x\|=1$.

Proof. It follows by Theorem 1 for $a=1$. □

Corollary 5. *With the assumptions of Theorem 1 we have*

$$(1\leq)\exp\left(\frac{1}{2M^2}\left\|Ax-\frac{m+M}{2}x\right\|^2\right)\tag{2.22}$$
$$\leq\frac{\frac{m+M}{2}\exp\left[\frac{\langle Ax,x\rangle}{\frac{m+M}{2}}-1\right]}{\Delta_x(A)}$$
$$\leq\exp\left(\frac{1}{2m^2}\left\|Ax-\frac{m+M}{2}x\right\|^2\right)\leq\exp\left(\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right)$$

and

$$(1\leq)\exp\left(\frac{1}{2M^2}\left\|Ax-\frac{m+M}{2}x\right\|^2\right)\tag{2.23}$$
$$\leq\frac{\Delta_x(A)}{\frac{m+M}{2}\exp\left(\left(\frac{m+M}{2}\right)^{-1}-\langle A^{-1}x,x\rangle\right)}$$
$$\leq\exp\left(\frac{1}{2m^2}\left\|Ax-\frac{m+M}{2}x\right\|^2\right)\leq\exp\left(\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right)$$

for all $x\in H$, $\|x\|=1$.

3. Related Results

We also have:

Lemma 2. *For any a, $b > 0$ we have*

$$(0 \leq) \frac{b-a}{a} - \ln b + \ln a \leq \frac{(b-a)^2}{ab} \tag{3.1}$$

and

$$(0 \leq) \ln b - \ln a - \frac{b-a}{b} \leq \frac{(b-a)^2}{ab}. \tag{3.2}$$

Proof. If $b > a$, then

$$\int_a^b \frac{b-t}{t^2} dt \leq (b-a) \int_a^b \frac{1}{t^2} dt = (b-a) \frac{b-a}{ab} = \frac{(b-a)^2}{ab}.$$

If $a > b$, then

$$\int_a^b \frac{b-t}{t^2} dt = \int_b^a \frac{t-b}{t^2} dt \leq (a-b) \int_b^a \frac{1}{t^2} dt = (a-b) \frac{a-b}{ab} = \frac{(b-a)^2}{ab}.$$

Therefore,

$$\int_a^b \frac{b-t}{t^2} dt \leq \frac{(b-a)^2}{ab}$$

for any a, $b > 0$ and by the representation (2.2) we get the desired result (3.1). □

Theorem 2. *For $A > 0$ and $a > 0$ we have the inequalities*

$$(1 \leq) \frac{a \exp\left[\frac{\langle Ax,x\rangle - a}{a}\right]}{\Delta_x(A)} \leq \exp\left(\frac{\langle Ax,x\rangle}{a} + a\left\langle A^{-1}x,x\right\rangle - 2\right) \tag{3.3}$$

and

$$(1 \leq) \frac{\Delta_x(A)}{a \exp\left(a^{-1} - \langle A^{-1}x,x\rangle\right)} \leq \exp\left(\frac{\langle Ax,x\rangle}{a} + a\left\langle A^{-1}x,x\right\rangle - 2\right) \tag{3.4}$$

for all $x \in H$, $\|x\| = 1$.

Proof. Using the continuous functional calculus for selfadjoint operators and the inequality (3.1) we get

$$(0 \leq) \frac{A - aI}{a} - \ln A + \ln aI \leq \frac{A}{a} + aA^{-1} - 2,$$

which is equivalent to

$$\frac{\langle Ax,x\rangle}{a} - 1 - \langle \ln Ax,x\rangle + \ln a \leq \frac{\langle Ax,x\rangle}{a} + a\left\langle A^{-1}x,x\right\rangle - 2$$

for all $x \in H$, $\|x\| = 1$.

If we take the exponential, then we get

$$\exp\left(\frac{\langle Ax,x\rangle}{a} - 1 - \langle \ln Ax,x\rangle + \ln a\right) \leq \exp\left(\frac{\langle Ax,x\rangle}{a} + a\left\langle A^{-1}x,x\right\rangle - 2\right),$$

for all $x \in H$, $\|x\| = 1$.

By utilizing the equality (2.10) we then obtain the desired result (3.3).

The inequality (3.4) follows by (3.2) in a similar way. □

Corollary 6. *For $A>0$ we have*

$$(1\leq)\,\frac{\langle Ax,x\rangle}{\Delta_x(A)}\leq\exp\left(\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right) \tag{3.5}$$

and

$$(1\leq)\,\frac{\Delta_x(A)}{\langle Ax,x\rangle\exp\left(\langle Ax,x\rangle^{-1}-\langle A^{-1}x,x\rangle\right)}\leq\exp\left(\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right) \tag{3.6}$$

for all $x\in H$, $\|x\|=1$.

The proof follows by Theorem 2 for $a=\langle Ax,x\rangle$, $x\in H$, $\|x\|=1$.

Corollary 7. *For $A>0$ we have*

$$(1\leq)\,\frac{\Delta_x(A)}{\langle A^{-1}x,x\rangle^{-1}}\leq\exp\left(\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle-1\right) \tag{3.7}$$

for all $x\in H$, $\|x\|=1$.

The proof follows by Theorem 2 for $a=\left\langle A^{-1}x,x\right\rangle^{-1}$, $x\in H$, $\|x\|=1$.

Remark 2. *If we use Kantorovich inequality, see for instance* [7, p. 24] *that holds for an operator A that satisfies the condition $0<mI\leq A\leq MI$, where m,M are positive numbers,*

$$\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle\leq\frac{(M+m)^2}{4mM}$$

then by Corollaries 6 and 7 we derive

$$(1\leq)\,\frac{\langle Ax,x\rangle}{\Delta_x(A)}\leq\exp\left(\frac{(M-m)^2}{4mM}\right), \tag{3.8}$$

$$(1\leq)\,\frac{\Delta_x(A)}{\langle Ax,x\rangle\exp\left(\langle Ax,x\rangle^{-1}-\langle A^{-1}x,x\rangle\right)}\leq\exp\left(\frac{(M-m)^2}{4mM}\right) \tag{3.9}$$

and

$$(1\leq)\,\frac{\Delta_x(A)}{\langle A^{-1}x,x\rangle^{-1}}\leq\exp\left(\frac{(M-m)^2}{4mM}\right) \tag{3.10}$$

for $x\in H$, $\|x\|=1$.

If we use the additive inequality, see for instance [7, p. 28]

$$\left\langle A^{-1}x,x\right\rangle-\langle Ax,x\rangle^{-1}\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm},$$

which implies by multiplying with $\langle Ax,x\rangle>0$ that

$$\left\langle A^{-1}x,x\right\rangle\langle Ax,x\rangle-1\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}\langle Ax,x\rangle\leq\left(\sqrt{\frac{M}{m}}-1\right)^2,$$

then by Corollaries 6 and 7 we obtain

$$(3.11) \qquad (1 \leq) \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq \exp\left(\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}\langle Ax, x\rangle\right) \leq \exp\left(\left(\sqrt{\frac{M}{m}}-1\right)^2\right),$$

$$(3.12) \qquad (1 \leq) \frac{\Delta_x(A)}{\langle Ax, x\rangle \exp\left(\langle Ax, x\rangle^{-1} - \langle A^{-1}x, x\rangle\right)} \leq \exp\left(\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}\langle Ax, x\rangle\right) \leq \exp\left(\left(\sqrt{\frac{M}{m}}-1\right)^2\right)$$

and

$$(3.13) \qquad (1 \leq) \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq \exp\left(\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{Mm}\langle Ax, x\rangle\right) \leq \exp\left(\left(\sqrt{\frac{M}{m}}-1\right)^2\right)$$

for $x \in H$, $\|x\| = 1$.

Corollary 8. *For* $A > 0$ *we have the inequalities*

$$(3.14) \qquad (1 \leq) \frac{\exp\left[\langle Ax, x\rangle - 1\right]}{\Delta_x(A)} \leq \exp\left(\langle Ax, x\rangle + \left\langle A^{-1}x, x\right\rangle - 2\right)$$

and

$$(3.15) \qquad (1 \leq) \frac{\Delta_x(A)}{\exp\left(1 - \langle A^{-1}x, x\rangle\right)} \leq \exp\left(\langle Ax, x\rangle + \left\langle A^{-1}x, x\right\rangle - 2\right)$$

for all $x \in H$, $\|x\| = 1$.

The proof follows by Theorem 2 for $a = 1$.

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